STUDIA MATHEMATICA, T. LXXXVIII. (1988)

On the differentiation of integrals of functions from $L\varphi(L)$

by

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Abstract. The following alternative is established for any translation invariant differentiation basis B of rectangles with sides parallel to the coordinate axes: either B differentiates the integral of every summable function, or for every class $L\varphi(L)$ with $\varphi(t) = o(\ln t)$ as $t \to \infty$ there is a function whose integral is not differentiated by B. A geometric characteristic is introduced which permits to decide which class, L or $L\log^+ L$, is precisely differentiated by a given basis. Also, a scale of non-translation invariant bases of rectangles with sides parallel to the axes is constructed which differentiate precisely the classes $L\varphi(L)$ intermediate between L and $L\log^+ L$. The results obtained, together with the theorems of Lebesgue and Jessen, Marcinkiewicz and Zygmund, yield a complete description of the behaviour of differentiation bases of rectangles with sides parallel to the axes. Applications to the theory of multiple Fourier series and extensions from $\mathbf{R}^2$ to the multidimensional case are also given.

1. Introduction. A *differentiation basis at a point* $x \in \mathbf{R}^N$ is a collection $B(x)$ of bounded open subsets of $\mathbf{R}^N$ containing x such that there is a sequence $\{R_k\} \subset B(x)$ with $\operatorname{diam} R_k \to 0$ as $k \to \infty$. The family $B = \{R\colon R \in B(x),\ x \in \mathbf{R}^N\}$ is then called a *differentiation basis* in $\mathbf{R}^N$. A differentiation basis is called *translation invariant* (briefly: a *TI-basis*) if it contains all translates of any of its elements.

If a basis B has the property that for each R in B, if $x \in R$ then $R \in B(x)$, then B is called a *Busemann–Feller basis* (a *BF-basis*).

We define the *upper* and *lower derivatives of the integral* of a locally integrable function f at a point x with respect to a basis B by

$$\bar{D}_B\left(\int f, x\right) = \sup_{\substack{\{R_k\} \subset B(x) \\ \operatorname{diam} R_k \to 0}} \limsup_{k \to \infty} |R_k|^{-1} \int_{R_k} f(y)\, dy,$$

$$\underline{D}_B\left(\int f, x\right) = \inf_{\substack{\{R_k\} \subset B(x) \\ \operatorname{diam} R_k \to 0}} \liminf_{k \to \infty} |R_k|^{-1} \int_{R_k} f(y)\, dy.$$

We say that a basis B *differentiates the integral of* f if $\bar{D}_B\left(\int f, x\right) = \underline{D}_B\left(\int f, x\right) = f(x)$ a.e. If B differentiates the integral of every function $f \in \Phi(L)$ (for the definition of the classes $\Phi(L)$, see e.g. [14, p. 650]) then we say that B *differentiates* $\Phi(L)$; if for every function g on $\mathbf{R}_+$ with $g(t) \downarrow 0$ as $t \to \infty$ there exists an $f \in g(L)\Phi(L)$ with $\bar{D}_B\left(\int f, x\right) = +\infty$ a.e., then we say that B *does not differentiate* $o(\Phi(L))$. Finally, B *differentiates precisely* $\Phi(L)$

(written $B \in D(\Phi(L))$) if B differentiates $\Phi(L)$ and does not differentiate $o(\Phi(L))$.

We denote by $\boldsymbol{B}_s(\boldsymbol{R}^N)$, $s = 1, \ldots, N$, the Busemann–Feller TI-basis consisting of all rectangular parallelepipeds R, where

$$R = \{(x_1, \ldots, x_N) \in \boldsymbol{R}^N\colon \alpha_i < x_i < \alpha_i + \gamma_i,\ i = 1, \ldots, N,\quad \text{with } \gamma_j = \gamma \text{ for } j = 1, \ldots, s\}.$$

The following classical results are fundamental in the theory of differentiation of integrals in $\boldsymbol{R}^N$:

$-\boldsymbol{B}_N(\boldsymbol{R}^N)$ differentiates $L(\boldsymbol{R}^N)$ (H. Lebesgue [7], 1910).

$-\boldsymbol{B}_1(\boldsymbol{R}^N)$ differentiates $L(\log^+ L)^{N-1}(\boldsymbol{R}^N)$ (B. Jessen, J. Marcinkiewicz and A. Zygmund [5], 1935).

$-\boldsymbol{B}_1(\boldsymbol{R}^N)$ does not differentiate $o\big(L(\log^+ L)^{N-1}\big)(\boldsymbol{R}^N)$ (S. Saks [9], 1935).

$-\boldsymbol{B}_s(\boldsymbol{R}^N)$ differentiates $L(\log^+ L)^{N-s}(\boldsymbol{R}^N)$, $s = 1, \ldots, N-1$ (A. Zygmund [16], 1967).

$-\boldsymbol{B}_s(\boldsymbol{R}^N)$ does not differentiate $o\big(L(\log^+ L)^{N-s}\big)(\boldsymbol{R}^N)$, $s = 1, \ldots, N-1$ (see e.g. [12, Theorem 2]; [8]).

It follows from the above results that the differentiation properties of a basis can be improved by making it sufficiently rare. A. Zygmund proposed the following rarefaction of the basis $\boldsymbol{B}_1(\boldsymbol{R}^2)$ (see [4, Ch. 6, § 4]).

Let B be the TI-basis consisting of the rectangles for which $D^2 \leqslant d \leqslant D \leqslant 1$, where d, D are the lengths of the smaller and larger sides respectively. Is it then true that B differentiates $L\sqrt{\log^+ L}$? R. Moriyón proved (see [4, App. IV]) that this is not the case: B does not differentiate $o(L\log^+ L)$. This shows that a rarefaction of this kind does not improve the differentiation properties of the basis.

It turns out that no rarefaction within the class of TI-bases permits the differentiation properties of bases to be improved in a continuous way. More precisely, if B is a TI-basis then either B differentiates L, or B does not differentiate $o(L\log^+ L)$.

In the present paper we prove the above alternative and show how to rarefy a basis in order to obtain a basis which differentiates precisely a given class $L\varphi(L)$ intermediate between L and $L\log^+ L$.

The main results of this paper were announced in [13].

The author would like to express his deep gratitude to Professor V. G. Krotov, under whose guidance this work was done, for formulating the problems, valuable advice and constant attention.

2. Main results. All differentiation bases considered in this paper are BF-bases consisting of rectangular parallelepipeds with edges parallel to the coordinate axes.

In order to clarify the idea of the problem, we formulate and prove our main results in the case of $\boldsymbol{R}^2$, which, for the most cases, is typical. N-dimensional versions are considered in Section 4.

First, we introduce a geometric characteristic of a basis consisting of rectangles. Let $B \subset \boldsymbol{B}_1$. For every rectangle $R \in B$ we denote by R^* the concentric rectangle of minimal measure containing R with side-lengths of the form 2^k, $k \in \boldsymbol{Z}$. Thus to every basis B we attach, in a natural way, another basis $B^* = \{R^*\colon R \in B\}$, called the basis *associated* to B.

Further, we will say that two rectangles R and R' are *comparable*, and write $R \sim R'$, if there is a translation placing one of them inside the other. Otherwise we call them *incomparable* and write $R \not\sim R'$.

We say that a basis B has *property* (S) if

$$\text{(S)}\qquad \forall \varepsilon > 0\ \forall k \in \boldsymbol{N}\ \exists \{R_i\}_{i=1}^k \subset B^*\colon\quad R_i \not\sim R_j\ (i \neq j),\quad \operatorname{diam} R_i < \varepsilon,\ i = 1, \ldots, k;$$

i.e. we can find an arbitrary number of arbitrarily small pairwise incomparable rectangles in the associated basis.

