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Singularities of triples of vector fields on R^4: the focusing stratum

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Abstract. We consider the space $\mathscr{H}^{4,3}$ of germs of triples of smooth vector fields on R^4, with the action of the group $H^{4,3}$ generated by germs of coordinate changes and multiplication of triples by 3×3 invertible matrices of germs of functions. The paper is an extension of [M1]. The stratification of $\mathscr{H}^{4,3}$ of Thom–Boardman type, invariant under $H^{4,3}$ and studied in the general case of $\mathscr{H}^{n,k}$ by Jakubczyk and Przytycki in [JP2], is used. Attention is focused on the pivotal codimension 4 stratum S^F. We show that S^F cannot be further divided with the use of the Boardman-like algebraic approach. We also note some striking (nongeneric) phenomena in the behaviour of triples in S^F.

Contents

1. Introduction. In [JP2] a general method of studying the Boardman–Thom type stratification of the space $\mathscr{H}^{n,k}$ of germs of k-tuples of smooth vector fields at a point of R^n was presented. Highly nonintegrable k-tuples are considered (integrable ones have codimension ∞). The strata are invariant under the action of the group $H^{n,k}$ generated by germs of coordinate changes and of multiplication by $k\times k$ invertible matrices of functions. The aim of the theory is to find a rich decomposition of $\mathscr{H}^{n,k}$ into $H^{n,k}$-invariant sets approximating $H^{n,k}$-orbits as closely as possible. (We use the notion of "stratification" in a weaker sense than usual, understanding by it just a partition into semialgebraic sets. In each case, it is a matter of proof whether such a stratification is a stratification into manifolds in the sense of Thom [T].)

In [M1] attention was focused on triples of smooth vector fields on $\boldsymbol{R}^4$. We described there a stratification of the above type outside a codimension 5 algebraic set Q_5.

For all notation and precise definitions we refer to [JP2] and [M1]; however, in some key places we shall briefly recall their intuitive meaning.

The aim of this paper is to consider carefully a pivotal stratum from the stratification of $\mathscr{H}^{4,3}\setminus Q_5$ mentioned above, namely, the one which corresponds (for generic triples) to isolated points in $\boldsymbol{R}^4$ (like the point P in Fig. 2).

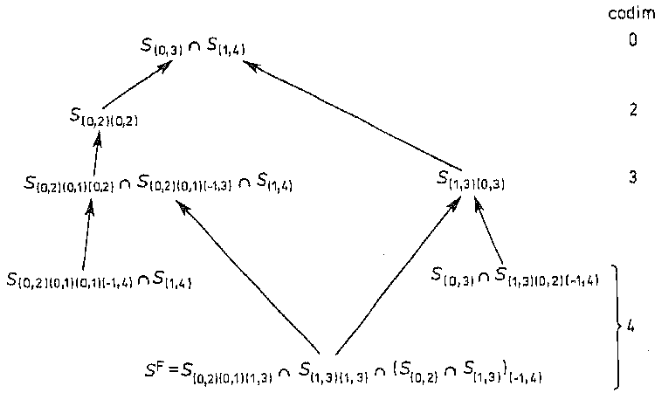

Fig. 1

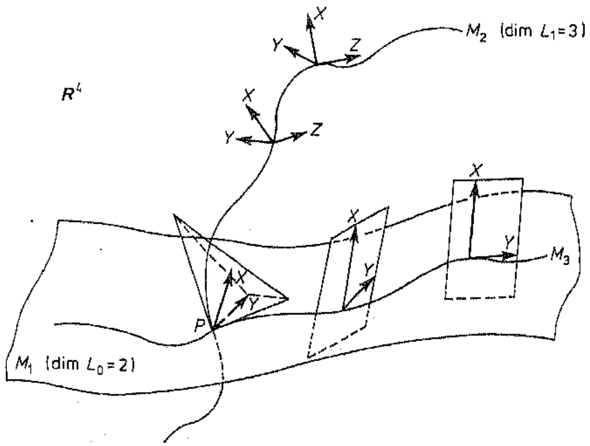

Fig. 2

These are intersection points of the traces M_1, M_2, M_3 of the singularity sets $S_{(0,2)}$, $S_{(1,3)}$, $S_{(0,2)(0,1)}$ respectively (see again Fig. 2). We call this curious stratum the "focusing stratum" and denote it by S^F (see Fig. 1). This stratum was formally defined by the formula (12) in [M1] (we recall that definition in the statement of Th. 1).

It was announced in [M1] that S^F is a (semialgebraic) manifold in $\mathscr{H}^{4,3}$.

Recall that the sets $S_{(0,m)}$ (as in Fig. 1) consist of those triples for which the dimension of the space spanned by the vector fields themselves (at, say, 0) is equal to m; $S_{(1,m)}$ refers to vector fields and their first Lie brackets.

The set $(S_{(0,2)} \cap S_{(1,3)})_{(-1,4)}$ is an example of the so-called generalized sets of type S. Sets of this kind used in this paper are not all possible generalizations of the sets S_I suggested by Remark 2.7 in [JP2]. This applies in particular to Theorem 1; on this point we refer to comments in 2.2.

The geometric idea underlying the rigorous algebraic definition of the sets S_I (I stands for admissible sequences of indices describing systems of consecutive jacobian extensions) consists in the following.

Consider, for a generic triple of vector fields, for example the set $\boldsymbol{M}_1$. (Recall that M_1 consists of points where the distribution $L_0(X, Y, Z)(\cdot) = \operatorname{span}(X, Y, Z)(\cdot)$ is 2-dimensional.) Generically M_1 is a smooth codimension 2 manifold in $\boldsymbol{R}^4$. One may ask how the distribution $L_0|_{M_1}$ behaves with respect to M_1 (i.e. the singularity set at the previous step). At the next step a singularity set $M_3 \subset M_1$ arises from investigation of singularities of this behaviour. M_3 consists of points where $L_0|_{M_1}$ has deficient codimension relative to TM_1 (i.e. has one direction in common with the tangent bundle). This procedure applied to the (generically!) smooth manifold M_3 determines the codimension of $L_0|_{M_3}$ relative to TM_3. In generic cases the procedure may be continued. Lie brackets of the vector fields may come into play too.

Note that the paper [MR] is devoted to the description of the behaviour of a triple of vector fields near a point in $M_1 \setminus M_3$. An explicit local model for the germ of such a triple is computed there. Observe that in [MR] (except for Fig. 1) the sets M_2, M_3 — in the notation of [M1] and of the present paper — are labelled M_3, M_2 respectively.

The geometric approach outlined above is in fact due to Thom (cf. his lectures [L]), who considered consecutive singularities of smooth mappings. The algebraic formalization is similar to that from Boardman's paper [B], which uses jacobian extensions of ideals. Nevertheless, in sharp contrast to Boardman's construction, in the case of k-tuples of vector fields there are many invariant subbundles of the tangent bundle (subbundles corresponding to the distributions $L_i(\cdot)$, $i \geqslant -1$, for a given k-tuple). This enriches the whole algebraic setting enormously and makes it possible to have nontrivial intersections of singularity sets, such as $S_{(0,2)} \cap S_{(1,3)} \subset \mathscr{H}^{4,3}$, containing the focusing stratum S^F.

From the algebraic point of view, the jacobian extension procedure over a given functional ideal describing the singularity set at some stage consists in differentiating the generators of the ideal in the directions of a tangent subbundle of a given order (working, as in [JP2], with functions on the germ space) and examining the rank r of the resulting matrix. (r is the codimension of the subbundle relative to the singularity set under consideration; cf. the original paper [JP1] and Remark 2.8 in [JP2].) One extends the given ideal by all $(r+1)$-minors of this matrix. The ideal thus obtained describes the singularity set at the next stage.

The ideals Δ_I used in defining the sets S_I are contained in the ring of smooth functions on the germ space $\mathscr{H}^{n,k}(U) = \mathscr{H}^{n,k} \times U$, U being a certain fixed neighbourhood of $0 \in \mathbf{R}^n$. Instead of dealing with Δ_I, we shall frequently formulate statements in terms of ideals related to a given k-tuple, $k = 3$, of vector fields (ideals in the ring of functions on U). For the purposes of this paper we recall briefly that, for a fixed triple (X, Y, Z) defined on U, we may express the fact that its germ at 0, $(\tilde{X}, \tilde{Y}, \tilde{Z})$, belongs to S_I in terms of the ideal

$$\operatorname{germ}_0(\Delta_I \circ (\tilde{\tilde{X}}, \tilde{\tilde{Y}}, \tilde{\tilde{Z}})),$$

where $(\tilde{\tilde{X}}, \tilde{\tilde{Y}}, \tilde{\tilde{Z}})\colon U \to \mathscr{H}^{4,3}(U)$ is the germ mapping induced by (X, Y, Z). This ideal is equal to $\partial_I(\tilde{X}, \tilde{Y}, \tilde{Z})$ (for the formal definition see [JP2], 1.4(e)). The ideals $\partial_I(\cdot)$ are described more simply than the ideals Δ_I (especially for triples of vector fields suitably chosen as in this paper). We recall briefly that one starts with a given triple (X, Y, Z) and defines $\partial_I(\tilde{X}, \tilde{Y}, \tilde{Z})$ by successive jacobian extensions in the ring $\mathscr{F}_0^4$ of germs at 0 of smooth functions on $\mathbf{R}^4$. If $(\tilde{X}, \tilde{Y}, \tilde{Z})$ is in $S_{I(i,j)}$, we shall say that $\partial_I(\tilde{X}, \tilde{Y}, \tilde{Z})$ satisfies the (i,j)-condition.

