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Extensions and selections of maps with decomposable values

by

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Abstract. Let X be a separable metric space and E a Banach space. Let μ be a nonatomic probability measure on a measurable space T, and let $L^1 = L^1(T; E)$ be the Banach space of μ-integrable functions u: $T \to E$. A subset K of L^1 is *decomposable* if, for any μ-measurable set $A \subseteq T$ and all $u, v \in K$, one has $u \cdot \chi_A + v \cdot \chi_{T \setminus A} \in K$. Using the property of decomposability as a substitute for convexity, the analogues of three theorems by Dugundji, Cellina and Michael are proved.

1) A continuous map f from a closed set $Y \subseteq X$ into a decomposable subset K of L^1 can be continuously extended to a map $\tilde{f}$: $X \to K$.

2) An upper semicontinuous multivalued map F: $X \to 2^{L^1}$ with decomposable values has a continuous ε-approximate selection, for any $\varepsilon > 0$.

3) A lower semicontinuous multifunction G: $X \to 2^{L^1}$ with closed decomposable values admits a continuous selection.

The compactness assumption on X, which appears in previous papers, is here never used. From 1) it follows that, if $L^1(T; E)$ is separable, then any closed decomposable subset $K \subseteq L^1$ is a retract of the whole space, hence it has the compact fixed point property.

1. Introduction. Consider a measure space $(T, \mathscr{F}, \mu)$, where $\mathscr{F}$ is a σ-algebra of subsets of T and μ is a nonatomic probability measure on $\mathscr{F}$. If E is a Banach space, let $L^1(T; E)$ be the Banach space of all functions u: $T \to E$ which are Bochner μ-integrable [17]. According to [10], a subset $K \subseteq L^1(T; E)$ is *decomposable* if, for every measurable set $A \in \mathscr{F}$,

$$u \cdot \chi_A + v \cdot \chi_{T \setminus A} \in K \quad \forall u, v \in K. \tag{1.1}$$

In several cases, the property of decomposability is a good substitute for convexity [15]. Three classical theorems, which make use of a convexity assumption, will be considered here.

THEOREM I (Dugundji [6, p. 188]). *Let A be a closed subset of a metric space X and let K be a convex subset of a Banach space Z. Then every continuous map f: $A \to K$ has a continuous extension $\tilde{f}$: $X \to K$.*

THEOREM II (Cellina [2, p. 84]). *Let X be a metric space and Z a Banach space. Let F: $X \to 2^Z$ be an upper semicontinuous map with convex values. Then, for every $\varepsilon > 0$, F admits a continuous ε-approximate selection,* *i.e. a continuous function* $f_\varepsilon\colon X \to Z$ *such that*

$$\mathrm{graph}(f_\varepsilon) \subseteq B(\mathrm{graph}(F), \varepsilon).$$

Here $B(V, \varepsilon)$ denotes the ε-neighborhood of a set V.

THEOREM III (Michael [13]). *Let X be a paracompact topological space and Z a Banach space. Then any lower semicontinuous multivalued map $F\colon X \to 2^Z$ with closed convex values admits a continuous selection.*

Aim of the present paper is to establish the analogues of the above results when Z is the Banach space $L^1(T; E)$ and, in the assumptions, convexity is replaced by decomposability.

The first result concerning the existence of a continuous selection, for a continuous multifunction with decomposable but not necessarily convex values, is due to Antosiewicz and Cellina [1]. Their selection theorem yields the existence of solutions for the differential inclusion $\dot{x} \in F(t, x)$ with Hausdorff-continuous right-hand side, by means of a classical fixed point argument. These results were extended to the lower semicontinuous case by Bressan [3] and Łojasiewicz [12]. More recently, Fryszkowski stated a general selection theorem for lower semicontinuous maps with decomposable values [7]. Approximate continuous selections for upper semicontinuous maps with decomposable values were constructed in [5]. We remark that the proofs of Theorems I–III rely on the paracompactness of the space X. In the decomposable case, however, all known results require that the space X be compact. This unnatural assumption is motivated only by a technical difficulty, which will be removed in the present paper.

In the analysis of decomposable sets, instead of taking convex combinations, one can continuously interpolate between different points following a well-established procedure. Consider an increasing family $\{A_\lambda;\ \lambda \in [0, 1]\}$ of measurable subsets of T with the property that $\mu(A_\lambda) = \lambda \cdot \mu(T)$ for every λ. The existence of such a family is proved in [9], Lemma 4. Let $u_1, \ldots, u_p$ be elements of a decomposable set $K \subseteq L^1(T; E)$, and let $\lambda_1, \ldots, \lambda_p$ be nonnegative numbers which add up to 1. Setting $\eta_0 = 0$, $\eta_i = \lambda_1 + \ldots + \lambda_i$ $(i = 1, \ldots, p)$, a combination of the u_i with the λ_i as parameters is given by

$$\Gamma(u, \lambda) = \sum_{i=1}^{p} u_i \cdot \chi_{A_{\eta_i} \setminus A_{\eta_{i-1}}}. \tag{1.2}$$

As in the case of convex combinations, the right-hand side of (1.2) lies inside K and varies continuously with each u_i and λ_i. Moreover, $\Gamma(u, \lambda) = u_i$ whenever $\lambda_i = 1$. Together with these analogies there is, however, a major difference. In Banach spaces, the metric and the algebraic structures are linked together by the fact that balls are convex. On the other hand, balls in $L^1(T; E)$ are not decomposable. The failure of this basic property is a primary source of technical difficulties. If $\bar{u} \in L^1$ and $\|u_i - \bar{u}\| \leqslant \varrho$ for all $i \in \{1, \ldots, p\}$, without additional assumptions on the sets A_λ the only available estimate for (1.2) is

$$\|\Gamma(u, \lambda) - \bar{u}\| \leqslant p\varrho. \tag{1.3}$$

This bound can be improved if the sets A_λ are more carefully chosen. In [7] the author defines the measures μ_i by setting

$$\mu_i(A) = \int_A \|u_i - \bar{u}\|_E \, d\mu \qquad (A \in \mathscr{F}).$$

By Lyapunov's Convexity Theorem [9], one can then choose a family of sets A_λ satisfying the additional conditions

$$\mu_i(A_\lambda) = \lambda \mu_i(T), \qquad (\lambda \in [0, 1],\ i = 1, \ldots, p).$$

If these special sets are used in (1.2), the stronger estimate $\|\Gamma(u, \lambda) - \bar{u}\| \leqslant \varrho$ holds. So far, this technique has been applied only in the case of finitely many functions u_i. Indeed, Lyapunov's theorem does not hold for an infinite family of measures μ_i [11]. In order to extend the results given in [5] and [7] from the compact to the paracompact case, one has to construct continuous combinations for an infinite family of functions u_i, taking advantage of the fact that only a finite number of u_i enter in a combination at any given time.

To do this, our key technical tool is Lemma 1 in § 4. It contains an extension of Lyapunov's theorem, valid for a countable set of measures, which is precisely fit for our purpose. Using this lemma, we can prove the analogues of Theorems I–III for the decomposable case, in a quite general setting. Indeed, a separability assumption is the only additional requirement. The statements of our main results are collected in § 3.

An interesting consequence of the extension theorem is that, in a separable space $L^1(T; E)$, any closed decomposable subset K is a retract of the whole space. Therefore, K has the compact fixed point property [16, p. 33]. This provides a further generalization of the fixed point theorems of Cellina [4] and Fryszkowski [8], which holds for L^1 spaces over any abstract measure space $(T, \mathscr{F}, \mu)$.