Property (S) permits us to formulate a criterion to decide which of the classes L or $L\log^+ L$ is precisely differentiated by a given TI-basis:

THEOREM 1. *Let B be a TI-basis with $B \subset \boldsymbol{B}_1$. Then if B has property* (S) *then it does not differentiate $o(L\log^+ L)$, and if B fails property* (S) *then it differentiates L.*

In connection with this theorem, the problem arises whether there exist at all bases of rectangles with sides parallel to the coordinate axes which differentiate precisely a given class $L\varphi(L)$ intermediate between L and $L\log^+ L$. An answer is given by the following theorem.

THEOREM 2. *Let $\varphi(t)$ be an increasing concave function with $\varphi(0) = 0$ and such that $\varphi(t)/\ln t$ is decreasing for $t \geqslant t_0 > 1$. Then there is a basis B with $B \subset \boldsymbol{B}_1$ such that $B \in D\big(L\varphi(L)\big)$.*

3. Proofs of the main results. The main tool in the proof of Theorem 1 is the following lemma.

LEMMA 1. *Suppose a basis B has property* (S). *Then for arbitrary $\varepsilon > 0$ and $k \in \boldsymbol{N}$ there are sets $\Theta = \Theta(\varepsilon, k)$ and $Y = Y(\varepsilon, k)$ such that*

$$\text{(1)}\qquad \Theta \subset Y, \qquad \operatorname{diam} Y < \varepsilon,$$

$$\text{(2)}\qquad |Y| \geqslant k 2^{k-1} |\Theta|,$$

$$\text{(3)}\qquad \forall x \in Y\ \exists R \in B^*(x)\colon\quad \operatorname{diam} R < \varepsilon, \qquad |R \cap \Theta|/|R| \geqslant 2^{-k}.$$

Proof. By property (S) we can find $k+1$ pairwise incomparable rectangles $\{R_\nu\}_{\nu=0}^k$ in B^* of diameter less than ε. Let $|\mathrm{pr}_1 R_\nu| = 2^{-m_\nu}$, $|\mathrm{pr}_2 R_\nu| =$

2^{-n_ν}, $\nu = 0, \ldots, k$, where $\mathrm{pr}_i E$, $i = 1, 2$, is the projection of the set E on the corresponding coordinate axis and $|\cdot|$ denotes the Lebesgue measure of the corresponding dimension. Assume that $m_0 > \ldots > m_k$, $n_0 < \ldots < n_k$. Define

$$J = [0, 2^{-m_k}] \times [0, 2^{-n_0}],$$
$$\Theta = \{x = (x^1, x^2): \forall i, j \ (0 \leqslant i, j \leqslant k): r_{m_i}(x^1) + r_{n_j}(x^2) = 2\} \cap J,$$
$$Y_\nu = \{x = (x^1, x^2): \forall i, j \ (0 \leqslant j \leqslant \nu \leqslant i \leqslant k): r_{m_i}(x^1) + r_{n_j}(x^2) = 2\} \cap J,$$
$$Y = \bigcup_{\nu=0}^{k} Y_\nu,$$

where $r_m(t)$ are the Rademacher functions.

As can easily be seen, Θ is a union of $2^{-2k} 2^{n_k - n_0} 2^{m_0 - m_k}$ disjoint rectangles whose projections are dyadic-rational intervals of length 2^{-n_k} and 2^{-m_0} respectively, and Y_ν is a union of $2^{-k} 2^{-n_0 + n_\nu} 2^{-m_k + m_\nu}$ disjoint rectangles whose projections are dyadic-rational intervals of lengths 2^{-n_ν} and 2^{-m_ν} respectively. Hence $|\Theta| = 2^{-2k}|J|$, $|Y_\nu| = 2^{-k}|J|$, $|J| = 2^{-m_k} 2^{-n_0}$.

Further, it is easily seen that

$$\Big|Y_\nu \cap \bigcup_{p=1}^{\nu-1} Y_p\Big| \leqslant \tfrac{1}{2}|Y_\nu|, \quad \nu = 2, \ldots, k.$$

Therefore

$$|Y| \geqslant \tfrac{1}{2} \sum_{\nu=0}^{k} |Y_\nu| = \tfrac{1}{2}(k+1) 2^{-k}|J| = \tfrac{1}{2}(k+1) 2^{k}|\Theta|,$$

and so (1) and (2) are proved.

Let now ν be any integer between 0 and k, and let R be any of the rectangles that form Y_ν. It follows from the definitions of Θ and Y_ν that $\Theta \subset Y_\nu$, $R \in B^*$ and

$$\frac{|R \cap \Theta|}{|R|} = \frac{|Y_\nu \cap \Theta|}{|Y_\nu|} = \frac{|\Theta|}{|Y_\nu|} = \frac{1}{2^k},$$

which implies (3) and completes the proof of the lemma.

Remark 1. Write

$$M^\varepsilon f(x) = \sup_{\substack{R \in \boldsymbol{B}_1(x) \\ \operatorname{diam} R < \varepsilon}} |R|^{-1} \int_R |f(y)|\, dy$$

for the truncated strong maximal operator. Then Lemma 1 essentially means that for arbitrary $\varepsilon > 0$ and $k \in N$ there are sets Θ and Y such that

$Y \subset \{x: M^\varepsilon \chi_\Theta(x) > 2^{-k}\}$ and so

$$|\{x: M^\varepsilon \chi_\Theta(x) > 2^{-k}\}| \geqslant c \int \frac{\chi_\Theta}{2^{-k}} \ln^+ \frac{\chi_\Theta}{2^{-k}} dx,$$

with c independent of ε, k, Θ. This inequality is a converse of the well-known weak type estimate for the strong maximal operator. Such inequalities are of great importance in differentiation theory and constitute the main tool both for proving positive results and for constructing counterexamples. For more details, see [1]–[3], [10].

We now turn to the proof of Theorem 1.

Suppose that a basis B has property (S), and let $g(t) \downarrow 0$ as $t \to \infty$. Denote by $\tilde{B}$ the basis obtained from B by dilation with coefficient $\frac{1}{2}$. Clearly, $\tilde{B}$ also has property (S), and applying Lemma 1 to it we obtain sequences of sets $\{\Theta_k\}_{k=1}^\infty$, $\{Y_k\}_{k=1}^\infty$ such that $\operatorname{diam} Y_k \downarrow 0$ as $k \to \infty$ and $\sum_{k=1}^\infty k 2^k |\Theta_k| < \infty$.

Then there are numbers $m_k \in N$, $k = 1, 2, \ldots$, such that

$$\sum_{k=1}^\infty k 2^k |\Theta_k|\, m_k = \infty, \qquad \sum_{k=1}^\infty g(2^k) k 2^k |\Theta_k|\, m_k < \infty. \tag{4}$$

Moreover, let α_k be numbers such that $\alpha_k \uparrow \infty$ as $k \to \infty$ and

$$\sum_{k=1}^\infty g(\alpha_k 2^k) k \alpha_k 2^k |\Theta_k|\, m_k < \infty. \tag{5}$$

Write

$$N_0 = 0, \quad N_j = \sum_{i=1}^{j} m_i, \quad j = 1, 2, \ldots$$

We define sequences of sets $\{E_k\}_{k=1}^\infty$, $\{G_k\}_{k=1}^\infty$ by

$$E_k = \Theta_j, \quad G_k = Y_j \quad \text{for } N_{j-1} < k \leqslant N_j.$$

Then by (4)

$$\sum_{k=1}^\infty |G_k| = \sum_{j=1}^\infty |Y_j|\, m_j \geqslant \tfrac{1}{4} \sum_{j=1}^\infty j 2^j |\Theta_j|\, m_j = \infty.$$

Since $\operatorname{diam} G_k \downarrow 0$ as $k \to \infty$, by the well-known Calderón lemma (see e.g. [15, Ch. XIII, Lemma (1.24)]) there are translations τ_k such that

$$|\limsup_{k \to \infty} \tau_k G_k| = 1. \tag{6}$$

Define

$$f_k(x) = \alpha_j 2^j \chi_{\tau_k E_k}(x), \quad N_{j-1} < k \leqslant N_j, \quad f(x) = \sup_{k \in N} f_k(x).$$

Putting $\Phi(t) = tg(t)\log^+ t$ we obtain by (5)

$$\int_{[0,1]^2} \Phi(f(x))\,dx \leqslant \sum_{k=1}^{\infty} \int_{[0,1]^2} \Phi(f_k(x))\,dx = \sum_{j=1}^{\infty} \Phi(\alpha_j 2^j)|\Theta_j|\, m_j < \infty,$$

and so $f \in g(L)L\log^+ L([0, 1]^2)$.