Throughout this paper, for the sake of simplifying the notation, we shall take the liberty to skip the tildes over triples of vector fields when working with their germs at $0 \in \mathbf{R}^4$.

The first objective of this work (dealt with in Sec. 3–7, 9) is to show the thinness of the focusing stratum S^{F}: it cannot be further divided using the sets S_I and generalized sets of type S. We shall include here a duality related to S^{F}, since this focusing stratum has also a second description involving no generalized sets of type S.

The second objective is to substantiate the so-called strange nongenericities manifested by triples of vector fields belonging to that stratum and by certain others lying in the boundary of S^{F}. In Sec. 7 we substantiate an important feature of this kind, formulated first in [M1], Sec. 4, (12). But there is also another example of the like, which is presented in Sec. 8. This second nongenericity (nontypical geometrical behaviour) concerns triples of vector fields not in the focusing stratum itself, but lying in its closure.

Acknowledgements. I would like to express my deep thanks to Professor F. Przytycki for his valuable criticism, steady attention to this work, and a number of indications (especially the advice to apply the technique of Lemma 4.4).

2. Statement of theorems. The sets S_I are defined for certain sequences of pairs of integers I which we call *admissible* (see [JP2], 1.4(d) and 2.1(g)). Generalized sets of type S are precisely defined in [M1], Sec. 3, (4). These are sets of type $(S_I \cap S_J)_{(i,j)}$ and $(S_I \underline{\cup} S_J)_{(i,j)}$ where I, J, (i,j) are admissible, $i \geqslant -1$. All sets of this form may be called *singularity sets of Boardman type*.

Thus we come to a thinness statement.

2.1. THEOREM 1. *The focusing stratum*

$$S^{\mathrm{F}} = S_{(0,2)(0,1)(1,3)} \cap S_{(1,3)(1,3)} \cap (S_{(0,2)} \cap S_{(1,3)})_{(-1,4)}$$

is a semialgebraic codimension 4 *manifold in the space* $\mathscr{H}^{4,3}$. *Moreover,* S^{F} *is included in, or disjoint from, every set* S_I *and every generalized set of type* S.

This focusing stratum cannot thus be subdivided by any singularity set of Boardman type.

2.2. *Comment on Theorem* 1. Generalizations of the sets S_I occurring in the formulation of Th. 1 are some of those suggested by Remark 2.7 in [JP2]. Their definition involves only one jacobian extension of a sum or an intersection of ideals defining the given sets S_I and S_J, so that we call them *one-step generalizations*. It does not seem reasonable to construct further generalizations in the way suggested there, since some one-step generalizations may lack geometric interpretation (i.e. the correspondence, mentioned in Sec. 1, between the algebraic and geometric aspects may no longer be valid).

One particular example of such a situation is presented and discussed in 7.6.

2.3. Note that we may consider the germ at 0 of the surface M_1 or the curve M_2 (cf. Sec. 1), corresponding to a germ of a triple of vector fields. Germs of triples of vector fields in the focusing stratum manifest a strange geometrical nongenericity. Namely, we have

THEOREM 2. *For a given germ belonging to* S^{F}, *vectors in the space spanned by the triple of vector fields at* $0 \in \mathbf{R}^4$ *cannot stick out of the three-dimensional vector space* $T_0 M_1 + T_0 M_2$, *i.e.*

$$L_0(X, Y, Z)(0) \subset T_0 M_1 + T_0 M_2.$$

In the proof of Theorem 1 we shall often use a certain technical tool concerning several smooth vector fields. The next section begins with that, and deals, as does Sec. 5, with consecutive eliminations of various thin strata (of codimension $\geqslant 5$) falling into the negligible nongeneric algebraic set Q_5 (cf. Sec. 1). This gives the necessary preparation for the explanation of striking geometric phenomena described in Secs. 7 and 8.

3. Elimination of nontransversal sets—the first encounter.

3.1. THE STRAIGHTENING LEMMA. *Let $X^1, \ldots, X^p$ be smooth vector fields defined in a neighbourhood of $0 \in \mathbf{R}^n$, and let Γ be an $(n-p)$-dimensional smooth manifold passing through $0 \in \mathbf{R}^n$ such that* $\operatorname{span}(X^1(0), \ldots, X^p(0)) \cap T_0\Gamma = \{0\}$. *Then there exists a new smooth coordinate system around $0 \in \mathbf{R}^n$ such that*

$$X^i|_{\{x_1=\ldots=x_{i-1}=0\}} \equiv \partial/\partial x_i, \quad i = 1, \ldots, p,$$
$$\Gamma = \{x_1 = \ldots = x_p = 0\}.$$

Proof. Suppose that Ψ: $(\mathbf{R}^{n-p}, 0) \to (\Gamma, 0)$ is a local smooth parametrization of Γ. Let φ_t^X denote the flow of the vector field X for the time t. The new coordinates around 0 are given by the following diffeomorphic mapping:

$$\varphi_{x_1}^{X^1} \varphi_{x_2}^{X^2} \ldots \varphi_{x_p}^{X^p}(\Psi(x_{p+1}, \ldots, x_n)) \mapsto (x_1, \ldots, x_n). \ \blacksquare$$

In proving Th. 1 we shall get rid of subsequent nongeneric parts of the singularity set

$$S_{(0,2)} \cap S_{(1,3)} \subset \mathscr{H}^{4,3} \tag{1}$$

(in order to reach S^F eventually). The set $S_{(0,2)} \cap S_{(1,3)}$ is a semialgebraic manifold of codimension 4 in $\mathscr{H}^{4,3}$, according to [JP2], 2.12(b).

3.2. *Restriction to* $S_{(0,2)(-1,2)}$. Since $S_{(0,2)}$ has codimension 2, we have $S_{(0,2)} = S_{(0,2)(-1,0)} \cup S_{(0,2)(-1,1)} \cup S_{(0,2)(-1,2)}$. The first two summands are of codimensions 10 and 5 respectively (cf. [JP2], 2.16(a)). We get rid of them, as of everything thin enough (having codimension greater than 4), keeping the remaining part a (semialgebraic) manifold.

3.3. *Restriction to* $S_{(0,2)(0,1)}$. A certain restrictive inclusion for the sets S_I holds in $\mathscr{H}^{4,3}$, namely

$$S_{(0,2)(0,2)} \subset S_{(1,4)}$$

(cf. [M1], (7)). It yields in turn the corresponding geometrical restriction relating the sets M_1, M_2 and M_3 for a given triple. (These sets need not be smooth geometric figures for this result.) Namely, M_2 may meet M_1 but at points of $M_3 \subset M_1$. (For generic triples we thus know in advance that the curve M_2 may cross the surface M_1 only at points of the curve M_3.) Hence, constructing S^F within the LHS of (1), we remain in $S_{(0,2)(0,0)} \cup S_{(0,2)(0,1)}$. Since $S_{(0,2)(0,0)}$ is a codimension 6 manifold (cf. [JP2], 2.16(b)), we consider in the sequel only the manifold

$$S^{(I)} = S_{(0,2)(-1,2)} \cap S_{(0,2)(0,1)} \cap S_{(1,3)}.$$

Here I stands for a Roman ordinal, not for a sequence of pairs of integers. The set $S^{(I)}$, resulting after subtraction of the thin parts, is still a—semialgebraic!—manifold, because the union of all thin parts subtracted up to now is closed. This argument will often be applied later on.

4. Excision of the $S_{(0,2)(1,1)}$-part. The use of the special forms technique.

We want to excise further from $S^{(I)}$ certain thin $H^{4,3}$-invariant sets expressing nontypical singularities. In this section we shall eliminate the $S_{(0,2)(1,1)}$-singularity (meaning, for the triples in $S^{(I)}$, that $T_0 M_1 \subset L_1(X, Y, Z)(0)$). We are going to describe a procedure of transforming germs in $S^{(I)}$ by means of elements of the group $H^{4,3}$, which generally brings the triples into a simpler form.