2. Notation and basic definitions. Throughout this paper, $(T, \mathscr{F}, \mu)$ denotes a measure space, where $\mathscr{F}$ is a σ-algebra of subsets of T and μ is a nonatomic probability measure on $\mathscr{F}$. Given a μ-integrable function $f\colon T \to \mathbf{R}$, we write $f \cdot \mu$ for the measure having density f w.r.t. μ. We denote by $\sigma\{A_\lambda;\ \lambda \in \Lambda\}$ the σ-algebra generated by a family of measurable sets $A_\lambda \in \mathscr{F}$. If E is a Banach space with norm $\|\cdot\|_E$, $L^1(T; E)$ denotes the Banach space of Bochner μ-integrable functions $u\colon T \to E$ [17, p. 132], with norm $\|u\|_1 = \int_T \|u\|_E \, d\mu$. Given two metric spaces X, Y with distances d_X, d_Y respectively, the distance on their product is $d_{X \times Y} = d_X + d_Y$. The open ε-neighborhood of a set $S \subseteq X$ is $B(S, \varepsilon) = \{x \in X;\ d(x, S) < \varepsilon\}$. The diameter of S is

$\operatorname{diam}(S) = \sup\{d(x, x');\ x, x' \in S\}$. The set-theoretic difference between two sets A, B is written $A \setminus B$; their symmetric difference is $A \triangle B = (A \setminus B) \cup (B \setminus A)$. $\# A$ stands for the cardinality of the set A, while χ_A is the characteristic function of A.

Following [10], we now introduce the main concept discussed in this paper.

DEFINITION. A set $K \subseteq L^1(T; E)$ is *decomposable* if

$$u \cdot \chi_A + v \cdot \chi_{T \setminus A} \in K \quad \text{whenever } u, v \in K,\ A \in \mathscr{F}.$$

The collection of all nonempty decomposable subsets of $L^1(T; E)$ is denoted by $D(L^1(T; E))$. For any set $H \subseteq L^1(T; E)$, the *decomposable hull* of H is

$$\operatorname{dec}[H] = \bigcap \{K \in D(L^1(T; E));\ H \subseteq K\}.$$

Clearly, $\operatorname{dec}[H]$ represents the smallest decomposable set which contains H.

At last, we recall two basic properties of multivalued mappings [2]. Let X, Y be metric spaces. A multifunction $F\colon X \to 2^Y$ is *lower semicontinuous* (*l.s.c.*) iff the set $\{x \in X;\ F(x) \subseteq C\}$ is closed for every closed set $C \subseteq Y$. A map $F\colon X \to 2^Y$ is *Hausdorff-upper semicontinuous* (*H-u.s.c.*) iff, for every $x_0 \in X$ and every $\varepsilon > 0$, there exists a neighborhood V of x_0 such that $F(x) \subseteq B(F(x_0), \varepsilon)$ for all $x \in V$.

3. Statement of the main results. Our first result is the counterpart of Dugundji's extension theorem, for maps taking values in a separable L^1 space, with the convex hull replaced by the decomposable hull.

THEOREM 1. *Let A be a closed subset of a metric space X. If either X or $L^1(T; E)$ is separable, then every continuous map $f\colon A \to L^1(T; E)$ has a continuous extension $\tilde{f}\colon X \to L^1(T; E)$ such that $\tilde{f}(X) \subseteq \operatorname{dec}[f(A)]$.*

COROLLARY 1. *If $L^1(T; E)$ is separable, then every closed decomposable subset $K \subseteq L^1(T; E)$ is a retract of the whole space.*

Following [16, p. 33], we say that a topological space K has the *compact fixed point property* if every continuous map $f\colon K \to K$ with relatively compact image has a fixed point. Theorem 1 yields a general fixed point theorem, which is valid for L^1 spaces over any abstract measure space $(T, \mathscr{F}, \mu)$ with a nonatomic probability measure μ.

COROLLARY 2. *Every closed decomposable set $K \subseteq L^1(T; E)$ has the compact fixed point property.*

Indeed, if $L^1(T; E)$ is separable, then Corollary 2 is an immediate consequence of Corollary 1. To cover the case where $L^1(T; E)$ is not separable, let $f\colon K \to K$ be a continuous map whose image is relatively compact, and let X be the closure of the convex hull of $f(K)$. Since X is compact, it is obviously separable. Using Theorem 1, extend the identity map $\imath$ on $X \cap K$ to a continuous map $\tilde{\imath}\colon X \to K$. The composition $f \circ \tilde{\imath}$ maps X into $X \cap K$. By Schauder's theorem, it has a fixed point $\bar{x} \in X \cap K$, which is then a fixed point of f.

The next two theorems are concerned with multivalued maps having decomposable values. They provide the analogues of the selection theorems by Cellina and Michael, respectively.

THEOREM 2. *Let X be a metric space and let $F\colon X \to D(L^1(T; E))$ be a H-u.s.c. multifunction with decomposable values. If either X or $L^1(T; E)$ is separable, then for every $\varepsilon > 0$ there exists a continuous map $f_\varepsilon\colon X \to L^1(T; E)$ such that*

$$\operatorname{graph}(f_\varepsilon) \subseteq B(\operatorname{graph}(F), \varepsilon).$$

Moreover, $f_\varepsilon(X) \subseteq \operatorname{dec}[F(X)]$.

THEOREM 3. *Let X be a separable metric space, and let $F\colon X \to D(L^1(T; E))$ be a l.s.c. multifunction with closed decomposable values. Then F has a continuous selection.*

Remark. For simplicity, the above results are stated in terms of a probability measure μ, but they all can be easily extended to the case where μ is any nonatomic, nonnegative, bounded measure on $(T, \mathscr{F})$. For this purpose, it suffices to consider the probability measure $\tilde{\mu} = [\mu(T)]^{-1} \cdot \mu$, which is equivalent to μ.

4. Three technical lemmas.

LEMMA 1. *Let $(T, \mathscr{F}, \mu)$ be a measure space with a σ-algebra $\mathscr{F}$ of subsets of T and a nonatomic probability measure μ on $\mathscr{F}$. Let $(g_n)_{n \geq 0}$ be a sequence of nonnegative functions in $L^1(T; \mathbf{R})$ with $g_0 \equiv 1$. Then there exists a map $\Phi\colon \mathbf{R}^+ \times [0, 1] \to \mathscr{F}$ with the following properties*:

(a) $\Phi(\tau, \lambda_1) \subseteq \Phi(\tau, \lambda_2)$ *if* $\lambda_1 \leq \lambda_2$,

(b) $\mu(\Phi(\tau_1, \lambda_1) \triangle \Phi(\tau_2, \lambda_2)) \leq |\lambda_1 - \lambda_2| + 2|\tau_1 - \tau_2|$,

(c) $\int_{\Phi(\tau,\lambda)} g_n\, d\mu = \lambda \int_T g_n\, d\mu \quad \forall n \leq \tau$,

for all $\lambda, \lambda_1, \lambda_2 \in [0, 1]$, $\tau, \tau_1, \tau_2 \geq 0$.

Proof. The lemma will be proved first in a special case, assuming that

$$\int_T g_n\, d\mu = 1 \quad \forall n \geq 0. \tag{4.1}$$

By induction on n, we shall define a sequence of families of measurable sets $\{A_\lambda^n;\ \lambda \in [0, 1]\}$, $n \geq 0$, and a decreasing sequence of σ-algebras $\mathscr{F}^n \subseteq \mathscr{F}$ with the following properties:

(i) $\mu(A_\lambda^n) = \lambda$,

(ii) $A_\lambda^n \in \mathscr{F}^{n-1}$,

(iii) $A_{\lambda_1}^n \subseteq A_{\lambda_2}^n$ whenever $\lambda_1 \leq \lambda_2$,

(iv) $\mathscr{F}^n = \sigma\{A_\lambda^n;\ \lambda \in [0, 1]\}$,

(v) $\mu(A) = \int_A g_i\, d\mu$ whenever $A \in \mathscr{F}^n$, $i \leq n$.