On the other hand, by (6) almost every point x belongs to an infinite sequence of sets $\tau_k G_k$. Since $\tau_k G_k$ and $\tau_k E_k$ satisfy a relation of the type (3), for any $x \in \tau_k G_k$ with $N_{j-1} < k \leqslant N_j$ we can find a rectangle $R \in \tilde{B}^*(x)$ such that

$$|R|^{-1} \int_R f(y)\,dy \geqslant |R|^{-1} \int_R f_k(y)\,dy = \frac{\alpha_j 2^j |R \cap \tau_k E_k|}{|R|} \geqslant \alpha_j.$$

Since $\alpha_j \uparrow \infty$ as $j \to \infty$ $(k \to \infty)$, we conclude that $\bar{D}_{\tilde{B}^*}(\int f, x) = +\infty$ a.e. on $[0, 1]^2$, and the obvious relation $\bar{D}_B(\int f, x) \geqslant \frac{1}{4}\bar{D}_{\tilde{B}^*}(\int f, x)$ completes the proof of the first part of Theorem 1.

In the proof of the second part we will use the following known facts. Let B' be a TI-basis generated by translates of rectangles from a monotonic family $\{R_\alpha\}_{\alpha \geqslant 1}$, $R_\alpha \subset R_\beta$ for $\alpha > \beta$. Then B' differentiates L, and almost every point is a Lebesgue point with respect to B'. Consequently, the integral of any summable function is differentiated at almost every point by any basis B'' regular with respect to B' (see [11, Ch. I, 5.3(d) and 1.8]). But it is not hard to see that if a basis B fails property (S), then B^* decomposes into a finite number of bases generated by monotonic families of rectangles, and into a collection of rectangles with diameters greater than some $\varepsilon_0 > 0$. Hence B^* differentiates L, and since B is regular with respect to B^* (see [11, Ch. I, 1.8]), it follows that B also differentiates L and the proof of Theorem 1 is complete.

We now turn to the proof of Theorem 2. As is known, the differentiation properties of a basis are closely related to its covering properties. In this connection we introduce the following property (V_Ψ) of weak overlapping, where Ψ is an increasing convex function with $\Psi(0) = 0$. We say that a basis B has *property* (V_Ψ) if there are constants $c_j > 0$ $(j = 1, 2, 3)$, $n_0 \in N$, $m_0 \in N$ $(n_0 \leqslant m_0)$ such that for any system $\{R_\alpha\}_{\alpha \in A} \subset B$ we can find a subsystem $\{R_{\alpha_i}\}$ satisfying

$$\int_W \Psi\big(c_1(\sum_i \chi_{R_{\alpha_i}}(x) - n_0)\big)\,dx \leqslant c_2 \sum_i |R_{\alpha_i}| \tag{7}$$

where $W = \{x\colon \sum_i \chi_{R_{\alpha_i}}(x) \geqslant m_0\}$,

$$\Big|\bigcup_{\alpha \in A} R_\alpha\Big| \leqslant c_3 \sum_i |R_{\alpha_i}| \tag{8}$$

(for the particular case of property $(\mathrm{V}_{\exp})$, see [4, App. II]). Without loss of generality we can assume that $c_2 \geqslant 1$.

We define the maximal operator corresponding to the basis B by

$$M_B f(x) = \sup_{R \in B(x)} |R|^{-1} \int_R |f(y)|\,dy.$$

LEMMA 2. *Let $\Phi(t)$, $\Psi(t)$ be Young conjugate convex functions with $\Phi(t)$ satisfying the Δ_2 condition. Assume that a basis B has property (V_Ψ). Then the maximal operator M_B is of weak type $L + \Phi(L)$:*

$$|\{x\colon M_B f(x) > \lambda\}| \leqslant c_4 \int_{\{M_B f > \lambda\}} (|f|/\lambda + \Phi(|f|/\lambda))\,dy \tag{9}$$

for all $f \in L \cap \Phi(L)$ and $\lambda > 0$.

Proof. Let

$$\{x\colon M_B f(x) > \lambda\} = \bigcup_x R_x, \qquad R_x \in B, \qquad |R_x|^{-1} \int_{R_x} |f(y)|\,dy > \lambda.$$

Take $\{R_{x_i}\}$ satisfying (7), (8). Define

$$U = \bigcup_i R_{x_i}, \qquad W = \{x\colon \sum_i \chi_{R_{x_i}}(x) \geqslant m_0\}.$$

Then

$$\begin{aligned}
\sum_i |R_{x_i}| &\leqslant \sum_i \int_{R_{x_i}} \frac{|f(y)|}{\lambda}\,dy = \int_U \big(\sum_i \chi_{R_{x_i}}\big)\frac{|f|}{\lambda}\,dy \\
&\leqslant \int_W \frac{c_1}{2c_2}\big(\sum_i \chi_{R_{x_i}} - n_0\big)\frac{2c_2}{c_1}\cdot\frac{|f|}{\lambda}\,dy + m_0 \int_U \frac{|f|}{\lambda}\,dy \\
&\equiv J_1 + J_2 \qquad (2c_2 > 1).
\end{aligned}$$

By the Young inequality

$$J_1 \leqslant \int_W \Psi\Big(\frac{c_1}{2c_2}\big(\sum_i \chi_{R_{x_i}} - n_0\big)\Big)\,dy + \int_U \Phi\Big(\frac{2c_2}{c_1}\cdot\frac{|f|}{\lambda}\Big)\,dy \equiv J_3 + J_4.$$

Since $2c_2 > 1$ and $\Psi(t)$ is convex, we obtain by (7)

$$J_3 \leqslant \frac{1}{2c_2} \int_W \Psi\big(c_1(\sum_i \chi_{R_{x_i}} - n_0)\big)\,dy \leqslant \tfrac{1}{2}\sum_i |R_{x_i}|,$$

and since $\Phi(t)$ satisfies the Δ_2 condition,

$$J_4 \leqslant c_5 \int_U \Phi(|f|/\lambda)\,dy.$$

It follows that

$$\tfrac{1}{2}\sum_i |R_{x_i}| \leqslant c_6 \int_U (|f|/\lambda + \Phi(|f|/\lambda))\,dy,$$

and so

$$|\{x\colon M_B f(x) > \lambda\}| = |\bigcup_x R_x| \leqslant c_3 \sum_i |R_{x_i}| \leqslant c_4 \int_{\bigcup_x R_x} (|f|/\lambda + \Phi(|f|/\lambda))\,dy,$$

which completes the proof of Lemma 2.

Using Lemma 2 and a standard technique (see e.g. [2, Ch. II, § 1]) it is easy to obtain the following lemma.

LEMMA 3. *Let $\Phi(t)$, $\Psi(t)$ be Young conjugate convex functions with $\Phi(t)$ satisfying the Δ_2 condition. Then if a basis B has property* (V_Ψ) *then B differentiates $\Phi(L)$.*

The following lemma gives a method of constructing bases which differentiate precisely $\Phi(L)$, provided certain covering properties of a simple collection of sets are known. In the present section we restrict our attention to the classes $L\varphi(L)$ which are close to L; more precisely, it will be assumed that the inverse function to $\varphi(t)$, denoted by $\Psi(t)$, satisfies the Δ_3 condition:

$$(\Delta_3) \qquad \exists c_7 > 0,\ t_0 > 0\ \forall t \geqslant t_0\colon \quad t\Psi(t) \leqslant \Psi(c_7 t).$$

(For the general version, see Section 4.)