4.1. *The special procedure* (I). For every triple in $S^{(I)}$, the set M_1 is a smooth 2-dimensional surface (by the geometric interpretation of $S_{(0,2)(-1,2)}$). Additionally, $T_0 M_1$ has codimension 1 in $T_0 M_1 + L_0(X, Y, Z)(0)$ (by the interpretation of $S_{(0,2)(0,1)}$). The announced procedure runs as follows:

1° Choose a triple in the $H^{4,3}$-orbit of a given triple such that $X(0) \notin T_0 M_1$, $Y(0) \in T_0 M_1$, $Y(0) \neq 0$.

2° Straighten simultaneously X and Y using a suitable plane Γ not transversal to M_1 at 0 (see Lemma 3.1, with the coordinates labelled x, y, z, w) and its parametrization such that in the outcome $\Gamma \cap T_0 M_1 = \operatorname{span}(\partial/\partial w)(0)$.

3° Replace the vector field $Y = [Y^1, Y^2, Y^3, Y^4]$ by $(1/Y^2)(Y - Y^1 X)$.

4° Replace Z by $Z - Z^1 X - Z^2 Y$.

We thus obtain a triple of the form

$$(\partial/\partial x,\ \partial/\partial y + \alpha\partial/\partial z + \beta\partial/\partial w,\ B\partial/\partial z + C\partial/\partial w). \tag{2}$$

4.1.1. Remark. For the obtained triple (2) we have

$$\alpha|_{x=0} = \beta|_{x=0} = 0, \tag{3}$$

$$B,\ C,\ B_y,\ C_y,\ B_w,\ C_w \text{ vanish at } 0 \in \mathbf{R}^4, \tag{4}$$

$$\begin{vmatrix} B_x & B_z \\ C_x & C_z \end{vmatrix}(0) \neq 0, \tag{5}$$

$$\begin{vmatrix} \alpha_x & B_x \\ \beta_x & C_x \end{vmatrix}(0) = 0. \tag{6}$$

Proof. (3) follows by applying Lemma 3.1; step 3° does not violate this property. The ideal $\partial_{(0,2)}(X, Y, Z) \subset \mathscr{F}_0^4$ is generated by B and C, thus M_1 is described by $B = C = 0$. But $Y(0) = (\partial/\partial y)(0) \in T_0 M_1$ (3° does not change this) and $(\partial/\partial w)(0) \in T_0 M_1$ by 2°, hence (4) follows.

(4), together with the triple being in $S_{(0,2)(-1,2)}$, implies (5). Now it follows from (5) that

$$\dim L_1(X, Y, Z)(0) \geqslant 3. \tag{7}$$

It is $[X, Z](0)$ that is linearly independent of $X(0)$, $Y(0)$. Within $S^{(1)} \subset S_{(1,3)}$ the remaining first Lie brackets must then belong to span $(X, Y, [X, Z])(0)$. Therefore the vectors $[Y, Z](0)$ and $[X, Y](0)$ ought to be proportional to $[X, Z](0)$, which means (6), since $[Y, Z](0) = 0$. ∎

4.1.2. Definition. A triple in the manifold $S^{(1)}$ will be called a *special form* (I) *triple* if it is of the form (2) and fulfils the conditions (3)–(6) (within $S^{(1)}$ the last two follow from (3) and (4)).

We denote the subset of $S^{(1)}$ consisting of all such triples by $\mathrm{Spec}^{\mathrm{I}}$.

4.1.3. Remark. $S_{(0,2)(-1,2)} \cap S_{(0,2)(0,1)} \subset S_{(1,3)} \cup S_{(1,4)}$.

Proof. The special procedure (I) may be applied in the whole LHS of the inclusion, and there (7) holds. ∎

$\mathrm{Spec}^{\mathrm{I}}$ is a very thin subset of $S^{(1)}$, it has infinite codimension there (in the sense that the codimensions of its projections into $j^r S^{(1)} \subset j^r \mathscr{H}^{4,3}$ tend to infinity as $r \to \infty$). Consequently, we shall have to work on some finite jet level if we want to apply the classical differential topology as a tool for the excisions.

4.2. *The simplified special procedure* (I) *in the vicinity of* $\mathrm{Spec}^{\mathrm{I}}$. We would like to describe the special procedure (I) in a uniform way with respect to a triple of vector fields, for triples in $S^{(1)}$ close to the special form (I) triples in the Whitney topology. We need a neighbourhood U of $\mathrm{Spec}^{\mathrm{I}}$ in $S^{(1)}$ so small that for $(X, Y, Z) \in U$:

(a) $X(0) \notin T_0 M_1$, $X^1(0) > 3\max(|X^2(0)|, |X^3(0)|, |X^4(0)|)$.

(b) $Y^2(0) > 3\max(|Y^1(0)|, |Y^3(0)|, |Y^4(0)|)$.

(c) $(\partial/\partial z)(0) \notin T_0 M_1$.

(d) For $W =$ the orthogonal projection of $(\partial/\partial w)(0)$ on $T_0 M_1$,

$$W^4 > 3\max(|W^1|, |W^2|, |W^3|).$$

(All this holds obviously in $\mathrm{Spec}^{\mathrm{I}}$.)

4.2.1. Remark. U is defined on the 1-jet level (i.e. in terms of 1-jets of germs).

4.2.2. Under conditions (a)–(d), the vectors $X(0)$, $Y(0)$, $(\partial/\partial z)(0)$, W form a basis of $T_0 \boldsymbol{R}^4$.

Idea of proof. Each of the above vectors has its characteristic coordinate, greatly predominant in size. Thus no linear combination of any three of them can be equal to the fourth one. ∎

Because of 4.2.2, for every constant c, the vectors $X(0)$, $Y(0)+cX(0)$, $(\partial/\partial z)(0)$, W are also a basis of $T_0 \boldsymbol{R}^4$.

4.2.3. The *uniform special procedure* (I), defined in U, denoted by $\tilde{\Phi}$, consists in:

$1^{\circ\prime}$ Performing a linear combination X, $Y+cX$, Z, with c depending smoothly on $j^1(X)$, $j^1(Y)$ such that $(Y+cX)(0) \in T_0 M_1$.

$2^{\circ\prime}$ A linear change of coordinates, depending smoothly on $j^1(X, Y, Z)$, preserving the versors $(\partial/\partial x)(0)$, $(\partial/\partial y)(0)$, $(\partial/\partial z)(0)$ and replacing $(\partial/\partial w)(0)$ by W (see condition (d) in the definition of U).

$3^{\circ\prime}$ The simultaneous straightening of X and Y with the help of $\Gamma = \mathrm{span}(\partial/\partial z, \partial/\partial w)(0)$ with its natural parametrization (Lemma 3.1).

$4^{\circ\prime}$ Executing steps 3° and 4° of the special procedure (I), 4.1.

Thanks to conditions (a)–(d) assumed to hold in U, images of $\tilde{\Phi}$ are indeed special form (I) triples in the sense of Def. 4.1.2.

4.2.4. Remark. $\tilde{\Phi}|_{\mathrm{Spec}^{\mathrm{I}}} = \mathrm{id}$.

Proof. Step $3^{\circ\prime}$ only requires some comments. For $(X, Y, Z) \in \mathrm{Spec}^{\mathrm{I}}$ the straightening diffeomorphism $\varphi_x^X \varphi_y^Y(0, 0, z, w) \mapsto (x, y, z, w)$ is the identity, for Y is already straightened at points $(0, 0, z, w)$, and hence $\varphi_y^Y(0, 0, z, w) = (0, y, z, w)$, and $X = \partial/\partial x$ everywhere. ∎

The described uniform special procedure $\tilde{\Phi}$ will serve in the next section to define a submersion allowing us to excise the thin part $S_{(0,2)(1,1)}$.

4.3. *Construction of a submersion on a chosen finite jet level.* We can construct, with the help of the mapping $\tilde{\Phi}$, a smooth submersion Φ on an arbitrary r-jet level, $r \geqslant 1$.