To do this, using Lyapunov's theorem [9], construct a family of sets $\{A_\lambda^0\}$ such that (i) and (iii) hold for $n=0$, and let $\mathscr{F}^0$ be the σ-algebra generated by the sets A_λ^0. Let now A_λ^m be defined for all $\lambda\in[0, 1]$ and all $m\leqslant n-1$ so that the properties (i)–(v) hold. Apply Lyapunov's theorem to the two nonatomic measures μ and $\mu_n = g_n\cdot\mu$, on the measurable space $(T, \mathscr{F}^{n-1})$. This yields a family of sets $\{A_\lambda^n;\ \lambda\in[0, 1]\}$ such that (i), (ii) and (iii) hold for n. Define $\mathscr{F}^n$ by (iv). We then have

$$\mu(A) = \int_A g_n\,d\mu \quad (A\in\mathscr{F}^n),$$

because the equality holds whenever $A = A_\lambda^n$ for some λ, and the family of sets $\{A_\lambda^n;\ \lambda\in[0, 1]\}$ is increasing and, by definition, generates $\mathscr{F}^n$. If $i<n$, then $A\in\mathscr{F}^n$ implies $A\in\mathscr{F}^i\supseteq\mathscr{F}^n$, hence (v) is a consequence of the inductive hypothesis.

We now define the sets $\Phi(\tau, \lambda)$ as follows.

If τ is an integer, $\Phi(\tau, \lambda) = A_\lambda^\tau$.

If $\tau = n+\delta$ with n integer, $0<\delta<1$, we consider two cases: when $\lambda\leqslant\delta$ we set $\Phi(\tau, \lambda) = A_\lambda^{n+1}$; when $\lambda>\delta$ we set $\Phi(\tau, \lambda) = A_\delta^{n+1}\cup A_\xi^n$, where ξ is the smallest number in $[0, 1]$ for which the equality $\mu(A_\delta^{n+1}\cup A_\xi^n) = \lambda$ holds.

Notice that for any n, δ the function

$$\xi\to\psi(\xi) = \mu(A_\delta^{n+1}\cup A_\xi^n)$$

is Lipschitz-continuous and nondecreasing, with $\psi(0)=\delta$, $\psi(1)=1$. In the case $\delta<\lambda\leqslant 1$, the set $\{\xi\in[0, 1];\ \psi(\xi)=\lambda\}$ is nonempty, closed and connected, hence it contains a minimal element. The map Φ is thus well defined.

The verification of (a) is elementary. By construction, we also have

$$\mu(\Phi(\tau, \lambda)) = \lambda \qquad \forall\tau\geqslant 0,\ \lambda\in[0, 1]. \tag{4.2}$$

Observe that on $\mathscr{F}^n$ the measures $g_1\cdot\mu, \ldots, g_n\cdot\mu$ all coincide with $\mu = g_0\cdot\mu$, because of (v). Since $\Phi(\tau, \lambda)\in\mathscr{F}^n$ whenever $\tau\geqslant n$, (4.2) implies (c). To prove (b) notice that (a) and (4.2) together yield

$$\mu(\Phi(\tau, \lambda_1)\,\triangle\,\Phi(\tau, \lambda_2)) = |\lambda_1-\lambda_2| \qquad \forall\tau, \lambda_1, \lambda_2.$$

Therefore, to establish (b), it suffices to prove the inequality

$$\mu(\Phi(\tau_1, \lambda)\,\triangle\,\Phi(\tau_2, \lambda))\leqslant 2|\tau_1-\tau_2|. \tag{4.3}$$

Moreover, we can assume that $\tau_1<\tau_2$ and that τ_1, τ_2 both belong to the same interval $[n, n+1]$. For $i=1, 2$, set $\delta_i = \tau_i - n$ and, if $\lambda>\delta_i$, let $\Phi(\tau_i, \lambda) = A_{\delta_i}^{n+1}\cup A_{\xi_i}^n$. Three cases must be considered.

1) If $\lambda\leqslant\delta_1<\delta_2$, then $\Phi(\tau_1, \lambda) = \Phi(\tau_2, \lambda) = A_\lambda^{n+1}$ and (4.3) holds trivially.

2) If $\delta_1\leqslant\lambda\leqslant\delta_2$, then

$$\begin{aligned}\mu(\Phi(\tau_1, \lambda)\,\triangle\,\Phi(\tau_2, \lambda)) &\leqslant \mu((A_{\delta_1}^{n+1}\cup A_{\xi_1}^n)\,\triangle\,A_{\delta_1}^{n+1}) + \mu(A_{\delta_1}^{n+1}\,\triangle\,A_{\delta_2}^{n+1})\\ &= (\lambda-\delta_1)+(\delta_2-\delta_1)\leqslant 2(\delta_2-\delta_1) = 2|\tau_1-\tau_2|.\end{aligned}$$

3) If $\delta_1<\delta_2\leqslant\lambda$, observe that $A_{\delta_1}^{n+1}\subseteq A_{\delta_2}^{n+1}$ and $A_{\xi_1}^n\supseteq A_{\xi_2}^n$. Using these relations, we obtain

$$\begin{aligned}\mu(\Phi(\tau_2, \lambda)\setminus\Phi(\tau_1, \lambda)) &+ \mu(\Phi(\tau_1, \lambda)\setminus\Phi(\tau_2, \lambda))\\ &\leqslant \mu(A_{\delta_2}^{n+1}\setminus A_{\delta_1}^{n+1}) + \mu(\Phi(\tau_1, \lambda)\setminus(A_{\delta_1}^{n+1}\cup A_{\xi_2}^n))\\ &= (\delta_2-\delta_1)+\mu(\Phi(\tau_1, \lambda)) - \mu(A_{\delta_1}^{n+1}\cup A_{\xi_2}^n)\\ &\leqslant (\delta_2-\delta_1)+\lambda-[\lambda-(\delta_2-\delta_1)] = 2|\tau_1-\tau_2|.\end{aligned}$$

Here the last inequality is deduced from the inclusion

$$(A_{\delta_2}^{n+1}\cup A_{\xi_2}^n)\setminus(A_{\delta_2}^{n+1}\setminus A_{\delta_1}^{n+1})\subseteq(A_{\delta_1}^{n+1}\cup A_{\xi_2}^n).$$

The above estimates complete the proof of Lemma 1 under the additional assumption (4.1).