LEMMA 4. *Let $\varphi(t)$ be an increasing concave function with $\varphi(0) = 0$, and suppose $\Psi(t)$, the inverse function to $\varphi(t)$, satisfies the Δ_3 condition. Moreover, suppose there are a collection of bounded open sets $\sigma = \{R_i^j\}$, $j = 1, 2, \ldots$, $i = 1, 2, \ldots, n_j$, and constants $c_s > 0$, m_0, $n_0 \in N$ such that*

$$(10) \qquad |R_i^j| = |R_\nu^j|, \quad j \geqslant 1,\ i, \nu = 1, \ldots, n_j,$$

$$(11) \qquad \forall j \geqslant 1\ \forall \{R_{i_k}^j\} \subset \{R_i^j\}\colon$$

$$\int_{W_j} \Psi\Big(c_8\big(\sum_k \chi_{R_{i_k}^j}(x) - n_0\big)\Big)\,dx \leqslant c_9 \sum_k |R_{i_k}^j|$$

$$\textit{where } W_j = \{x\colon \sum_k \chi_{R_{i_k}^j}(x) \geqslant m_0\},$$

and such that there are measurable sets E_j and numbers λ_j, $\lambda_j \uparrow \infty$ as $j \to \infty$, satisfying

$$(12) \qquad \frac{|R_i^j \cap E_j|}{|R_i^j|} \geqslant \frac{c_{10}}{\lambda_j}, \quad j \geqslant 1, \quad i = 1, \ldots, n_j,$$

$$(13) \qquad |\bigcup_{i=1}^{n_j} R_i^j| \geqslant c_{11} \lambda_j \varphi(\lambda_j) |E_j|, \quad j \geqslant 1.$$

Then there is a basis B whose every element is a dilation of some member of σ and such that B differentiates precisely $L\varphi(L)$.

Proof. Since (10)–(13) are dilation invariant, we can assume without loss of generality that

$$E_j \subset \bigcup_{i=1}^{n_j} R_i^j \subset [0, 1]^2, \quad j \geqslant 1,$$

and introduce the notation

$$(14) \qquad I = [0, 1]^2, \quad X^j = \bigcup_{i=1}^{n_j} R_i^j,$$

$$\omega_j = [(\lambda_j \varphi(\lambda_j) |E_j|)^{-1}], \quad S_k = \sum_{j=1}^{k} \omega_j.$$

Clearly,

$$(15) \qquad \sum_{j=1}^{\infty} \omega_j |X^j| = \infty.$$

We now start constructing the basis in question. Let $\{m_n\}_{n=1}^{\infty}$ be an increasing sequence of positive integers tending to infinity, to be defined later. Let $S_0 = 0$ and let $n \in N$, $S_{k-1} < n \leqslant S_k$. We divide I into m_n^2 equal squares I_n^ν:

$$I = \bigcup_{\nu=1}^{m_n^2} I_n^\nu, \quad |I_n^\nu| = m_n^{-2}, \quad \nu = 1, \ldots, m_n^2.$$

Denote by H_n^ν dilation with coefficient m_n^{-1} taking X^k into I_n^ν and let

$$R_n^{\nu,i} = H_n^\nu(R_i^k), \quad H_n = \bigcup_{\nu=1}^{m_n^2} H_n^\nu(X^k), \quad \beta_n = |R_n^{\nu,i}|.$$

In this notation

$$H_n^\nu(X^k) = \bigcup_{i=1}^{n_k} H_n^\nu(R_i^k) = \bigcup_{i=1}^{n_k} R_n^{\nu,i}.$$

We now show that the numbers m_n can be chosen so rapidly increasing that $|\limsup H_n| = 1$.

At the jth step of our construction we add to the collection B the sets $R_j^{\nu,i}$, $\nu = 1, \ldots, m_j^2$, $i = 1, \ldots, n_k$, where $S_{k-1} < j \leqslant S_k$. The numbers n_k depend in general on j. Therefore to avoid additional indexation, from now on we write $\bar{n}_j$ in place of n_k.

Let $m_1 = 1$. Then $\bigcup_{i=1}^{\bar{n}_1} R_1^{1,i} \subset I$. Choose m_2 so large that

$$(16) \qquad |\{\bigcup I_2^\nu\colon I_2^\nu \cap (\bigcup_{i=1}^{\bar{n}_1} \partial R_1^{1,i}) \neq \emptyset\}| \leqslant \min(2^{-1}\beta_1, 2^{-1}(1 - |H_1|)),$$

and let $A_1 = \{\nu\colon I_2^\nu \cap H_1 = \emptyset\}$, $A_2 = \{\nu\colon I_2^\nu \cap \partial H_1 \neq \emptyset\}$. Obviously

$$I \setminus H_1 \subset \Big(\bigcup_{\nu \in A_1} I_2^\nu\Big) \cup \Big(\bigcup_{\nu \in A_2} I_2^\nu\Big),$$

and therefore

$$\Big|\bigcup_{\nu \in A_1} I_2^\nu\Big| \geqslant (1-|H_1|) - \Big|\bigcup_{\nu \in A_2} I_2^\nu\Big|.$$

But it is easily seen that

$$\bigcup_{\nu \in A_2} I_2^\nu \subset \Big\{\bigcup I_2^\nu\colon I_2^\nu \cap \Big(\bigcup_{i=1}^{\bar{n}_1} \partial R_1^{1,i}\Big) \neq \emptyset\Big\},$$

so that from (16) we obtain

$$\Big|\bigcup_{\nu \in A_2} I_2^\nu\Big| \leqslant \tfrac{1}{2}(1-|H_1|)$$

and hence

$$\Big|\bigcup_{\nu \in A_1} I_2^\nu\Big| \geqslant \tfrac{1}{2}(1-|H_1|).$$

Moreover, it is clear that $|H_2^\nu(X^1)|/|I_2^\nu| = |H_2|/|I| = |H_2|$, and so $|H_2^\nu(X^1)| = |H_2| \cdot |I_2^\nu|$. Consequently,

$$\begin{aligned}|(I\setminus H_1)\setminus H_2| &\leqslant \Big|(I\setminus H_1)\setminus \bigcup_{\nu \in A_1} H_2^\nu(X^1)\Big| \\ &= 1-|H_1|-|H_2|\cdot\Big|\bigcup_{\nu\in A_1} I_2^\nu\Big| \leqslant (1-\tfrac{1}{2}|H_1|)(1-\tfrac{1}{2}|H_2|).\end{aligned}$$

Suppose that we have already chosen $m_1, \ldots, m_{n-1}$ in such a way that

$$\begin{aligned}(17)\quad &\Big|\Big\{\bigcup I_q^\nu\colon I_q^\nu \cap \bigcup_{p=1}^{q-1} \bigcup_{i=1}^{\bar{n}_p} \bigcup_{\nu=1}^{m_p^2} \partial R_p^{\nu,i} \neq \emptyset\Big\}\Big| \\ &\qquad \leqslant \min\Big(2^{-q}\beta_{q-1},\, 2^{-1}\Big|I\setminus \bigcup_{s=1}^{q-1} H_s\Big|\Big), \quad q = 3, \ldots, n-1,\end{aligned}$$

$$(18)\qquad \Big|I \setminus \bigcup_{s=p}^{r} H_s\Big| \leqslant \prod_{s=p}^{r} (1-\tfrac{1}{2}|H_s|), \quad 1 \leqslant p \leqslant r \leqslant n-1.$$