Every point p in U is mapped into $\mathrm{Spec}^{\mathrm{I}}$ with the help of a certain $h_p \in H^{4,3}$:

$$\tilde{\Phi}(p) = h_p(p), \tag{8}$$

where h_p consists in executing operations $1^{\circ\prime}$–$4^{\circ\prime}$ on an *arbitrary* triple of vector fields (so that it acts on the whole $\mathscr{H}^{4,3}$), with the operations being defined by the triple p (and its consecutive images under the earlier operations). The $H^{4,3}$-group action on $\mathscr{H}^{4,3}$ induces the jet action of $j^{r+1} H^{4,3}$ on $j^r \mathscr{H}^{4,3}$ ($(r+1)$-jets of function matrices act with an effective use of their r-jet truncations only). So a definition we intend to give could read

$$\Phi(j^r p) = (j^{r+1} h_p)(j^r p), \tag{9}$$

provided that different representatives of the considered r-jet of a triple, $j^r p = j^r p'$, yield the same induced element of $j^{r+1} H^{4,3}$, i.e., $j^{r+1} h_p = j^{r+1} h_{p'}$. For the uniform special procedure $\tilde{\Phi}$ this is really the case:

4.3.1. Lemma. *If the triples* (X, Y, Z) *and* (X', Y', Z') *in* $U \subset S^{(1)}$ *have the same r-jets at* 0, *then the straightening diffeomorphisms at step* $3^{\circ\prime}$ *in* 4.2.3 *have the same* $(r+1)$*-jets at* 0.

Sketch of proof. Estimating the distance between $q_t = \varphi_t^Y(0, 0, z, w)$ and $q_t' = \varphi_t^{Y'}(0, 0, z, w)$, $0 \leqslant |t| \leqslant |y|$, notice that both points remain within an $O(\max(|x|, |y|, |z|, |w|))$-distance of $0 \in \boldsymbol{R}^4$. Write $M = \max(|x|, |y|, |z|, |w|)$. Because

$$\|Y(q_t) - Y'(q_t')\| \leqslant \|Y(q_t) - Y(q_t')\| + \|Y(q_t') - Y'(q_t')\|$$
$$= O(M) + O(M^{r+1}) = O(M),$$

it is possible to estimate the distance between q_y and q_y' by a higher order quantity.

Writing $q_y - q_0$ as the vector-valued integral $\int_0^y Y(q_t)\,dt$, and using the fact that $q_0 = q_0' = (0, 0, z, w)$, we have

$$\|q_y - q_y'\| = \Big\|\int_0^y Y(q_t)\,dt - \int_0^y Y'(q_t')\,dt\Big\| \leqslant \int_0^y \|Y(q_t) - Y'(q_t')\|\,dt$$
$$= \int_0^y O(M)\,dt = O(M^2).$$

So one can repeat the above estimate for $\|Y(q_t) - Y'(q_t')\|$ with a higher accuracy $O(M^2)$. Proceeding inductively, one obtains an $O(M^{r+1})$-estimate for this deviation of velocity vectors. So finally $\|q_y - q_y'\| = O(M^{r+2})$. Estimating now

$$\|\varphi_x^X(q_y) - \varphi_x^{X'}(q_y')\| \leqslant \|\varphi_x^X(q_y) - \varphi_x^X(q_y')\| + \|\varphi_x^X(q_y') - \varphi_x^{X'}(q_y')\|,$$

we have, as above, an $O(M^{r+2})$-bound for the second term. The first term is $O(M^{r+2})$ as well, being the distance between the images of q_y, q_y' under the locally Lipschitzian mapping $\varphi_x^X(\cdot)$. Thus the inverse diffeomorphisms to the discussed straightening ones have the same $(r+1)$-jets at 0. ∎

4.3.2. Remark. It is important that the straightening diffeomorphism in 4.2.3 does not involve any Lie brackets of vector fields in the considered triple. Otherwise Lemma 4.3.1 would not hold; Φ would not be defined by (9). Only a mapping between jet spaces of different orders would then be induced by $\tilde{\Phi}$ (cf. (8)). Consequently, Lemma 4.4 below would have to be replaced by a technically more complicated statement.

4.3.3. Proposition. *Fix $r \geqslant 1$. Then Φ: $j^r U \to j^r \mathrm{Spec}^l$, restricted to a certain open $\tilde{U} \supset j^r \mathrm{Spec}^l$, is a smooth surjective submersion.*

Proof. The definition (9) of Φ is correct:

(i) The $H^{4,3}$-action at step $1^{o'}$ in 4.2.3 depends only on $j^1 p$, and so does the action's $(r+1)$-jet, coinciding with its 0-jet (there acts a constant matrix if p is fixed).

(ii) The $H^{4,3}$-action at step $2^{o'}$ depends on the position of the plane $T_0 M_1$ only, i.e. on the 1-jet of the image of p under step $1^{o'}$, and so does the action's $(r+1)$-jet, coinciding with its 1-jet (this is a linear change of coordinates).

(iii) Step $3^{o'}$ is sound by Lemma 4.3.1; the $(r+1)$-jet of the diffeomorphism g_p acting on triples at $3^{o'}$ depends on the r-jet of the image of p under the action $\bar{h}_p := 2^{o'} \circ 1^{o'}$.

(iv) The entries of the function matrix acting on triples at $4^{o'}$ are rational in coordinate functions of the image of p under the action $\bar{\bar{h}}_p := g_p \cdot \bar{h}_p$ $(= 3^{o'} \circ 2^{o'} \circ 1^{o'})$. Therefore

(*) The r-jet of the matrix acting at $4^{o'}$ smoothly depends on $j^r \bar{\bar{h}}_p(p)$.

According to a remark preceding the definition (9), only the r-jet truncations of the $(r+1)$-jets of matrices act on $j^r \mathscr{H}^{4,3}$.

Thus Φ is well defined. To prove the smoothness of Φ let us choose the simplest possible smooth section s: $j^r U \to U$ of π_r: $U \to j^r U$, namely the one assigning to a set of r-jet coordinates the triple of polynomial vector fields with these coordinates as coefficients.

We claim the mapping $j^r U \to j^{r+1} H^{4,3}$, $j^r p \mapsto j^{r+1} h_{s(j^r p)}$ is smooth.

That will finish the proof, since by (i)–(iv) this mapping is nothing but the well-defined mapping $j^r p \mapsto j^{r+1} h_p$.

To prove the claim, note that

(**) The mapping $j^r p \mapsto j^{r+1} \bar{h}_{s(j^r p)}$ is smooth,

because the dependences mentioned in (i) and (ii) are rational in the $j^1 p$-coordinates, and $r \geqslant 1$.

Let $(\bar{X}, \bar{Y}, \bar{Z}) := \bar{h}_{s(j^r p)}(s(j^r p))$. The diffeomorphism $g^{-1}_{s(j^r p)}$ carries (x, y, z, w) to $\varphi_x^{\bar{X}} \varphi_y^{\bar{Y}}(0, 0, z, w)$. Thanks to (i), (ii)

(***) $\bar{X}, \bar{Y}, \bar{Z}$ are polynomial vector fields of order $\leqslant r$

(whose coefficients are rational functions of $j^1 p$).

Let us consider another mapping

$$(x, y, z, w, \text{ coefficients of } \bar{X}, \bar{Y}) \mapsto \varphi_x^{\bar{X}} \varphi_y^{\bar{Y}}(0, 0, z, w).$$

Its smoothness is due to the smooth dependence of the solution to a smooth ordinary differential equation on parameters and initial values (the coefficients of $\bar{Y}$ influence the initial values of solutions of the equation determined by $\bar{X}$). Thus each mapping

$$(x, y, z, w, \text{ coefficients of } \bar{X}, \bar{Y}) \mapsto \frac{\partial^l(\varphi_x^{\bar{X}} \varphi_y^{\bar{Y}}(0, 0, z, w))}{\partial x_{i_1} \ldots \partial x_{i_l}},$$

$x_{i_1}, \ldots, x_{i_l} \in \{x, y, z, w\}$, is smooth too. In particular, all the

$$\frac{\partial^l(\varphi_x^{\bar{X}} \varphi_y^{\bar{Y}}(0, 0, z, w))}{\partial x_{i_1} \ldots \partial x_{i_l}}(0, 0, 0, 0, \text{ coefficients of } \bar{X}, \bar{Y})$$

are smooth. Considering $l \leqslant r+1$ yields that $j^{r+1}(g^{-1}_{s(j^r p)})$ depends smoothly on the coefficients of $\bar{X}$, $\bar{Y}$, and so does $j^{r+1} g_{s(j^r p)}$. By (***), the latter depends smoothly on

$$j^r\left(\bar{h}_{s(j^r p)}(s(j^r p))\right) = (j^{r+1}\bar{h}_{s(j^r p)})(j^r s(j^r p)) = (j^{r+1}\bar{h}_{s(j^r p)})(j^r p).$$

This together with (**) yields that

$$j^r p \mapsto (j^{r+1} g_{s(j^r p)})(j^{r+1}\bar{h}_{s(j^r p)})$$

is smooth, which means that

$$j^r p \mapsto j^{r+1}\bar{\bar{h}}_{s(j^r p)}$$

is smooth. Via (*) we know, at long last, that $j^r p \mapsto j^{r+1} h_{s(j^r p)}$ is smooth too. The claim is thus proved.