To treat the general case, for each $n\geqslant 0$, set $\hat{g}_n\equiv 1$ if $g_n = 0$ μ-almost everywhere; otherwise define $\hat{g}_n = [\int_T g_n\,d\mu]^{-1}\cdot g_n$. If $\{\Phi(\tau, \lambda)\}$ is a family of sets which satisfy (a)–(c) for the sequence $(\hat{g}_n)$, one can easily check that these same sets satisfy (a)–(c) for the sequence (g_n) as well. ∎

LEMMA 2. *Let X be a separable metric space, and let φ_n: $X\to L^1(T;\boldsymbol{R})$, h_n: $X\to[0, 1]$ $(n\geqslant 1)$ be two sequences of continuous functions, with $\varphi_n(x)(t)\geqslant 0$ $\forall x\in X$, $\forall t\in T$, and such that $\{\mathrm{supp}(h_n);\ n\geqslant 1\}$ is a locally finite (closed) covering of X. Then, for every $\varepsilon>0$ and every continuous strictly positive function l: $X\to\boldsymbol{R}^+$, there exist a continuous function τ: $X\to\boldsymbol{R}^+$ and a map Φ: $\boldsymbol{R}^+\times[0, 1]\to\mathscr{F}$ which satisfy conditions* (a), (b) *in Lemma* 1 *together with*

(c′) *For all $x\in X$, $\lambda\in[0, 1]$ and $n\geqslant 1$, if $h_n(x) = 1$ then*

$$\Big|\int_{\Phi(\tau(x), \lambda)}\varphi_n(x)\,d\mu - \lambda\int_T\varphi_n(x)\,d\mu\Big| < \varepsilon/(4l(x)).$$

Proof. Let $\varepsilon>0$ and l be given. For every $x\in X$, choose an open neighborhood U_x of x which intersects the supports of finitely many functions h_n, so that the set of indices $I_x = \{n;\ U_x\cap\mathrm{supp}(h_n)\neq\emptyset\}$ is finite. Set $\psi_n(x) = h_n(x)\,\varphi_n(x)\in L^1(T;\boldsymbol{R})$ and define

$$V_x = \{x'\in U_x;\ \|\psi_n(x')-\psi_n(x)\|_1 < \varepsilon/(8l(x))\ \forall n\in I_x\}. \tag{4.4}$$

The family $\{V_x;\ x\in X\}$ is an open covering of the paracompact separable space X. Hence, there exists a sequence of functions k_m: $X\to[0, 1]$ such that the family $\{\mathrm{supp}(k_m);\ m\geqslant 1\}$ is a countable nbd-finite refinement of $\{V_x\}$ and the sets $W_m = \{x \in X;\ k_m(x) = 1\}$ still cover X. For all $m \geq 1$, select x_m such that $W_m \subseteq V_{x_m}$. Define the sequence $(g_j)_{j\geq 0}$ in $L^1(T; \boldsymbol{R})$ by setting: $g_j = \psi_n(x_m)$ if $j = 2^m 3^n$ for some integers $m, n \geq 1$; $g_j \equiv 1$ otherwise. Moreover, set

$$\tau(x) = \sum_{m,n\geq 1} k_m(x)\, h_n(x)\, 2^m 3^n. \tag{4.5}$$

The function τ is continuous, because the summation in (4.5) is locally finite. Using Lemma 1, construct a map Φ which satisfies (a)–(c) for the sequence $(g_j)_{j\geq 0}$. We claim that (c′) holds as well.

To see this, fix $x \in X$, $n \geq 1$ and $\lambda \in [0, 1]$. For some index m, $x \in W_m$. If $h_n(x) = 1$, then

$$\begin{aligned}
&\Big|\int_{\Phi(\tau(x),\lambda)} \varphi_n(x)\, d\mu - \lambda \int_T \varphi_n(x)\, d\mu\Big| \\
&\leq \int_{\Phi(\tau(x),\lambda)} |\psi_n(x) - \psi_n(x_m)|\, d\mu + \Big|\int_{\Phi(\tau(x),\lambda)} \psi_n(x_m)\, d\mu - \lambda \int_T \psi_n(x_m)\, d\mu\Big| \\
&\quad + \lambda \int_T |\psi_n(x_m) - \psi_n(x)|\, d\mu \\
&\leq 2\|\psi_n(x) - \psi_n(x_m)\|_1 + \Big|\int_{\Phi(\tau(x),\lambda)} g_{2^m 3^n}\, d\mu - \lambda \int_T g_{2^m 3^n}\, d\mu\Big|.
\end{aligned}$$

By (4.4), since $x \in V_{x_m}$, the first term of this last expression is less than $\varepsilon/(4l(x))$, while the second term vanishes because $\tau(x) \geq 2^m 3^n$, by (4.5). ■

LEMMA 3. *Let X be a paracompact topological space. For every $x \in X$, let U_x be an open neighborhood of x and let $M(x)$ be an integer number. Then there exists a continuous function $\tau\colon X \to \boldsymbol{R}$ such that $\tau(x) \geq \min\{M(x');\ x \in U_{x'}\}$ for every $x \in X$.*

Proof. Let $\{V_i;\ i \in I\}$ be an open nbd-finite refinement of the covering $\{U_x\}$, and let $\{p_i(\cdot);\ i \in I\}$ be a continuous partition of unity subordinate to $\{V_i\}$. For each i, select a point x_i such that $V_i \subseteq U_{x_i}$. Define $\tau(x) = \sum_{i\in I} p_i(x)\, M(x_i)$. Clearly, τ is continuous. Moreover,

$$\begin{aligned}
\tau(x) &\geq \min\{M(x_i);\ p_i(x) \neq 0\} \geq \min\{M(x_i);\ x \in U_{x_i}\} \\
&\geq \min\{M(x');\ x \in U_{x'}\}. \ ■
\end{aligned}$$

5. Proof of Theorem 1. We assume first that $L^1(T; E)$ is separable. For each $x \in X \setminus A$, take an open ball $B(x, r_x)$ with radius $r_x < \frac{1}{2} d(x, A)$. The family $\{B(x, r_x);\ x \in X \setminus A\}$ is an open covering of the paracompact space $X \setminus A$, hence it admits an open nbd-finite refinement $\{V_i;\ i \in I\}$. Here I is a possibly uncountable set of indices. For each i, choose two points $x_i \in V_i$ and $y_i \in A$ such that $d(x_i, y_i) < 2d(x_i, A)$. Using the separability assumption, select a countable subset $D = \{u_n;\ n \geq 1\}$ of $f(A)$ which is dense in $f(A)$.

Define the sequence $(g_k)_{k\geq 0}$ in $L^1(T; \boldsymbol{R})$ by setting

$$g_k(t) = \begin{cases} \|u_m(t) - u_n(t)\|_E & \text{whenever } k = 2^m 3^n \text{ for some } m, n \geq 1; \\ 1 & \text{otherwise.} \end{cases}$$

Applying Lemma 1 to this sequence, we obtain a family $\{\Phi(\tau, \lambda)\}$ of measurable subsets of T with the properties (a)–(c). For each $i \in I$, choose $u_{\nu(i)} \in D$ such that $\|u_{\nu(i)} - f(y_i)\| < d(x_i, y_i)$. Let $\{p_i(\cdot);\ i \in I\}$ be a continuous partition of unity subordinate to the covering $\{V_i\}$. For every $n \geq 1$, define the open set $W_n = \bigcup\{V_i;\ \nu(i) = n\}$ and let $q_n(x) = \sum_{\nu(i)=n} p_i(x)$. Clearly, $\{q_n(\cdot);\ n \geq 1\}$ is a continuous partition of unity subordinate to the locally finite open covering $\{W_n\}$. Construct a sequence of continuous functions $(h_n)_{n\geq 1}$ such that $h_n \equiv 1$ on $\operatorname{supp}(q_n)$ and $\operatorname{supp}(h_n) \subseteq W_n$. For every $x \in X \setminus A$, define $\lambda_n(x) = \sum_{m\leq n} q_m(x)$, $n \geq 0$, and consider the function

$$\tau(x) = \sum_{m,n\geq 1} h_m(x)\, h_n(x)\, 2^m 3^n.$$

Notice that τ is continuous on $X \setminus A$ and that

$$\tau(x) \geq 2^m 3^n \quad \forall x \in \operatorname{supp}(q_m) \cap \operatorname{supp}(q_n). \tag{5.1}$$

We can now extend the map f to the whole space X by setting

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in A, \\ \sum_{n\geq 1} u_n \cdot \chi_n(x) & \text{if } x \in X \setminus A, \end{cases}$$

where

$$\chi_n(x) = \chi_{\Phi(\tau(x),\lambda_n(x))\setminus\Phi(\tau(x),\lambda_{n-1}(x))}.$$

It is clear that $\tilde{f}$ maps X into $\operatorname{dec}[f(A)]$. Moreover, $\tilde{f}$ is continuous on $X \setminus A$, because the functions $\tau(\cdot)$ and $\lambda_n(\cdot)$ $(n \geq 0)$ are continuous, the characteristic function of the set $\Phi(\tau, \lambda)$ varies continuously in $L^1(T; \boldsymbol{R})$ w.r.t. the parameters τ and λ, and because the summation defining $\tilde{f}$ is locally finite.