Choose m_n large enough that

$$(19)\quad \Big|\Big\{\bigcup I_n^\nu\colon I_n^\nu \cap \bigcup_{p=1}^{n-1} \bigcup_{i=1}^{\bar{n}_p} \bigcup_{\nu=1}^{m_p^2} \partial R_p^{\nu,i} \neq \emptyset\Big\}\Big| \leqslant \min\Big(2^{-n}\beta_{n-1},\, 2^{-1}\Big|I\setminus\bigcup_{s=1}^{n-1} H_s\Big|\Big)$$

and put

$$A_1^n = \Big\{\nu\colon I_n^\nu \cap \bigcup_{s=q}^{n-1} H_s = \emptyset\Big\}, \qquad A_2^n = \Big\{\nu\colon I_n^\nu \cap \partial\Big(\bigcup_{s=q}^{n-1} H_s\Big) \neq \emptyset\Big\}$$

where q is a fixed integer between 1 and n. Obviously,

$$\bigcup_{\nu\in A_2^n} I_n^\nu \subset \Big\{\bigcup I_n^\nu\colon I_n^\nu \cap \bigcup_{p=1}^{n-1} \bigcup_{i=1}^{\bar{n}_p} \bigcup_{\nu=1}^{m_p^2} \partial R_p^{\nu,i} \neq \emptyset\Big\}.$$

It follows from (19) that

$$\Big|\bigcup_{\nu\in A_2^n} I_n^\nu\Big| \leqslant \tfrac{1}{2}\Big|I\setminus\bigcup_{s=1}^{n-1} H_s\Big| \leqslant \tfrac{1}{2}\Big|I\setminus\bigcup_{s=q}^{n-1} H_s\Big|.$$

Moreover,

$$I\setminus\bigcup_{s=q}^{n-1} H_s \subset \Big(\bigcup_{\nu\in A_1^n} I_n^\nu\Big) \cup \Big(\bigcup_{\nu\in A_2^n} I_n^\nu\Big),$$

and hence

$$\Big|\bigcup_{\nu\in A_1^n} I_n^\nu\Big| \geqslant \Big|I\setminus\bigcup_{s=q}^{n-1} H_s\Big| - \Big|\bigcup_{\nu\in A_2^n} I_n^\nu\Big| \geqslant \tfrac{1}{2}\Big|I\setminus\bigcup_{s=q}^{n-1} H_s\Big|.$$

Since $|H_n^\nu(X^k)| = |H_n|\cdot|I_n^\nu|$, we therefore obtain

$$\begin{aligned}\Big|I\setminus\bigcup_{s=q}^{n} H_s\Big| &\leqslant \Big|\Big(I\setminus\bigcup_{s=q}^{n-1} H_s\Big)\setminus \bigcup_{\nu\in A_1^n} H_n^\nu(X^k)\Big| \\ &= \Big|I\setminus\bigcup_{s=q}^{n-1} H_s\Big| - |H_n|\cdot\Big|\bigcup_{\nu\in A_1^n} I_n^\nu\Big| \leqslant \prod_{s=q}^{n}(1-\tfrac{1}{2}|H_s|).\end{aligned}$$

The choice of the numbers m_n is thus fully described, and clearly

$$\Big|I\setminus\bigcup_{s=q}^{\infty} H_s\Big| \leqslant \prod_{s=q}^{\infty}(1-\tfrac{1}{2}|H_s|), \qquad \forall q \geqslant 1.$$

The infinite product on the right diverges to zero provided $\sum_{n=1}^\infty |H_n| = \infty$, and this follows easily from (15). Indeed, by dilation, $|H_n| = |X^k|$ with $S_{k-1} < n \leqslant S_k$, and so

$$\sum_{n=1}^\infty |H_n| = \sum_{k=1}^\infty \sum_{n=S_{k-1}+1}^{S_k} |H_n| = \sum_{k=1}^\infty \omega_k |X^k| = \infty.$$

But then we have $|\limsup H_n| = 1$, and therefore almost every point $x \in I$ belongs to an infinite sequence $\{R_{k_n}^{\nu_{m},i_l}\}$ with diameters tending to zero.

We will show that the collection $\{R_n^{\nu,i}\}$ has property (V_Ψ), and so if we adjoin to it all dilated copies of $R_1^{1,1}$, we obtain a basis B which differentiates $L\varphi(L)$.

We introduce additional notation to simplify the writing. Instead of $R_k^{\nu,i}$ we will write R_α where $\alpha = (k, i, \nu)$ is a multiindex. In this notation, extracting a subsequence $\{R_{k_n}^{\nu_m,i_l}\}$ from a sequence $\{R_k^{\nu,i}\}$ means that we take a subsystem $\{R_\alpha\}_{\alpha\in A'}$, $A' \subset A$, of a system $\{R_\alpha\}_{\alpha\in A}$ with some set of indices A.

Below we use both notations, which should cause no confusion.

Let therefore $\{R_\alpha\}_{\alpha\in A}\subset B$. Without loss of generality we may assume that A is countable and write $\{R_\alpha\}_{\alpha\in A}$ in the form $\{R_{l_k}^{q_\nu,h_s}\}$. Put

$$Y_k^\nu=\bigcup_s R_{l_k}^{q_\nu,h_s},\qquad Y_k=\bigcup_\nu Y_k^\nu.$$

We will show how to choose a subsystem of $\{R_\alpha\}_{\alpha\in A}$ with the weak overlapping property (V_Ψ). Let

$$\bar{Y}_1=Y_1,\qquad \bar{Y}_k=\{Y_k\setminus Y_k^\nu\colon\ Y_k^\nu\subset\bigcup_{i=1}^{k-1}\bar{Y}_i\},\qquad k\geqslant 2.$$

Obviously, $\left|\bigcup_{k=1}^\infty Y_k\right|=\left|\bigcup_{k=1}^\infty \bar{Y}_k\right|$. Furthermore, put

$$\tilde{Y}_1=Y_1,\qquad \tilde{Y}_k=\{\bar{Y}_k\setminus Y_k^\nu\colon\ Y_k^\nu\cap\partial\Big(\bigcup_{i=1}^{k-1}\bar{Y}_i\Big)\neq\emptyset\},\qquad k\geqslant 2,$$

and, finally, let

$$\tilde{Y}_i=\bigcup_j Y_{k_i}^{\nu_j},\qquad Y_{k_i}^{\nu_j}=\bigcup_{\alpha\in A_i^j}R_\alpha,\qquad A'=\bigcup_{i,j}A_i^j.$$

Define

$$v(\tilde{Y}_i)=\{\bigcup Y_i^\nu\colon\ Y_i^\nu\subset\bar{Y}_i,\ Y_i^\nu\cap\partial\Big(\bigcup_{j=1}^{i-1}\tilde{Y}_j\Big)\neq\emptyset\},\qquad i\geqslant 2.$$

It follows from (17) that $|v(\tilde{Y}_i)|\leqslant 2^{-i}\left|\bigcup_{j=1}^{i-1}\tilde{Y}_j\right|$ and clearly

$$\Big|\bigcup_{i=1}^\infty\tilde{Y}_i\Big|+\Big|\bigcup_{i=2}^\infty v(\tilde{Y}_i)\Big|\geqslant\Big|\bigcup_{i=1}^\infty\bar{Y}_i\Big|.$$

But

$$\Big|\bigcup_{i=2}^\infty v(\tilde{Y}_i)\Big|\leqslant\sum_{i=2}^\infty|v(\tilde{Y}_i)|\leqslant\sum_{i=2}^\infty 2^{-i}\Big|\bigcup_{j=1}^{i-1}\tilde{Y}_j\Big|\leqslant\Big|\bigcup_{i=1}^\infty\tilde{Y}_i\Big|.$$