To show that the smooth mapping Φ is—locally around $j^r\mathrm{Spec}^{\mathrm{I}}$—an onto submersion, note that for $j^r q \in j^r\mathrm{Spec}^{\mathrm{I}}$ and a representative p of $j^r q$ such that $p \in \mathrm{Spec}^{\mathrm{I}}$ we have, by Remark 4.2.4, $h_p = \mathrm{id}$. In particular, $j^{r+1} h_p = \mathrm{id}$, and $\Phi|_{j^r\mathrm{Spec}^{\mathrm{I}}} = \mathrm{id}$. Thus $D\Phi(j^r q)$ is an epimorphism (being the identity when restricted to $T_{j^r q} j^r\mathrm{Spec}^{\mathrm{I}}$). Hence there exists a neighbourhood $\tilde{U}$ of $j^r\mathrm{Spec}^{\mathrm{I}}$ in $j^r U$ where $D\Phi(\cdot)$ is an epimorphism. ∎

4.4. THE DIFFERENTIAL TOPOLOGY LEMMA. *Suppose S is a finite-dimensional smooth manifold, $S^{\mathrm{spec}} \subset S$ is its regular smooth submanifold and there is a family $H \subset \mathrm{Diff}(S)$ such that $\forall p \in S\ \exists h \in H,\ h(p) \in S^{\mathrm{spec}}$.*

Let $Q \subset S$ be H-invariant such that $Q \cap S^{\mathrm{spec}}$ is a regular smooth codimension d submanifold of S^{spec}. Let additionally Q and $S \setminus Q$ be preserved by a certain family of local submersions, namely assume $\forall p \in S^{\mathrm{spec}}\ \exists$ a smooth submersion $\Psi\colon V \to S^{\mathrm{spec}}$ ($p \in V \subset S$, V open) such that $\Psi(V \cap Q) \subset S^{\mathrm{spec}} \subset Q$, $\Psi(V \setminus Q) \subset S^{\mathrm{spec}} \setminus Q$.

Then Q is a regular smooth codimension d submanifold of S.

This lemma has some similarity and plays an analogous role to Lemma 6 in [R], p. 76.

Proof. Take an arbitrary $q \in S$ and $h \in H$ with $h(q) \in S^{\mathrm{spec}}$. Take a submersion pair Ψ, V for $h(q)$. Then

$$q' \in h^{-1}(V) \cap Q \iff h(q') \in V \cap Q \iff \Psi(h(q')) \in S^{\mathrm{spec}} \cap Q.$$

So $h^{-1}(V) \cap Q$ is the counterimage of $S^{\mathrm{spec}} \cap Q$ under the smooth submersion $\Psi \circ h$ (its image is an open subset of S^{spec}, for each submersion is an open mapping). ∎

4.4.1. Remark. When applying Lemma 4.4 to the situation from Prop. 4.3.3, we will have one global submersion Ψ of $V \supset S^{\mathrm{spec}}$ onto S^{spec}. Then also $\Psi \circ h$ in the above proof will have the whole S^{spec} as its image. This will not mean that the entire Q is the counterimage of $Q \cap S^{\mathrm{spec}}$ under any such $\Psi \circ h$ — $h^{-1}(V)$ need not cover Q. (For a given $q \in Q$, the diffeomorphism $h = h_q$ does not send—in general—other $q' \in Q$ into S^{spec}. Similarly, $h_q(q') \in V$ need not hold.)

4.5. *Achieving the excision.* Since the manifold $S^{(\mathrm{I})}$ and the singularity set $S_{(0,2)(1,1)}$ are both defined in $j^1\mathscr{H}^{4,3}$, we are going to apply Lemma 4.4 for $S = j^1 S^{(\mathrm{I})}$, $Q = j^1(S_{(0,2)(1,1)} \cap S^{(\mathrm{I})})$, $S^{\mathrm{spec}} = j^1\mathrm{Spec}^{\mathrm{I}}$, $H = j^2 H^{4,3}$, $\Psi = \Phi$ defined by (9) for $r = 1$, $V = \tilde{U}$ for $r = 1$ ($\tilde{U}$ is defined in the proof of 4.3.3).

4.5.1. Observation. In this interpretation $Q \cap S^{\mathrm{spec}}$ is a regular smooth codimension 1 submanifold of S^{spec}.

Proof. Take an arbitrary representative $(X, Y, [Z^1, Z^2, B, C])$ of a given $p \in Q \cap S^{\mathrm{spec}}$. We have $Z^1, Z^2 \in \mathfrak{m}^2$ and by Cramer's formulas we may find smooth functions $f, g \in \mathfrak{m}^2$ such that $Z' = Z + fX + gY$ has its 1st and 2nd coordinates vanishing (cf. step 4° in 4.1). Then $Z'^3 \equiv B \bmod \mathfrak{m}^3$, $Z'^4 \equiv C \bmod \mathfrak{m}^3$, so that

$$\partial_{(0,2)}(X, Y, Z) = \partial_{(0,2)}(X, Y, Z') = \langle B + \text{sth 2-flat},\ C + \text{sth 2-flat}\rangle$$

(cf. (31)). $(X, Y, Z) \in S_{(0,2)(1,1)}$ means that all 2-minors of the matrix of the derivatives of generators of $\partial_{(0,2)}(X, Y, Z)$ in the directions included in $L_1(X, Y, Z)(0)$ vanish. This last space is spanned by the vectors $X(0)$, $Y(0)$, $[X, Z](0)$, but $Y(0)$—by (4)—differentiates that ideal to 0 ((X, Y, Z) has properties (4)–(7) of special form (I) triples, which concern 1-jets only). Hence the sole condition reads

$$\begin{vmatrix} X(B + \text{sth 2-flat}) & [X, Z](B + \text{sth 2-flat}) \\ X(C + \text{sth 2-flat}) & [X, Z](C + \text{sth 2-flat}) \end{vmatrix}(0) = 0.$$

Computing and taking account of (4) and (5), we obtain the equation

$$B_x(0) = 0, \tag{10}$$

expressed in terms of $j^1(X, Y, Z) = j^1(X, Y, Z') = p$. The equation (10) describes a regular, smooth, codimension 1 submanifold in S^{spec}. Indeed, $C_x(0) \neq 0$ then and $B_x(0)$ is a regular smooth (coordinate!) function on $j^1\mathrm{Spec}^{\mathrm{I}}$; its regularity at p manifests itself e.g. on the line

$$p(t) = j^1\left(X,\ Y + t\frac{Y_x^4}{C_x}(0)\, x\partial/\partial z,\ Z + tx\partial/\partial z\right),$$

$p(0) = p$, included in S^{spec} for small $|t|$ (cf. (3)–(6)). ∎

All the assumptions of Lemma 4.4 are fulfilled, including $\Phi(V \cap Q) \subset S^{\mathrm{spec}} \cap Q$, $\Phi(V \setminus Q) \subset S^{\mathrm{spec}} \setminus Q$ (the submersion Φ is defined with the help of elements of $j^2 H^{4,3}$, and Q, $j^1\mathscr{H}^{4,3} \setminus Q$ are $j^2 H^{4,3}$-invariant).

Elimination of the $S_{(0,2)(1,1)}$-part follows: now that we know that it constitutes a thin, regular part of $S^{(I)}$ (its closure is an algebraic set of codimension 5), we excise it, obtaining the smooth, regular, semialgebraic, still 4-codimensional manifold

(11) $$S^{(II)} = S^{(I)} \cap S_{(0,2)(1,2)} = S_{(0,2)(1,2)} \cap S_{(1,3)}.$$

5. Excision of further nongeneric parts. Refinements of the special forms technique. Applying the special procedure (I) (cf. 4.1) to a triple in $S^{(II)}$ we get a special form (I) triple in $\mathrm{Spec}^{I} \cap S^{(II)}$ ($S^{(II)}$ is $H^{4,3}$-invariant). Thus (10) is *not* valid, $B_x(0) \neq 0$.

5.1. *The special procedure* (II). For every triple in $S^{(II)}$ we extend the special procedure (I) by adding one more step to 4.1:

5° Take new coordinates $\left(x,\ y,\ \dfrac{1}{B_x(0)}z,\ w - \dfrac{C_x(0)}{B_x(0)}z\right)$.

The resulting triple is still of the form (2); let us retain the names α, β, B, C for the appropriate coordinates of the outcome triple.

The properties (3)–(6) remain valid and additionally we now have $B_x(0) = 1$, $C_x(0) = 0$, i.e.

(12) $$[X, Z](0) = [0,\ 0,\ B_x,\ C_x](0) = (\partial/\partial z)(0).$$

5.1.1. Remark. The description $T_0 M_1 = \mathrm{span}(\partial/\partial y,\ \partial/\partial w)(0)$ valid for special form (I) triples still holds after step 5°.

5.1.2. Definition. A triple in $S^{(II)}$ will be called a *special form* (II) *triple* if it is of the form (2) and fulfils conditions (3)–(6) and (12).