To prove that $\tilde{f}$ is continuous on A, let $a \in A$ and $\varepsilon > 0$ be given. Choose $\delta > 0$ such that $\delta < \varepsilon/12$ and $\|f(y) - f(a)\|_1 < \varepsilon/2$ whenever $y \in A$, $d(y, a) < 12\delta$. If $d(x, a) < \delta$ and $x \in V_i$ for some $i \in I$, then $\operatorname{diam}(V_i) < 2\delta$, $d(x_i, A) < 3\delta$ and $d(x_i, y_i) < 6\delta$. Therefore, $p_i(x) \neq 0$ implies that $d(y_i, a) < 9\delta$, $\|f(y_i) - f(a)\|_1 < \varepsilon/2$ and $\|u_{\nu(i)} - f(a)\|_1 < \varepsilon$. From the last inequality, it follows that

$$\|u_n - f(a)\|_1 < \varepsilon \quad \forall n \text{ such that } q_n(x) \neq 0. \tag{5.2}$$

For any $x \in X \setminus A$ with $d(x, a) < \delta$, fix an integer j for which $q_j(x) \neq 0$. Using (5.1), (5.2) and the property (c) of the sets $\Phi(\tau, \lambda)$, we obtain the estimate

$$\|f(a) - \tilde{f}(x)\|_1 \leqslant \|f(a) - u_j\|_1 + \|u_j - \tilde{f}(x)\|_1$$

$$\leqslant \varepsilon + \sum_{n=1}^{\infty} \int_T \|u_j - u_n\|_E \cdot \chi_n(x)\, d\mu$$

$$= \varepsilon + \sum_{n=1}^{\infty} \int_T g_{2^j 3^n} \cdot \chi_n(x)\, d\mu$$

$$= \varepsilon + \sum_{n=1}^{\infty} q_n(x) \int_T g_{2^j 3^n}\, d\mu = \varepsilon + \sum_{n=1}^{\infty} q_n(x) \|u_j - u_n\|_1 \leqslant 3\varepsilon.$$

Since ε was arbitrary, this completes the proof in the case where $L^1(T; E)$ is separable.

When X is separable, only minor modifications of the above arguments are needed. Consider again the open covering $\{B(x, r_x);\ x \in X \setminus A\}$ and a locally finite refinement $\{V_i;\ i \in I\}$. Notice that in this case the set I is necessarily countable, since $X \setminus A$ is separable. For each i, choose $x_i \in V_i$, $y_i \in A$ such that $d(x_i, y_i) < 2d(x_i, A)$. It now suffices to define the countable set $D = \{f(y_i);\ i \in I\} \subset L^1(T; E)$ and arrange its elements into a sequence, say $D = \{u_n;\ n \geqslant 1\}$. From this point on, the proof runs exactly as in the previous case.

6. Proof of Theorem 2. The following proof is an adaptation of the arguments given in [5].

Assume first that $L^1(T; E)$ is separable. Fix $\varepsilon > 0$. For every $x \in X$, choose a number $\delta(x) \in]0, \varepsilon/6[$ such that $F(x') \subseteq B(F(x), \varepsilon/6)$ whenever $x' \in B(x, \delta(x))$. Let $\{V_i;\ i \in I\}$ be an open nbd-finite refinement of the covering $\{B(x, \delta(x)/2);\ x \in X\}$ of X. For each i, choose $x_i \in X$ such that $V_i \subseteq B(x_i, \delta(x_i)/2)$ and select $u_i \in F(x_i)$. For $i, j \in I$, choose also $v_{i,j} \in F(x_j)$ such that

$$\|u_i - v_{i,j}\|_1 \leqslant \varepsilon/6 + \inf\{\|u_i - v\|_1;\ v \in F(x_j)\} = \varepsilon/6 + d_{L^1}(u_i, F(x_j)). \tag{6.1}$$

Let $D = \{y_n;\ n \geqslant 1\}$ be a countable dense subset of $F(X)$. For every $i \in I$ select a $y_{\nu(i)} \in D$ for which $\|u_i - y_{\nu(i)}\|_1 < \varepsilon/6$. The set D' of all functions $g \in L^1(T; \boldsymbol{R})$ of the form $g(t) = \|y_m(t) - y_n(t)\|$, $m, n \geqslant 1$, is countable. Arrange its elements into a sequence, say $D' = \{g_k;\ k \geqslant 1\}$. Let $\{p_i(\cdot);\ i \in I\}$ be a continuous partition of unity subordinate to the covering $\{V_i\}$. For every $n \geqslant 1$, define the open set $W_n = \bigcup \{V_i;\ \nu(i) = n\}$ and let $q_n(x) = \sum_{\nu(i)=n} p_i(x)$. Clearly, $\{q_n(\cdot);\ n \geqslant 1\}$ is a continuous partition of unity subordinate to the nbd-finite open covering $\{W_n\}$. Define

$$\lambda_n(x) = \sum_{m \leqslant n} q_m(x) \quad (n \geqslant 0,\ x \in X). \tag{6.2}$$

For every $x \in X$, take an open neighborhood U_x of x which intersects finitely many sets V_i. Setting $I(U_x) = \{i \in I;\ U_x \cap V_i \neq \emptyset\}$, this of course means that $N(x) = \# I(U_x)$ is a finite integer. For every couple of indices $i, j \in I(U_x)$, choose a $y_{\nu(i,j,x)} \in D$ such that

$$\|y_{\nu(i,j,x)} - v_{i,j}\|_1 < \varepsilon/(6N(x)). \tag{6.3}$$

Let $M(x)$ be an integer so large that the set $\{g_k;\ 1 \leqslant k \leqslant M(x)\}$ contains the finite set of functions $\{\|y_{\nu(i)} - y_{\nu(i,j,x)}\|_E;\ i, j \in I(U_x)\} \subseteq D'$. Applying Lemma 3 to the collection of neighborhoods $\{U_x;\ x \in X\}$ and integers $M(x)$, we get the existence of a continuous function $\tau\colon X \to \boldsymbol{R}^+$ such that

$$\tau(x) \geqslant \min\{M(x');\ x \in U_{x'}\}. \tag{6.4}$$

Recalling (6.2), the map $f_\varepsilon\colon X \to L^1(T; E)$ can now be defined by setting

$$f_\varepsilon(x) = \sum_{n \geqslant 1} y_n \cdot \chi_n(x). \tag{6.5}$$

Here $\{\Phi(\tau, \lambda)\}$ is the family of sets constructed in Lemma 1 relative to the sequence $(g_k)_{k \geqslant 1}$ of elements of D', and

$$\chi_n(x) = \chi_{\Phi(\tau(x), \lambda_n(x)) \setminus \Phi(\tau(x), \lambda_{n-1}(x))}.$$

It is easily checked that f_ε is continuous and takes values inside $\operatorname{dec}[F(X)]$.