Therefore

$$\Big|\bigcup_{i=1}^\infty\tilde{Y}_i\Big|\geqslant\Big|\bigcup_{i=1}^\infty\bar{Y}_i\Big|-\Big|\bigcup_{i=2}^\infty v(\tilde{Y}_i)\Big|\geqslant\Big|\bigcup_{i=1}^\infty\bar{Y}_i\Big|-\Big|\bigcup_{i=1}^\infty\tilde{Y}_i\Big|,$$

that is,

$$2\Big|\bigcup_{i=1}^\infty\tilde{Y}_i\Big|\geqslant\Big|\bigcup_{i=1}^\infty\bar{Y}_i\Big|=\Big|\bigcup_{i=1}^\infty Y_i\Big|.$$

This means that $\left|\bigcup_{\alpha\in A}R_\alpha\right|\leqslant 2\left|\bigcup_{\alpha\in A'}R_\alpha\right|$, and an inequality of the type (8) is proved. Further, it is easily seen that the sets $\tilde{Y}_k$ are pairwise disjoint and on each of them we have a (V_Ψ) estimate by condition (11) of the lemma. Hence

$$\int_W\Psi\Big(c_8\Big(\sum_{\alpha\in A'}\chi_{R_\alpha}(x)-n_0\Big)\Big)dx=\sum_{i,j\geqslant 1}\int_{U_{i,j}}\Psi\Big(c_8\Big(\sum_{\alpha\in A_i^j}\chi_{R_\alpha}(x)-n_0\Big)\Big)dx$$

$$\leqslant c_9\sum_{i,j\geqslant 1}\Big|\bigcup_{\alpha\in A_i^j}R_\alpha\Big|\leqslant c_9\sum_{\alpha\in A'}|R_\alpha|,$$

where $W=\{x\colon \sum_{\alpha\in A'}\chi_{R_\alpha}(x)\geqslant m_0\}$, $U_{i,j}=\bigcup_{\alpha\in A_i^j}R_\alpha$.

We have thus established estimates (7) and (8), i.e. property (V_Ψ). Moreover, since Ψ satisfies the Δ_3 condition, its conjugate function is equivalent to $t\varphi(t)$ (see [6, Th. 6.1]), and we conclude by Lemma 3 that B differentiates $L\varphi(L)$.

We now show that B does not differentiate $o(L\varphi(L))$. Let $g(t)\downarrow 0$ as $t\to\infty$. Then there are numbers $w_k\in N$, $0\leqslant w_k\leqslant\omega_k$, such that

$$\sum_{k=1}^\infty w_k|X^k|=\infty,\tag{20}$$

$$\sum_{k=1}^\infty w_k|X^k|\sqrt{g(\lambda_k)}<\infty,\tag{21}$$

where the λ_j are taken from condition (12) of the lemma. Write $\varepsilon_k=\gamma_k\varphi(\gamma_k\lambda_k)/\varphi(\lambda_k)$, where $\gamma_k\uparrow\infty$ slowly enough that

$$\varepsilon_k=o\big(g(\lambda_k)^{-1/2}\big)\quad\text{as } k\to\infty.\tag{22}$$

In every I_n^ν with $S_{k-1}<n\leqslant S_{k-1}+w_k$ we place $H_n^\nu(E_k)$, a delated copy of E_k (the dilation H_n^ν has been defined at the beginning of the proof). Put

$$Q_n=\begin{cases}\bigcup_{\nu=1}^{m_n^2}H_n^\nu(E_k), & S_{k-1}<n\leqslant S_{k-1}+w_k,\\ \emptyset, & S_{k-1}+w_k<n\leqslant S_k,\end{cases}$$

and define the functions

$$f_n(x)=\gamma_k\lambda_k\chi_{Q_n}(x),\qquad S_{k-1}<n\leqslant S_k,\qquad f(x)=\sup_{n\in N}f_n(x).$$

Write $F(t)=g(t)t\varphi(t)$, $t\geqslant 0$. Obviously,

$$\int_I F(f(x))\,dx\leqslant\sum_{k=1}^\infty\int_I F(f_k(x))\,dx=\sum_{k=1}^\infty\sum_{n=S_{k-1}+1}^{S_{k-1}+w_k}F(\gamma_k\lambda_k)|Q_n|$$

$$=\sum_{k=1}^\infty g(\gamma_k\lambda_k)\gamma_k\lambda_k\varphi(\gamma_k\lambda_k)w_k|E_k|$$

$$\leqslant\sum_{k=1}^\infty w_k\varepsilon_k g(\lambda_k)\lambda_k\varphi(\lambda_k)|E_k|.$$

By (21), (22), (13) we obtain

$$\int_I F(f(x))\,dx \leqslant c_{12}\sum_k w_k |X^k| \sqrt{g(\lambda_k)} < \infty .$$

Thus $f \in g(L)\,L\varphi(L)([0, 1]^2)$.

We now show that $\bar{D}_B(\int f, x) = +\infty$ a.e. on $[0, 1]^2$. By repeating the considerations used in the construction of the basis it is not difficult to show that almost every $x \in I$ belongs to an infinite sequence $\{R_{l_n}^{q_\nu, h_s}\}$. If $S_{k-1} < l_n \leqslant S_{k-1} + w_k$, then we obtain by (12) (writing $R = R_{l_n}^{q_\nu, h_s}$ for simplicity)

$$|R|^{-1}\int_R f(y)\,dy \geqslant |R|^{-1}\int_R f_{l_n}(y)\,dy \geqslant \frac{\lambda_k \gamma_k |R \cap H_{l_n}^{q_\nu}(E_k)|}{|R|} \geqslant c_{13}\gamma_k .$$

Since $\gamma_k \uparrow \infty$ as $k \to \infty$, it follows that $\bar{D}_B(\int f, x) = +\infty$ a.e. on $[0, 1]^2$, i.e. B does not differentiate $o(L\varphi(L))([0, 1]^2)$. Decomposing $\boldsymbol{R}^2$ into a union of unit squares we obtain a basis in $\boldsymbol{R}^2$. Thus the proof of Lemma 4 is complete.

Using Lemma 4 it is not difficult to prove Theorem 2. First, we need some additional information about the function $\varphi(t)$. We show that

$$\varphi(ab) \leqslant \varphi(a) + \varphi(b), \qquad \forall\, a, b \geqslant t_0 . \tag{23}$$

Indeed, we obviously have $\varphi(ab)/\ln(ab) \leqslant \varphi(a)/\ln a$ and $\varphi(ab)/\ln(ab) \leqslant \varphi(b)/\ln b$, $\forall a, b \geqslant t_0$, i.e. $\ln a/\ln(ab) \leqslant \varphi(a)/\varphi(ab)$ and $\ln b/\ln(ab) \leqslant \varphi(b)/\varphi(ab)$. Adding the last two inequalities gives (23). For the inverse function $\Psi(t)$ we then obtain

$$\Psi(a)\,\Psi(b) \leqslant \Psi(a+b), \qquad \forall\, a, b \geqslant t_0 . \tag{24}$$

Further, let $c_{14} = \varphi(t_0)/t_0$. By the concavity of $\varphi(t)$ we have

$$\varphi(t) \leqslant c_{14} t, \qquad \forall t \geqslant t_0 . \tag{25}$$

Hence

$$t\Psi(t) = \Psi(\varphi(t))\,\Psi(t) \leqslant \Psi(\varphi(t) + t) \leqslant \Psi((c_{14}+1)\,t), \qquad \forall t \geqslant t_0,$$

i.e. $\Psi(t)$ satisfies the Δ_3 condition:

$$t\Psi(t) \leqslant \Psi(c_{15} t), \qquad \forall t \geqslant t_0 . \tag{26}$$

We introduce the function

$$F(t) = \begin{cases} t^{-2}\,\Psi(t), & t > 0, \\ 0, & t \leqslant 0. \end{cases}$$