We denote the subset of such triples in $S^{(II)}$ by Spec^{II}.

5.2. *The simplified special procedure* (II) *near* Spec^{II}. We define a neighbourhood of Spec^{II} in $S^{(II)}$, $U^{II} = U \cap S^{(II)}$ (U^{II} is defined on the 1-jet level, cf. Remark 4.2.1). Over every germ in U^{II} we may perform a *uniform special procedure* (II), written $\tilde{\Phi}^{(II)}$, which is the extension of $\tilde{\Phi}$ (cf. 4.2.3) by the same step as in 5.1:

5°′ The linear change of coordinates $\left(x,\ y,\ \dfrac{1}{B_x(0)}z,\ w - \dfrac{C_x(0)}{B_x(0)}z\right)$, depending smoothly on $j^1(Z)$.

(Since $\tilde{\Phi}$ yields special form (I) triples, and $S^{(II)}$ is preserved under $\tilde{\Phi}$, we have $B_x(0) \neq 0$, as noted before the definition 5.1 of the special procedure (II).)

Obviously, images of $\tilde{\Phi}^{(II)}$ are special form (II) triples.

5.2.1. Remark. $\tilde{\Phi}^{(II)}|_{\mathrm{Spec}^{II}} = \mathrm{id}$.

Proof. See Remark 4.2.4 and note that step 5°′ does not change coordinates because $B_x(0) = 1$ on Spec^{II}. ∎

5.3. *Excision of the* $S_{(0,2)(0,1)(0,1)}$*-part.*

5.3.1. *Definition of the submersion* $\Phi^{(II)}$. Observe that the construction of 4.3 may be carried out with the mapping $\tilde{\Phi}^{(II)}$ instead of $\tilde{\Phi}$ (on an arbitrary r-jet level as well, $r \geqslant 1$). Indeed, represent $\tilde{\Phi}^{(II)}$ as in (8) and define $\Phi^{(II)}$ using $\tilde{\Phi}^{(II)}$ as in (9). This definition is correct, an argument analogous to 4.3.3 applies. To this effect one should only remark that

(v) The $H^{4,3}$-action at step 5°′ depends (smoothly) on $j^1(h_p(p))$ (see (8)), and so does its $(r+1)$-jet (this action is a linear change of coordinates; $r \geqslant 1$ is important).

5.3.2. Proposition. *Fix* $r \geqslant 1$. *Then* $\Phi^{(II)}\colon j^r U^{II} \to j^r \mathrm{Spec}^{II}$ *is a smooth onto submersion when restricted to a certain open* $\tilde{U}^{II} \supset j^r \mathrm{Spec}^{II}$.

Proof. Taking into account Prop. 4.3.3, one should only verify that step 5°′ does not violate smoothness. According to (v) above, the operation at 5°′, observed on the r-jet level and applied to $j^r(h_p(p))$, means a linear transformation of the coordinates of $j^r(h_p(p))$, the coefficients of the transformation being products of powers of $B_x(0)$ and $C_x(0)$ (with sums of exponents $\leqslant r$), some of them multiplied by $1/B_x(0)$ or $(C_x/B_x)(0)$. Remark 5.2.1 implies that the open set $\tilde{U}^{II}$ where $D\Phi^{(II)}(\cdot)$ is an epimorphism contains $j^r \mathrm{Spec}^{II}$. ∎

Because the singularity set $S_{(0,2)(0,1)(0,1)}$ is defined in 2-jets of germs of triples, Lemma 4.4 will be applied for $S = j^2 S^{(II)}$,

$$Q = j^2(S_{(0,2)(0,1)(0,1)} \cap S^{(II)}), \qquad S^{\mathrm{spec}} = j^2 \mathrm{Spec}^{II},$$
$$H = j^3 H^{4,3}, \qquad \Psi = \Phi^{(II)} \quad \text{for } r = 2 \text{ in } 5.3.1,$$
$$V = \tilde{U}^{II} \quad \text{for } r = 2 \text{ in } 5.3.2.$$

5.3.3. Observation. In this case the assumptions of Lemma 4.4 are fulfilled with $d = 1$.

Proof. We look for the description of $Q \cap S^{\mathrm{spec}}$ within S^{spec}. Let $(X, Y, [Z^1, Z^2, B, C])$ be an arbitrary representative of a given $p \in Q \cap S^{\mathrm{spec}}$. We follow the proof of 4.5.1; now $Z^1, Z^2 \in \mathfrak{m}^3$, $f, g \in \mathfrak{m}^3$,

(14) $$Z'^3 \equiv B \mod \mathfrak{m}^4, \qquad Z'^4 \equiv C \mod \mathfrak{m}^4,$$

$$\partial_{(0,2)}(X, Y, Z) = \partial_{(0,2)}(X, Y, Z') = \langle B + \text{sth 3-flat},\ C + \text{sth 3-flat}\rangle.$$

Since $\partial_{(0,2)}(X, Y, Z') = \langle Z'^3, Z'^4\rangle$, we have a simple description of $\partial_{(0,2)(0,1)}(X, Y, Z) = \partial_{(0,2)(0,1)}(X, Y, Z') = \langle Z'^3, Z'^4, \det\rangle$, where

$$\det := \begin{vmatrix} XZ'^3 & YZ'^3 \\ XZ'^4 & YZ'^4 \end{vmatrix}.$$

(Two other 2-minors involving the derivatives of Z'^3, Z'^4 in the Z'-direction fall into $\partial_{(0,2)}(X, Y, Z')$ already!)

$(X, Y, Z') \in S_{(0,2)(0,1)(0,1)}$ means that det has derivatives in the directions of $L_0(X, Y, Z')(0)$ proportional to the respective ones of Z'^3, Z'^4, i.e. in view of (4) and (14), $\det_y(0) = 0$. We compute it directly:

$$\det_y(0) = \frac{\partial}{\partial y}\left(\begin{vmatrix} X(B+\text{sth 3-flat}) & Y(B+\text{sth 3-flat}) \\ X(C+\text{sth 3-flat}) & Y(C+\text{sth 3-flat}) \end{vmatrix}\right)(0)$$

$$= \frac{\partial}{\partial y}\left(\begin{vmatrix} B_x+\text{sth 2-flat} & B_y+\alpha B_z+\beta B_w+\text{sth 2-flat} \\ C_x+\text{sth 2-flat} & C_y+\alpha C_z+\beta C_w+\text{sth 2-flat} \end{vmatrix}\right)(0)$$

$$= \begin{vmatrix} B_{xy}(0) & 0 \\ C_{xy}(0) & 0 \end{vmatrix} + \begin{vmatrix} B_x & B_{yy}+\alpha_y B_z+\alpha B_{zy}+\beta_y B_w+\beta B_{wy} \\ C_x & C_{yy}+\alpha_y C_z+\alpha C_{zy}+\beta_y C_w+\beta C_{wy} \end{vmatrix}(0)$$

$$= C_{yy}(0), \quad \text{by (3), (4) and (12).}$$

The equation

$$C_{yy}(0) = 0, \tag{15}$$

expressed in terms of $j^2(X, Y, Z') = j^2(X, Y, Z) = p$, is the sole condition for p to be in $Q \cap S^{\text{spec}}$. It describes a regular, smooth, codimension 1 submanifold of S^{spec}. In effect, $C_{yy}(0)$ is a smooth regular (coordinate) function on $j^2\,\text{Spec}^{\text{II}}$. Its regularity at p manifests itself e.g. on the line $p(t) = j^2(X, Y, Z + ty^2\,\partial/\partial w)$, $p(0) = p$, lying entirely in S^{spec} (for every t, (3)–(6) and (12) are fulfilled). ∎

The assumptions of Lemma 4.4 are fulfilled (Q and $j^2\mathscr{H}^{4,3}\setminus Q$ are now $j^3H^{4,3}$-invariant). We excise the thin $S_{(0,2)(0,1)(0,1)}$-part of $S^{(\text{II})}$, obtaining the smooth, semialgebraic, codimension 4 manifold

$$S^{(\text{IIa})} = S^{(\text{II})} \cap S_{(0,2)(0,1)(0,2)} \tag{16}$$
$$= S_{(0,2)(1,2)} \cap S_{(0,2)(0,1)(0,2)} \cap S_{(1,3)}.$$

Note that property (5) of special forms (I) means, for a special form (II) triple, that

$$C_z(0) \neq 0 \tag{17}$$

(cf. (12)). This allows us to notice (see (11)) that

5.4. $L_2(\cdot)(0)$ *has maximal dimension.* Indeed, the following inclusion holds:

$$S^{(\text{II})} \subset S_{(2,4)}.$$

Proof. It suffices, by the $H^{4,3}$-invariance of Jakubczyk–Przytycki's sets, to consider special forms (II) only. Then we know that $L_1(X, Y, Z)(0)$ $= \text{span}(\partial/\partial x, \partial/\partial y, \partial/\partial z)(0)$. Computing

$$[[X, Z], Z](0) = [0, 0, (B_x B_z + C_x B_w)(0), (B_x C_z + C_x C_w)(0)]$$
$$= [0, 0, B_z(0), C_z(0)],$$

and using (17) yield that this vector and $L_1(X, Y, Z)(0)$ span $T_0\mathbf{R}^4$. ∎

The singularity set $S^{(\text{IIa})}$ admits a shorter description. Within this set, we have

5.5. $L_1(\cdot)(0)$ *is transversal to* M_3.