To show that f_ε is an ε-approximate selection, fix $x \in X$ and define $I(x) = \{i \in I;\ p_i(x) \neq 0\}$, $J(x) = \{n \geqslant 1;\ q_n(x) \neq 0\}$. Notice that $\# J(x) \leqslant \# I(x) < +\infty$. Since $I(x)$ is finite, there exists an $\hat{\imath} \in I(x)$ such that $\hat{\delta} = \delta(x_{\hat{\imath}}) = \max\{\delta(x_i);\ i \in I(x)\}$. For every $i \in I(x)$ we have $x_i \in B(x_{\hat{\imath}}, \hat{\delta})$, hence

$$F(x_i) \subseteq B(F(x_{\hat{\imath}}), \varepsilon/6). \tag{6.6}$$

Take a point $z \in X$ such that $x \in U_z$ and $M(z) = \min\{M(x');\ x \in U_{x'}\}$. For every $n \in J(x)$, select an index $i_n \in I(x) \subseteq I(U_z)$ such that $\nu(i_n) = n$. Define

$$w = \sum_{n \geqslant 1} y_{\nu(i_n, \hat{\imath}, z)} \cdot \chi_n(x), \qquad w' = \sum_{n \geqslant 1} v_{i_n, \hat{\imath}} \cdot \chi_n(x).$$

Notice that $w' \in F(x_{\hat{\imath}})$. For every $n \in J(x)$, using (6.1), (6.3) and (6.6) we obtain

$$\|y_n - y_{\nu(i_n, \hat{\imath}, z)}\|_1 \leqslant \|y_n - u_{i_n}\|_1 + \|u_{i_n} - v_{i_n, \hat{\imath}}\|_1 + \|v_{i_n, \hat{\imath}} - y_{\nu(i_n, \hat{\imath}, z)}\|_1$$
$$\leqslant \varepsilon/6 + [\varepsilon/6 + d_{L^1}(u_{i_n}, F(x_{\hat{\imath}}))] + \varepsilon/(6N(z)) \leqslant \tfrac{2}{3}\varepsilon. \tag{6.7}$$

Relying on the properties of the sets $\Phi(\tau, \lambda)$ and recalling that by (6.4), $\tau(x) \geqslant M(z)$, from (6.7) we deduce the estimates

$$\|f_\varepsilon(x) - w\|_1 = \sum_{n \geqslant 1} \int_T \|y_n - y_{\nu(i_n, \hat{\imath}, z)}\|_E \cdot \chi_n(x)\, d\mu$$
$$= \sum_{n \geqslant 1} q_n(x) \|y_n - y_{\nu(i_n, \hat{\imath}, z)}\|_1 \leqslant \tfrac{2}{3}\varepsilon, \tag{6.8}$$

(6.9) $$\|w-w'\|_1 = \sum_{n\geq 1} \int_T \|y_{v(i_n,\hat{\imath},z)} - v_{i_n,\hat{\imath}}\|_E \cdot \chi_n(x)\, d\mu$$

$$\leq \sum_{n\in J(x)} \|y_{v(i_n,\hat{\imath},z)} - v_{i_n,\hat{\imath}}\|_1 \leq \frac{\# J(x)\cdot\varepsilon}{6N(z)} \leq \frac{\# I(U_z)\cdot\varepsilon}{6N(z)} = \varepsilon/6.$$

Putting together (6.8) and (6.9), one has

$$d_{X\times L^1}\big((x, f_\varepsilon(x)), (x_{\hat{\imath}}, w')\big) \leq d_X(x, x_{\hat{\imath}}) + \|f_\varepsilon(x)-w\|_1 + \|w-w'\|_1 < \varepsilon/6+2\varepsilon/3+\varepsilon/6 = \varepsilon.$$

Hence $(x, f_\varepsilon(x)) \in B(\mathrm{graph}(F), \varepsilon)$. This completes the proof in the case where $L^1(T;E)$ is separable.

When X is separable, a slight modification of the above arguments is needed. The nbd-finite open covering $\{V_i;\ i\in I\}$ of X is now countable, because of the separability assumption. It is therefore possible to define the countable set $D = \{u_i;\ i\in I\} \cup \{v_{i,j};\ i, j\in I\}$ and arrange it into a sequence, say $D = \{y_n;\ n\geq 1\}$. After this choice of the set D, the rest of the proof goes exactly as in the previous case.

7. Proof of Theorem 3. In what follows, the main arguments are taken from [7]. We list first some preliminary results.

PROPOSITION 1. *For every family $\mathscr{K}$ of nonnegative measurable functions $u\colon T\to \mathbf{R}^+$, there exists a measurable function $v\colon T\to\mathbf{R}^+$ such that*

(i) *$v\leq u$ μ-a.e. for all $u\in\mathscr{K}$,*

(ii) *if w is a measurable function such that $w\leq u$ μ-a.e. for all $u\in\mathscr{K}$, then $w\leq v$ μ-a.e.*

Furthermore, there exists a sequence (u_n) in $\mathscr{K}$ such that

$$v(t) = \inf\{u_n(t);\ n\geq 1\} \quad \text{for a.e. } t \text{ in } T.$$

If the family $\mathscr{K}$ is directed downwards (i.e., if for any $u, u'\in\mathscr{K}$ there exists $w\in\mathscr{K}$ such that $w\leq u$ and $w\leq u'$ μ-a.e.), then the sequence (u_n) can be chosen to be decreasing.

For the proof, see Neveu [14, p. 121].

By (ii), the function v is unique up to μ-equivalence. It represents the greatest lower bound of $\mathscr{K}$ in the sense of μ-a.e. inequality, and is denoted by $\operatorname{ess\,inf}\{u;\ u\in\mathscr{K}\}$.

PROPOSITION 2. *Let K be a nonempty closed decomposable subset of $L^1(T;E)$ and let $\psi(t) = \operatorname{ess\,inf}\{\|u(t)\|_E;\ u\in K\}$. Then, for every $v_0\in L^1(T;\mathbf{R})$ such that $v_0(t)>\psi(t)$ μ-a.e., there exists an element $u_0\in K$ such that*

(7.1) $$\|u_0(t)\|_E < v_0(t) \quad \mu\text{-a.e.}$$

Proof. Notice that the set $\mathscr{K} = \{\|u(\cdot)\|_E;\ u\in K\}$ is a decomposable subset of $L^1(T;\mathbf{R})$. Therefore, it is directed downwards. Using Proposition 1, take a sequence $(u_n)_{n\geq 1}$ in K such that

$$\|u_m(t)\|_E \geq \|u_n(t)\|_E \quad \forall\, m<n,\ t\in T,$$
$$\psi(t) = \lim_{n\to\infty} \|u_n(t)\|_E \quad \mu\text{-a.e.}$$

Let now v_0 be given, with $v_0(t)>\psi(t)$ μ-a.e., and define the increasing sequence of sets: $T_0=\emptyset$, $T_n = \{t\in T;\ \|u_n(t)\|_E < v_0(t)\}$, $n\geq 1$. Observe that $\mu(T\setminus\bigcup_{n\geq 0} T_n) = 0$. Define the sequence (w_n) by setting

$$w_n(t) = \begin{cases} u_k(t) & \text{if } t\in T_k\setminus T_{k-1},\ k = 1,\ldots,n-1,\\ u_n(t) & \text{if } t\in T\setminus \bigcup_{k<n} T_k.\end{cases}$$

Since K is decomposable, each w_n belongs to K. Moreover, the sequence $w_n(t)$ is eventually constant for a.e. $t\in T$, and $\|w_n(t)\|_E \leq \|u_1(t)\|_E$ μ-a.e.; hence, by the Dominated Convergence Theorem, w_n converges in $L^1(T;E)$ to some function u_0. Clearly, $u_0\in K$ because K is closed. Finally, if $t\in T_n\setminus T_{n-1}$ for some n, then $\|u_0(t)\|_E = \|u_n(t)\|_E < v_0(t)$. Therefore, u_0 satisfies (7.1). ■

PROPOSITION 3. *Let X be a metric space and let $F\colon X\to \mathscr{D}(L^1(T;E))$ be a l.s.c. map with closed decomposable values. For all $x\in X$, set $\psi_x(t) = \operatorname{ess\,inf}\{\|u(t)\|_E;\ u\in F(x)\}$. Then the multivalued map $P\colon X\to L^1(T;\mathbf{R})$ defined as*

(7.2) $$P(x) = \{v\in L^1(T;\mathbf{R});\ v(t)>\psi_x(t)\ \mu\text{-a.e.}\}$$

is lower semicontinuous.