By (26), $t^2\,\Psi(t) \leqslant \Psi(c_{15}^2 t)/c_{15}$, and so

$$\Psi(t) \leqslant c_{16} F(c_{17} t), \qquad t \geqslant t_0 . \tag{27}$$

Put

$$m_0 = [c_{17} t_0 + 2]. \tag{28}$$

Without loss of generality we assume that $\Psi(m_0) > 1$. For $m \geqslant 2m_0$ set

$$R_i^m = \left[0, \frac{1}{\Psi(i)}\right] \times \left[0, \frac{\Psi(i)}{\Psi(m_0+m)}\right], \qquad i = m_0, \ldots, m_0+m,$$

$$E_m = \left[0, \frac{1}{\Psi(m_0+m)}\right] \times \left[0, \frac{\Psi(m_0)}{\Psi(m_0+m)}\right], \qquad \lambda_m = \Psi(m_0+m).$$

We prove that $\{R_i^m\}_{i=m_0}^{m_0+m}$ satisfies relations of the form (10)–(13). The equality (10) is obvious, (11) will be proved later, and now we establish (12) and (13). It is easily seen that

$$\frac{|R_i^m \cap E_m|}{|R_i^m|} = \frac{|E_m|}{|R_i^m|} = \frac{\Psi(m_0)}{\Psi(m_0+m)} \geqslant \frac{1}{\Psi(m_0+m)} = \frac{1}{\lambda_m}.$$

Further, it follows from (24) that

$$\frac{\Psi(i+m_0)}{\Psi(i)} \geqslant \Psi(m_0) > 1, \qquad \forall i \geqslant t_0 .$$

Thus there exists a constant $c_{18} > 0$ such that

$$\sum_{i=m_0}^{m_0+m} |R_i^m| \leqslant 2m_0 \sum_{i=1}^{[m/m_0-1]} |R_{i\cdot m_0}^m| \leqslant c_{18} \Big|\bigcup_{i=m_0}^{m_0+m} R_i^m\Big|,$$

and hence

$$\Big|\bigcup_{i=m_0}^{m_0+m} R_i^m\Big| \geqslant \frac{1}{c_{18}} m |R_i^m| = \frac{m}{c_{18}\Psi(m_0+m)} \geqslant \frac{m_0+m}{2c_{18}\Psi(m_0+m)}$$
$$= \frac{(m_0+m)\,\Psi(m_0+m)}{2c_{18}\Psi(m_0)} \cdot \frac{\Psi(m_0)}{\Psi(m_0+m)^2}$$
$$= \frac{(m_0+m)\,\Psi(m_0+m)}{2c_{18}\Psi(m_0)} |E_m| \geqslant c_{19}\lambda_m \varphi(\lambda_m)\,|E_m| .$$

We have thus proved (12) and (13). To prove (11), take $n \in N$, $m_0 < n \leqslant m_0+m$, $\{i_j\}_{j=1}^n \subset \{m_0, \ldots, m_0+m\}$, $i_0 = 0$, $1/\Psi(i_{n+1}) = 0$. Set

$$J = \int_{W_m} F\Big(\sum_{j=1}^n \chi_{R_{i_j}^m}(x) - 1\Big)\,dx, \qquad W_m = \Big\{x\colon \sum_{j=1}^n \chi_{R_{i_j}^m}(x) \geqslant m_0+1\Big\},$$

$$E_{\nu,k} = \left[\frac{1}{\Psi(i_{\nu+1})}, \frac{1}{\Psi(i_\nu)}\right] \times \left[\frac{\Psi(i_{\nu-k-1})}{\Psi(m_0+m)}, \frac{\Psi(i_{\nu-k})}{\Psi(m_0+m)}\right].$$

By the definition of $F(t)$ we obtain

$$J = \int_U F\Big(\sum_{j=1}^{n} \chi_{R^m_{i_j}}(x)-1\Big)\,dx, \qquad U = \bigcup_{\nu=m_0+1}^{n} \bigcup_{k=m_0}^{\nu-1} E_{\nu,k}.$$

Clearly, $|E_{\nu,k}| \leqslant \Psi(i_{\nu-k})/(\Psi(i_\nu)\,\Psi(m_0+m))$, and it can easily be seen that

$$\sum_{j=1}^{n} \chi_{R^m_{i_j}}(x) = k+1, \qquad \forall x\in E_{\nu,k}.$$

Hence

$$J \leqslant \sum_{\nu=m_0+1}^{n} \sum_{k=m_0}^{\nu-1} \int_{E_{\nu,k}} F\Big(\sum_{j=1}^{n} \chi_{R^m_{i_j}}(x)-1\Big)\,dx$$
$$= \sum_{\nu=m_0+1}^{n} \sum_{k=m_0}^{\nu-1} F(k)\,|E_{\nu,k}| \leqslant \frac{1}{\Psi(m_0+m)} \sum_{\nu=m_0+1}^{n} \sum_{k=m_0}^{\nu-1} \frac{\Psi(k)\,\Psi(i_{\nu-k})}{k^2\,\Psi(i_\nu)}.$$

But $k \geqslant m_0 \geqslant t_0$, $i_{\nu-k} \geqslant i_1 \geqslant m_0 \geqslant t_0$, and so $\Psi(i_{\nu-k})\,\Psi(k) \leqslant \Psi(i_{\nu-k}+k)$, and since it is easily seen that $i_{\nu-k}+k \leqslant i_\nu$, we finally obtain

$$J \leqslant \frac{1}{\Psi(m_0+m)} \sum_{\nu=m_0+1}^{n} \sum_{k=m_0}^{\nu-1} \frac{\Psi(i_\nu)}{k^2\,\Psi(i_\nu)} \leqslant \frac{2n}{\Psi(m_0+m)}.$$

On the other hand, $\sum_{j=1}^{n} |R^m_{i_j}| = n/\Psi(m_0+m)$. By (27) and (28) we obtain for $m \geqslant 2m_0$

$$\forall \{R^m_{i_j}\} \subset \{R^m_i\}_{i=m_0}^{m_0+m}: \qquad \int_{W_m} \Psi\Big(c_{17}^{-1}\Big(\sum_{j=1}^{n} \chi_{R^m_{i_j}}(x)-1\Big)\Big)\,dx \leqslant c_{20} \sum_{j=1}^{n} |R^m_{i_j}|$$

$$\text{where } W_m = \Big\{x:\ \sum_{j=1}^{n} \chi_{R^m_{i_j}}(x) \geqslant m_0+1\Big\},$$

and (11) is proved. All conditions of Lemma 4 are therefore satisfied, and the proof of Theorem 2 is completed by using the conclusion of that lemma.

Remark 2. It follows from Lemma 2 that the maximal operator corresponding to the constructed basis has weak type $L+\varphi(L)$.

4. N-dimensional analogues and some generalizations. All definitions introduced in Sections 1–3 carry over without change to the case of several variables. Theorems 1 and 2 also remain valid, with $\boldsymbol{B}_1$ understood as $\boldsymbol{B}_1(\boldsymbol{R}^N)$. The proof can be reduced to a two-dimensional argument by considering projections on two-dimensional coordinate hyperplanes.

The important and essentially new element in our proof of Theorem 2 is Lemma 4. The same method yields the following more general result which is of independent interest:

LEMMA A. *Let $\Phi(t)$ and $\Phi^*(t)$ be Young conjugate convex functions with $\Phi(t)$ satisfying the Δ_2 condition. Let $\sigma = \{R^n_k\}$, $k = 1, \ldots, m_n$, $n\in \boldsymbol{N}$, and $\{E_n\}_{n=1}^{\infty}$ be collections of bounded open sets in $\boldsymbol{R}^N$ and λ_n a sequence of numbers with $\lambda_n \uparrow \infty$ as $n\to\infty$ such that there are constants $c_i > 0$, $i = 0, \ldots, 6$, satisfying the following conditions for all $n\in \boldsymbol{N}$:*

(i) $$|R^n_k| = |R^n_j|, \qquad k, j = 1, \ldots, m_n,$$

(ii) $\forall \{R^n_{k_\nu}\}_{\nu=1}^{s} \subset \{R^n_k\}_{k=1}^{m_n}$:

$$\int_{W_n} \Phi^*\Big(c_1\Big(\sum_{\nu=1}^{s} \chi_{R^n_{k_\nu}}(x)-c_2\Big)\Big)\,dx \leqslant c_4 \sum_{\nu=1}^{s} |R^n_{k_\nu}|$$

$$\textit{where } W_n = \Big\{x:\ \sum_{\nu=1}^{s} \chi_{R^n_{k_\nu}}(x) \geqslant c_3\Big\},$$

(iii) $$\frac{|R^n_k \cap E_n|}{|R^n_k|} \geqslant \frac{c_5}{\lambda_n}, \qquad k = 1, \ldots, m_n,$$

(iv) $$\Big|\bigcup_{k=1}^{m_n} R^n_k\Big| \geqslant c_6\, \Phi(\lambda_n)\,|E_n|.$$

Then there is a basis $B(\boldsymbol{R}^N)$ whose every element is a dilation of some member of σ and such that $B(\boldsymbol{R}^N)$ differentiates precisely $\Phi(L)(\boldsymbol{R}^N)$.