5.5.1. Proposition. $S_{(0,2)(0,1)(0,2)} \cap S_{(0,2)(1,2)} \subset S_{(0,2)(0,1)(1,3)}$.

Proof. We may reduce any triple in the LHS, via the $H^{4,3}$-action, to the special form (2) except that (6) need not hold (the assumptions yield (7) only). Then

$$\partial_{(0,2)}(X, Y, Z) = \langle B, C\rangle, \qquad \partial_{(0,2)(0,1)}(X, Y, Z) = \langle B, C, \det\rangle,$$
$$Y(0) \in \ker dB(0) \cap \ker dC(0)$$

(cf. the proof of 5.3.3). But $Y(0) \notin \ker d(\det)(0)$ ($(X, Y, Z) \notin S_{(0,2)(0,1)(0,2)}$ otherwise), so that

$$\ker d(\det)(0) \not\supset \ker dB(0) \cap \ker dC(0).$$

Hence B, C, det are differentially independent at 0; the set M_3 — having the regular description

$$B = C = \det = 0 \tag{18}$$

— is a smooth curve. Once the smoothness of M_1 and M_3 is established, the inclusion in question follows from the geometric position of the spaces T_0M_1, T_0M_3, $L_0(X, Y, Z)(0)$ and $L_1(X, Y, Z)(0)$. ∎

5.5.2. Corollary. $S^{(\text{IIa})} = S_{(0,2)(0,1)(1,3)} \cap S_{(1,3)}$ (cf. (16)).

Proof. The inclusion LHS $\supset$ RHS is implied by the independence of the functions B, C, det (when using the special forms) in the directions of the 3-dimensional $L_1(X, Y, Z)(0)$, which contains the 2-dimensional $L_0(X, Y, Z)(0)$. ∎

By this corollary the manifold $S^{(\text{IIa})}$ really gets a simpler description which suggests focusing attention on the $S_{(1,3)}$- term. This is purposeful, for we need to get rid of nontransversal parts of $S_{(1,3)}$ within $S^{(\text{IIa})}$.

5.6. *Excision of the* $S_{(1,3)(1,2)}$*-part.* Within $S^{(\text{IIa})}$ we have one additional piece of information about special forms (II):

$$C_{yy}(0) \neq 0 \tag{19}$$

((15) does not hold after the excision in 5.3). For every special form (II) triple

(X, Y, Z) the ideal $\partial_{(1,3)}(X, Y, Z)$ is generated by the functions

$$F_1 := \begin{vmatrix} B_x & B \\ C_x & C \end{vmatrix}, \quad F_2 := \begin{vmatrix} B_x & \alpha_x \\ C_x & \beta_x \end{vmatrix},$$

$$F_3 := \begin{vmatrix} B_x & B_y + \alpha B_z + \beta B_w - \alpha_z B - \alpha_w C \\ C_x & C_y + \alpha C_z + \beta C_w - \beta_z B - \beta_w C \end{vmatrix}.$$

This is because the matrix $(X, Y, [X, Z], Z, [X, Y], [Y, Z])$ has the nonzero 3-minor

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \alpha & B_x \end{vmatrix}.$$

All 4-minors of this matrix (only 6 of them being, possibly, not identically zero) are functional combinations of the three 4-minors which are extensions of that invertible 3-minor (see e.g. [S], Ch. I, § 1, Example 4). This description of $\partial_{(1,3)}(X, Y, Z)$ allows us to note

5.6.1. Remark. $S^{(\mathrm{IIa})} \subset S_{(1,3)(1,2)} \cup S_{(1,3)(1,3)}$.

Proof. For a special form (II) triple, the 2-minor

$$\begin{vmatrix} YF_1 & [X, Z] F_1 \\ YF_3 & [X, Z] F_3 \end{vmatrix}(0)$$

turns out to be nonzero:

$$YF_1(0) = (\partial/\partial y) F_1(0) = \begin{vmatrix} B_{xy} & B \\ C_{xy} & C \end{vmatrix}(0) + \begin{vmatrix} B_x & B_y \\ C_x & C_y \end{vmatrix}(0) = 0,$$

$$YF_3(0) = 0 + \begin{vmatrix} B_x & (B_y + *)_y \\ C_x & (C_y + *)_y \end{vmatrix}(0) = \begin{vmatrix} 1 & * \\ 0 & C_{yy}(0) \end{vmatrix} \neq 0,$$

$$[X, Z] F_1(0) = (\partial/\partial z) F_1(0) = 0 + \begin{vmatrix} B_x & B_z \\ C_x & C_z \end{vmatrix}(0) \neq 0$$

(cf. (3), (4), (17), (19)). ∎

5.6.2. *Deriving the equation for the special forms* (II) *in the* $S_{(1,3)(1,2)}$*-part.* We want to excise from $S^{(\mathrm{IIa})}$ the first summand on the RHS in Remark 5.6.1. This singularity set is defined, as well as $S^{(\mathrm{IIa})}$, in 2-jets. We shall use Lemma 4.4 as in 5.3.

Let this time

$$S = j^2 S^{(\mathrm{IIa})}, \quad Q = j^2 (S_{(1,3)(1,2)} \cap S^{(\mathrm{IIa})}),$$

$$S^{\mathrm{spec}} = j^2 (\mathrm{Spec}^{\mathrm{II}} \cap S^{(\mathrm{IIa})}), \quad H = j^3 H^{4,3}, \quad \Psi = \Phi^{(\mathrm{II})}|_{\partial^{\mathrm{II}} \cap j^2 S^{(\mathrm{IIa})}}$$

(see Prop. 5.3.2 for $r = 2$).

Take any representative $(X, [Y^1, Y^2, \alpha, \beta], [Z^1, Z^2, B, C])$ of an arbitrary $p \in Q \cap S^{\mathrm{spec}}$. The upper left 3-minor of the matrix $(X, Y, [X, Z], Z, [X, Y], [Y, Z])$ is obviously nonzero ($[X, Z] \equiv B_x \partial/\partial z + C_x \partial/\partial w \bmod \mathfrak{m}^2$), so that $\partial_{(1,3)}(X, Y, Z) = \langle F_1, F_2, F_3 \rangle$, where the F_i are the same 4-minors as those written explicitly at the beginning of 5.6 for a special form (II) triple. F_1, F_2, F_3 are differentially dependent in the directions of $L_1(X, Y, Z)(0) = \mathrm{span}(\partial/\partial x, \partial/\partial y, \partial/\partial z)(0)$. Noticing that

$$F_1 \equiv \begin{vmatrix} B_x & B \\ C_x & C \end{vmatrix} \bmod \mathfrak{m}^4, \quad F_2 \equiv \begin{vmatrix} B_x & \alpha_x \\ C_x & \beta_x \end{vmatrix} \bmod \mathfrak{m}^2,$$

$$F_3 \equiv \begin{vmatrix} B_x & B_y + \alpha B_z + \beta B_w - \alpha_z B - \alpha_w C \\ C_x & C_y + \alpha C_z + \beta C_w - \beta_z B - \beta_w C \end{vmatrix} \bmod \mathfrak{m}^3;$$

taking account of Remark 5.6.1, and computing: $F_{1x}(0) = 0$,

$$F_{2x}(0) = \begin{vmatrix} B_{xx} & \alpha_x \\ C_{xx} & \beta_x \end{vmatrix}(0) + \begin{vmatrix} B_x & \alpha_{xx} \\ C_x & \beta_{xx} \end{vmatrix}(0) = \beta_{xx}(0) + \begin{vmatrix} B_{xx} & \alpha_x \\ C_{xx} & \beta_x \end{vmatrix}(0),$$

$$F_{2y}(0) = \beta_{xy}(0) + \begin{vmatrix} B_{xy} & \alpha_x \\ C_{xy} & \beta_x \end{vmatrix}(0), \quad F_{3x}(0) = C_{yx}(0) + (\alpha_x C_z)(0),$$

we arrive at the equation in 2-jets expressing the mentioned differential dependence:

$$C_z \left(C_{yy} \left(\beta_{xx} + \begin{vmatrix} B_{xx} & \alpha_x \\ C_{xx} & \beta_x \end{vmatrix} \right) - (C_{xy} + \alpha_x C_z) \left(\beta_{xy} + \begin{vmatrix} B_{xy} & \alpha_x \\ C_{xy} & \beta_x \end{vmatrix} \right) \right) = 0. \tag{20}$$

(Simplifying the notation, we have omitted in (20) all symbols of evaluation at 0.)