Proof. Let C be an arbitrary closed subset of $L^1(T;\mathbf{R})$. It suffices to show that, if $P(x_n)\subseteq C$ for some sequence $(x_n)_{n\geq 1}$ converging to x_0, then also $P(x_0)\subseteq C$. To this purpose, fix any $v_0\in P(x_0)$ and take, by Proposition 2, a function $u_0\in F(x_0)$ such that $\|u_0(t)\|_E < v_0(t)$ μ-a.e. Because of the lower semicontinuity of F, there exists a sequence $u_n\in F(x_n)$ such that $u_n\to u_0$ in $L^1(T;E)$. Then, for every $n\geq 1$, the function $v_n = \|u_n\|_E + v_0 - \|u_0\|_E$ belongs to $P(x_n)$ which is contained in C. Since the sequence (v_n) converges to v_0 in the norm of $L^1(T;\mathbf{R})$ and C is closed, this implies $v_0\in C$. ■

PROPOSITION 4. *Let X be a metric space and let $G\colon X\to\mathscr{D}(L^1(T;E))$ be a l.s.c. map with closed decomposable values. Assume that $g\colon X\to L^1(T;E)$ and $\varphi\colon X\to L^1(T;\mathbf{R})$ are continuous functions such that, for every $x\in X$, the set*

$$H(x) = \{u\in G(x);\ \|u(t)-g(x)(t)\|_E < \varphi(x)(t)\ \mu\text{-a.e.}\}$$

is nonempty. Then the map $H\colon X\to\mathscr{D}(L^1(T;E))$ is l.s.c. with decomposable values.

Proof. For every $x\in X$, $H(x)$ is the intersection of two decomposable sets, hence it is decomposable. To check the lower semicontinuity of H, let C be any closed subset of $L^1(T;E)$. It suffices to show that, for any sequence (x_n) in X converging to a point x_0, if $H(x_n)\in C$ for all $n\geqslant 1$, then $H(x_0)\subseteq C$.

To this purpose, fix any $u_0\in H(x_0)$. Because of the lower semicontinuity of G, there exists a sequence $u_n\in G(x_n)$ such that $u_n\to u_0$ in $L^1(T;E)$. By possibly taking a subsequence, we can assume that $u_n(t)$, $g(x_n)(t)$, $\varphi(x_n)(t)$ converge to $u_0(t)$, $g(x_0)(t)$, $\varphi(x_0)(t)$ respectively, μ-a.e. in T. Applying Egorov's theorem to these sequences w.r.t. the measure $\varphi(x_0)\cdot\mu$, for each $i\geqslant 1$ we obtain a measurable set $T_i\subseteq T$ such that u_n, $g(x_n)$ and $\varphi(x_n)$ converge uniformly on T_i and $\int_{T\setminus T_i}\varphi(x_0)\,d\mu<1/i$. For each $k\geqslant 1$, consider the sets

$$T_i^k=\{t\in T_i;\ \|u_0(t)-g(x_0)(t)\|_E<\varphi(x_0)(t)-1/k\}.$$

Notice that $\bigcup_{k\geqslant 1}T_i^k=T_i$ and $T_i^k\subseteq T_i^{k+1}$. Hence, for every $i\geqslant 1$, there exists a $k(i)$ such that

$$\int_{T_i\setminus T_i^{k(i)}}\varphi(x_0)\,d\mu<1/i.$$

Define $T_i'=T_i^{k(i)}$. The sets T_i' have the following properties:

$$\int_{T\setminus T_i'}\varphi(x_0)\,d\mu<2/i,\tag{7.3}$$

$$\|u_0(t)-g(x_0)(t)\|_E<\varphi(x_0)-1/k(i)\qquad\forall t\in T_i'.\tag{7.4}$$

By (7.4) and by the uniform convergence on T_i', for all $i\geqslant 1$ there exists some n_i such that

$$\|u_n(t)-g(x_n)(t)\|_E<\varphi(x_n)(t)\qquad\forall t\in T_i',\ n\geqslant n_i.\tag{7.5}$$

We can also assume that the sequence $(n_i)_{i\geqslant 1}$ is strictly increasing. For each n, choose an arbitrary $w_n\in H(x_n)$ and set, for $n_i\leqslant n<n_{i+1}$,

$$v_n=u_n\cdot\chi_{T_i'}+w_n\cdot\chi_{T\setminus T_i'}.$$

Since $H(x_n)$ is decomposable, $v_n\in H(x_n)$. We claim that $v_n\to u_0$ in $L^1(T;E)$, which implies $u_0\in C$.

Indeed, for $n_i\leqslant n<n_{i+1}$, (7.3) and (7.5) yield

$$\begin{aligned}\|v_n-u_0\|_1&\leqslant\int_{T\setminus T_i'}\|w_n-g(x_n)\|_E\,d\mu+\int_{T\setminus T_i'}\|g(x_n)-g(x_0)\|_E\,d\mu\\&\quad+\int_{T\setminus T_i'}\|g(x_0)-u_0\|_E\,d\mu+\int_{T_i'}\|u_n-u_0\|_E\,d\mu\\&\leqslant\int_{T\setminus T_i'}\varphi(x_n)\,d\mu+\|g(x_n)-g(x_0)\|_1+\int_{T\setminus T_i'}\varphi(x_0)\,d\mu+\|u_n-u_0\|_1\\&\leqslant[2/i+\|\varphi(x_n)-\varphi(x_0)\|_1]+\|g(x_n)-g(x_0)\|_1+2/i+\|u_n-u_0\|_1.\end{aligned}$$

As $n\to+\infty$, we also have $i\to+\infty$, hence our claim is proved. ∎

The next result, concerning the existence of approximate selections, is the core of the whole proof of Theorem 3.

Proposition 5. *Let X be a separable metric space and let $G\colon X\to D(L^1(T;E))$ be a l.s.c. map with closed decomposable values. Then, for every $\varepsilon>0$, there exist continuous maps $f_\varepsilon\colon X\to L^1(T;E)$ and $\varphi_\varepsilon\colon X\to L^1(T;\mathbf{R})$ such that f_ε is an ε-approximate selection of G, in the sense that, for each $x\in X$, the set*

$$G_\varepsilon(x)=\{u\in G(x);\ \|u(t)-f_\varepsilon(x)(t)\|_E<\varphi_\varepsilon(x)(t)\ \mu\text{-a.e.}\}\tag{7.6}$$

is nonempty, and $\|\varphi_\varepsilon(x)\|_1<\varepsilon$. Moreover, the map $x\to G_\varepsilon(x)$ is l.s.c. with decomposable values.