Lemma A gives a method of constructing bases in $\boldsymbol{R}^N$ consisting of elements of a given type and differentiating precisely the classes $\Phi(L)(\boldsymbol{R}^N)$.

The results obtained have applications in the theory of multiple Fourier series. Let $\{n_k\}$, $\{m_k\}$ be two sequences of positive integers tending to infinity, and let $\sigma_{n_k,m_k}(f, x)$ be the $(C, 1)$ means of the Fourier series of a function $f(x)$ on the rectangle $[0, n_k]\times[0, m_k]$. From the method of proof of Theorem (2.14) in [15, Ch. XVII] it follows that if $\int f$ is not differentiated by the TI-basis consisting of rectangles of the form $[0, n_k^{-1}]\times[0, m_k^{-1}]$ then

$$\limsup_{k\to\infty} \sigma_{n_k,m_k}(f, x) = +\infty \qquad \text{a.e. on } [0, 2\pi]^2.$$

On the other hand, if $f\in L\log^+ L([0, 2\pi]^2)$ then

$$\lim_{n,m\to\infty} \sigma_{n,m}(f, x) = f(x) \qquad \text{a.e. on } [0, 2\pi]^2$$

(see [5]). Proceeding by analogy with the proof of Theorem (3.1) in [15, Ch. XVII] it is not difficult to show that if $\{[0, n_n]\times[0, m_k]\}_{k=1}^{\infty}$ is a monotonic family of rectangles then for all f in $L([0, 2\pi]^2)$

$$\lim_{k\to\infty} \sigma_{n_k,m_k}(f, x) = f(x) \qquad \text{a.e. on } [0, 2\pi]^2.$$

Combining the above and the proof of Theorem 1 we obtain the following alternative for any sequences $\{n_k\}$, $\{m_k\}$ of positive integers tending to infinity: either $\sigma_{n_k,m_k}(f, x)$ converges to $f(x)$ a.e. for all $f \in L([0, 2\pi]^2)$, or for any function $g(t)$ with $g(t) \downarrow 0$ as $t \to \infty$ there is an $f \in g(L) L\log^+ L([0, 2\pi]^2)$ such that

$$\limsup_{k \to \infty} \sigma_{n_k,m_k}(f, x) = +\infty \quad \text{a.e. on } [0, 2\pi]^2.$$

Analogous results hold for (C, α, β) summability $(0 < \alpha \leqslant 1,\ 0 < \beta \leqslant 1)$.

References

[1] C. P. Calderón, *Some remarks on the multiple Weierstrass transform and Abel summability of Fourier–Hermite series*, Studia Math. 32 (1969), 119–148.
[2] N. A. Fava, *Weak inequalities for product operators*, ibid. 49 (1974), 184–197.
[3] M. de Guzmán, *An inequality for the Hardy–Littlewood maximal operator with respect to the product of differentiation bases*, ibid. 42 (1972), 265–286.
[4] –, *Differentiation of Integrals in R^n*, Lecture Notes in Math. 481, Springer, 1975.
[5] B. Jessen, J. Marcinkiewicz and A. Zygmund, *Note on the differentiability of multiple integrals*, Fund. Math. 25 (1935), 217–234.
[6] M. A. Krasnosel'skii and Ya. B. Rutitskii, *Convex Functions and Orlicz Spaces*, Noordhoff, Groningen 1961 (transl. from Russian).
[7] H. Lebesgue, *Sur l'intégration des fonctions discontinues*, Ann. Sci. École Norm. Sup. 27 (1910), 361–450.
[8] B. Mélero, *A negative result in differentiation theory*, Studia Math. 72 (1982), 173–182.
[9] S. Saks, *On the strong derivatives of functions of intervals*, Fund. Math. 25 (1935), 235–252.
[10] E. M. Stein, *Note on the class $L\log^+ L$*, Studia Math. 32 (1969), 305–310.
[11] –, *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press, 1970.
[12] A. M. Stokolos, *An inequality for equimeasurable rearrangements and its application in the theory of differentiation of integrals*, Analysis Math. 9 (1983), 133–146.
[13] –, *On the differentiation of multiple integrals by bases of rectangles*, Soobshch. Akad. Nauk Gruzin. SSR 114 (1984), 477–480 (in Russian).
[14] P. L. Ul'yanov, *Embedding of some function classes H_p^ω*, Izv. Akad. Nauk SSSR Ser. Mat. 32 (1968), 649–686 (in Russian).
[15] A. Zygmund, *Trigonometric Series*, vol. 2, Cambridge Univ. Press, 1959.
[16] –, *A note on the differentiability of multiple integrals*, Colloq. Math. 16 (1967), 199–204.

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Received March 27, 1986 (2159)

STUDIA MATHEMATICA, T. LXXXVIII. (1988)

On subspaces of H^1 isomorphic to H^1

by

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Abstract. We show that any subspace of H^1 which is isomorphic to H^1 contains a complemented copy of H^1. H^1 is proved to be primary.

Introduction. This work is best regarded as an appendix to the book *Symmetric Structures in Banach Spaces* by W. Johnson, B. Maurey, G. Schechtman and L. Tzafriri ([JMST]), where the result analogous to our Theorem 1 is proved for L^p spaces $(1 < p < \infty)$.

We use their notation and follow their arguments rather closely.

I feel obliged to indicate at which point the treatment of H^1 spaces requires different tools than that for L^p spaces $(1 < p < \infty)$:

In trying to find complemented subspaces in the range of embeddings on L^p, JMST rely on the following martingale inequality due to E. M. Stein: Given an increasing sequence of σ-fields $(\mathscr{F}_n)_{n \in N}$ in $[0, 1]$ with corresponding conditional expectations $(E_n)_{n \in N}$, for any $1 < p < \infty$ there exists $C_p \in R^+$ such that for any sequence of measurable functions $(f_n)_{n \in N}$ the following holds:

$$\int \Big(\sum_{n=1}^{\infty} |E_n f_n|^2\Big)^{p/2} \leqslant C_p \int \Big(\sum_{n=1}^{\infty} |f_n|^2\Big)^{p/2}.$$

There exist examples (cf. [St], p. 105) showing that this inequality does not hold for $p = 1$ or $p = \infty$.

Here we modify the selection process of [JMST] in such a way that projections can be constructed which are bounded on H^1. At this point the third component of the vector measure used below becomes crucial.

Definitions and notation. Recall that H^1 is the closed linear span of the L^∞-normalized Haar system

$$\{h_{ni}\colon (ni) \in \mathscr{A}\} \quad \text{where } \mathscr{A} = \{(ni)\colon n \in N,\ 0 \leqslant i \leqslant 2^n - 1\}$$

under the norm

$$\|f\|_{H^1} = \int S(f), \quad S(f) = \Big(\sum a_{ni}^2 h_{ni}^2\Big)^{1/2},$$

with $f = \sum a_{ni} h_{ni}$.