Proof. Fix $\varepsilon>0$. For every $\bar x\in X$ and $\bar u\in G(\bar x)$, the multivalued map Q defined as

$$Q(x)=\{v\in L^1(T;\mathbf{R});\ v(t)\geqslant\operatorname{ess\,inf}\{\|u(t)-\bar u(t)\|_E;\ u\in G(x)\}\ \text{for a.e. } t\in T\}\tag{7.7}$$

is l.s.c. with closed convex values.

To see this, define $F(x)=\{u-\bar u;\ u\in G(x)\}$. Then the map F is also l.s.c. with closed decomposable values. By Proposition 3, the multivalued map P defined in (7.2) is l.s.c. Hence Q is also l.s.c., because $Q(x)$ is the closure of $P(x)$, for all $x\in X$.

It is therefore possible to apply Michael's theorem to Q and obtain a continuous selection $\varphi_{\bar x,\bar u}$ such that $\varphi_{\bar x,\bar u}(x)\in Q(x)$ for all $x\in X$ and $\varphi_{\bar x,\bar u}(\bar x)\equiv 0$. The family of sets

$$\{\{x\in X;\ \|\varphi_{\bar x,\bar u}(x)\|_1<\varepsilon/4\};\ \bar x\in X,\ \bar u\in G(\bar x)\}$$

is an open covering of the separable metric space X, therefore it has a countable nbd-finite open refinement $\{V_n;\ n\geqslant 1\}$. Let $\{p_n(\cdot)\}$ be a continuous partition of unity subordinate to the covering $\{V_n\}$ and let $\{h_n(\cdot)\}$ be a family of continuous functions from X into $[0,1]$ such that $h_n\equiv 1$ on $\operatorname{supp}(p_n)$ and $\operatorname{supp}(h_n)\subset V_n$. For every $n\geqslant 1$, choose x_n, u_n such that $V_n\subseteq\{x;\ \|\varphi_{x_n,u_n}(x)\|_1<\varepsilon/4\}$ and set $\varphi_n=\varphi_{x_n,u_n}$. The functions φ_n have the following properties:

$$\varphi_n(x)(t)\geqslant\operatorname{ess\,inf}\{\|u(t)-u_n(t)\|_E;\ u\in G(x)\},\tag{7.8}$$

$$p_n(x)\|\varphi_n(x)\|_1\leqslant p_n(x)\cdot\varepsilon/4\qquad(x\in X,\ n\geqslant 1).\tag{7.9}$$

Lemma 2, applied to the sequences $\{\varphi_n\}$ and $\{h_n\}$, and to the function l: $l(x)=\sum_{n\geqslant 1}h_n(x)$, yields a continuous function $\tau\colon X\to\mathbf{R}^+$ and a family $\{\Phi(\tau,\lambda)\}$ of measurable subsets of T satisfying (a), (b) and (c').

It is now possible to construct the functions f_ε and φ_ε. Set $\lambda_0 \equiv 0$, $\lambda_n(x) = \sum_{m \leqslant n} p_m(x)$, and define

$$f_\varepsilon(x) = \sum_{n \geqslant 1} u_n \cdot \chi_n(x), \qquad \varphi_\varepsilon(x) = \varepsilon/4 + \sum_{n \geqslant 1} \varphi_n(x) \cdot \chi_n(x),$$

where

$$\chi_n(x) = \chi_{\Phi(\tau(x), \lambda_n(x)) \setminus \Phi(\tau(x), \lambda_{n-1}(x))}.$$

Clearly, f_ε and φ_ε are continuous, because the above summations are locally finite.

Let G_ε be defined by (7.6). To check that the values of G_ε are nonempty, fix any $x \in X$. For every $n \geqslant 1$, use Proposition 2 and select $u_x^n \in G(x)$ such that

(7.10) $$\|u_x^n(t) - u_n(t)\|_E < \varepsilon/4 + \operatorname{ess\,inf}\{\|u(t) - u_n(t)\|_E;\ u \in G(x)\}$$

μ-a.e. in T. Then

$$u_x = \sum_{n \geqslant 1} u_x^n \cdot \chi_n(x)$$

lies in $G(x)$, because $G(x)$ is decomposable. We claim that $u_x \in G_\varepsilon(x)$. Indeed, (7.8) and (7.10) yield

$$\|u_x(t) - f_\varepsilon(x)(t)\|_E \leqslant \sum_{n \geqslant 1} \|u_x^n(t) - u_n(t)\|_E \cdot \chi_n(x)(t) < \varphi_\varepsilon(x)(t) \quad \mu\text{-a.e. in } T.$$

Hence $G_\varepsilon(x) \neq \emptyset$. Being the intersection of two decomposable sets, $G_\varepsilon(x)$ is also decomposable. The lower semicontinuity of G_ε follows from Proposition 4.

To conclude the proof of Proposition 5, it now suffices to show that $\|\varphi_\varepsilon(x)\|_1 < \varepsilon$ for every x. Set $I(x) = \{n \geqslant 1;\ p_n(x) > 0\}$ and notice that $1 \leqslant \# I(x) \leqslant l(x)$. From (c′) in Lemma 2 and (7.9) we deduce

$$\|\varphi_\varepsilon(x)\|_1 = \varepsilon/4 + \sum_{n \geqslant 1} \int_T \varphi_n(x) \cdot \chi_n(x)\, d\mu$$

$$< \varepsilon/4 + \sum_{n \in I(x)} [p_n(x) \|\varphi_n(x)\|_1 + \varepsilon/(2l(x))] \leqslant \varepsilon/4 + \left[\varepsilon/4 + \frac{\# I(x) \cdot \varepsilon}{2l(x)}\right] \leqslant \varepsilon. \ \blacksquare$$

At this stage, everything is ready for the completion of the proof of Theorem 3.

Let the function F be given. Construct two sequences of continuous maps f_n: $X \to L^1(T; E)$ and φ_n: $X \to L^1(T; \mathbf{R})$, and a sequence of l.s.c. multifunctions G_n with decomposable values, such that, for all $x \in X$ and $n \geqslant 1$,

(i) $G_n(x) = \{u \in F(x);\ \|u(t) - f_n(x)(t)\|_E < \varphi_n(x)(t)\ \mu\text{-a.e.}\} \neq \emptyset$,

(ii) $\|f_n(x)(t) - f_{n-1}(x)(t)\|_E \leqslant \varphi_n(x)(t) + \varphi_{n-1}(x)(t)$ μ-a.e. in T $(n \geqslant 2)$,

(iii) $\|\varphi_n(x)\|_1 < 2^{-n}$.

To do this, define f_1 and φ_1 by applying Proposition 5 with $G = F$, $\varepsilon = 1/2$. Let now f_m, φ_m and G_m be defined so that (i)–(iii) hold for all $m = 1, \ldots, n-1$. To construct f_n and φ_n, apply again Proposition 5 with $\varepsilon = 2^{-n}$, defining $G(x)$ to be the closure of $G_{n-1}(x)$, for all x. By induction, the maps f_n, φ_n and G_n can be defined for all $n \geqslant 1$. By (ii), the sequence $(f_n)_{n \geqslant 1}$ is Cauchy in the L^1-norm, hence it converges uniformly to some continuous function f: $X \to L^1(T; E)$. By (i) and (iii), $d_{L^1}(f_n(x), F(x)) < 2^{-n}$. Since $F(x)$ is closed, this implies that $f(x) \in F(x)$ for all $x \in X$, hence f is a selection of F. $\blacksquare$

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