This is the condition, in terms of p, for p to be in $Q \cap S^{\mathrm{spec}}$.

5.6.3. Remark. The functions det and F_3 (cf. 5.3.3 and 5.6.2) have the same derivative ($= C_{yy}(0)$ for special forms (II) mod $\mathfrak{m}^3$) in the direction $Y(0)$. This is not accidental, because for special forms

$$\det - F_3 \in \langle B, C \rangle. \tag{21}$$

5.6.4. *Completing the excision.* The computation in 5.6.2 shows that $Q \cap S^{\mathrm{spec}}$ is described within S^{spec} by a regular function on S^{spec} — that on the LHS of (20). (Its regularity at $p = j^2(X, Y, Z)$ manifests itself, for instance, on the line $p(t) = j^2(X, Y + tx^2 \partial/\partial w, Z)$ included in S^{spec}. (17) and (19) are essential for this regularity.) The assumptions of Lemma 4.4 are fulfilled with $d = 1$.

After the excision of the thin $S_{(1,3)(1,2)}$-part there remains a smooth (semialgebraic) manifold:

$$S^{(\mathrm{III})} = S^{(\mathrm{IIa})} \cap S_{(1,3)(1,3)} = S_{(0,2)(0,1)(1,3)} \cap S_{(1,3)(1,3)}. \tag{22}$$

5.7. *Ensuring regular intersection of M_2 and M_3.* We intend to ensure that M_1 and M_2 be in general position with respect to each other. For this it will suffice to ensure the same for the curves M_2 and M_3 only – a surprise constituting the second strange nongenericity mentioned in Sec. 1.

5.7.1. *Examples of nontypical behaviour.* Germs belonging to the manifold (22) may still behave very nontypically. For instance, the curves M_2 and M_3 can even become identical, as for the triple

$$(23)\qquad (\partial/\partial x,\ [0,\ 1,\ 0,\ x^2],\ [0,\ 0,\ x,\ z+y^2]).$$

Here $F_1 = C$, $F_2 = \beta_x = 2x \sim B = x$, $F_3 = 2y = \det$ (cf. 5.3.3 and 5.6) and obviously $\partial_{(0,2)(0,1)}(X,\ Y,\ Z) = \partial_{(1,3)}(X,\ Y,\ Z)$.

These curves may also have an arbitrarily high degree of tangency to each other at $0 \in R^4$. Namely, all terms of the following sequence belong to $S^{(\mathrm{III})}$:

$$(24)\qquad (\partial/\partial x,\ [0,\ 1,\ 0,\ x^2+xw^k],\ [0,\ 0,\ x,\ z+y^2]),\qquad k = 1,\ 2,\ \ldots$$

For the kth term of this sequence

$$F_1 = C,\qquad F_2 = 2x+w^k = 2B+w^k,$$

$$F_3 = 2y-(z+y^2)kxw^{k-1} = \det - Ckxw^{k-1},$$

hence $\partial_{(1,3)}(X,\ Y,\ Z) = \langle 2B+w^k,\ C,\ \det\rangle$. Recalling that for special forms $\partial_{(0,2)(0,1)}(X,\ Y,\ Z) = \langle B,\ C,\ \det\rangle$, we see that the announced tangency is clear, except for $k = 1$ in (24), i.e. for the triple

$$(25)\qquad (\partial/\partial x,\ [0,\ 1,\ 0,\ x^2+xw],\ [0,\ 0,\ x,\ z+y^2]).$$

We shall make reference to this specimen triple in 7.8; this triple is quoted in [M1], (12). It supplies the model situation we strive for. The remaining mutual geometrical positions of M_2 and M_3 will be eliminated with the use of the technique of 4.3–4.4.

5.7.2. *The special procedure* (III). This procedure is nothing but a particular special procedure (II) (cf. 5.1) restricted to $S^{(\mathrm{III})}$. The particularity consists in a more thorough specification of a special procedure (I) applied.

Now that we are in $S_{(0,2)(0,1)(0,2)}$, we may require at step 2° in 4.1 that the curve $M_3 \subset M_1$ be tangent to Γ at 0. As a consequence, M_3 will have at 0 the $(\partial/\partial w)(0)$-direction after the straightening. Notice, as an extension of Remark 5.1.1, that this description, $T_0 M_3 = \mathrm{span}(\partial/\partial w)(0)$, remains valid after step 5° in 5.1.

We shall call special form (II) triples in $S^{(\mathrm{III})}$ having M_3 tangent at 0 to the w-axis *special form* (III) *triples*, and denote their set by $\mathrm{Spec}^{\mathrm{III}}$.

5.7.3. *The uniform special procedure* (III) *in the vicinity of* $\mathrm{Spec}^{\mathrm{III}}$. Define a neighbourhood of $\mathrm{Spec}^{\mathrm{III}}$ in $S^{(\mathrm{III})}$, called U^{III}, by requiring conditions (a)–(c) from 4.2 and

(d′) For W' = the orthogonal projection of $(\partial/\partial w)(0)$ on $T_0 M_3$,

$$W'^4 > 3\max(|W'^1|,\ |W'^2|,\ |W'^3|).$$

These conditions are obviously fulfilled in $\mathrm{Spec}^{\mathrm{III}}$.

U^{III} is defined on the 2-jet level because the description of $T_0 M_3$ involves the second jets of germs of triples.

The *uniform special procedure* (III), defined in $U^{(\mathrm{III})}$ and written $\tilde{\Phi}^{(\mathrm{III})}$, consists of the same steps as $\tilde{\Phi}^{(\mathrm{II})}$ (see 5.2), but with step 2°′ (cf. 4.2.3) replaced by

2°″ A linear change of coordinates, depending smoothly on $j^2(X,\ Y,\ Z)$, preserving the vectors $(\partial/\partial x)(0)$, $(\partial/\partial y)(0)$, $(\partial/\partial z)(0)$ and replacing $(\partial/\partial w)(0)$ by W' (see (d′) above).

Images of $\tilde{\Phi}^{(\mathrm{III})}$ are special form (III) triples and $\tilde{\Phi}^{(\mathrm{III})}$ is the identity on $\mathrm{Spec}^{\mathrm{III}}$.

5.7.4. *The submersion* $\Phi^{(\mathrm{III})}$. We define, as in 5.3.1 (on an arbitrary r-jet level, $r \geq 2$), the mapping $\Phi^{(\mathrm{III})}$ induced by $\tilde{\Phi}^{(\mathrm{III})}$. The arguments of 4.3.3, 5.3.1 and 5.3.2 apply here again so that one can check the correctness of this definition and the smoothness of $\Phi^{(\mathrm{III})}\colon j^r U^{\mathrm{III}} \to j^r \mathrm{Spec}^{\mathrm{III}}$, $r \geq 2$. $\Phi^{(\mathrm{III})}$, being a submersion, like $\Phi^{(\mathrm{II})}$ and Φ, on a certain open $\tilde{U}^{\mathrm{III}} \supset j^r\mathrm{Spec}^{\mathrm{III}}$, will serve to eliminate the degeneracies presented in 5.7.1.

5.7.5. *Restriction to the case of M_2, M_3 regularly intersecting.* Nontypical mutual positions of M_2, M_3 imply an irregular, to-be-eliminated, description of $M_2 \cap M_3$. Namely, in the language of ∂-ideals,

$$(26)\qquad \partial_{(0,2)(0,1)}(X,\ Y,\ Z) + \partial_{(1,3)}(X,\ Y,\ Z)$$

should not then contain 4 functions differentially independent at 0. In terms of generalized sets of type S this means

$$(X,\ Y,\ Z) \in (S_{(0,2)(0,1)} \cap S_{(1,3)})_{(-1,3)}.$$

This set is defined on the 2-jet level. We intend to apply Lemma 4.4 for $S = j^2 S^{(\mathrm{III})}$,

$$Q = j^2\big((S_{(0,2)(0,1)} \cap S_{(1,3)})_{(-1,3)} \cap S^{(\mathrm{III})}\big),\qquad S^{\mathrm{spec}} = j^2\,\mathrm{Spec}^{\mathrm{III}},$$

$$H = j^3 H^{4,3},\qquad V = \tilde{U}^{\mathrm{III}},\qquad \Psi = \Phi^{(\mathrm{III})}\qquad (\text{for } r = 2 \text{ in } 5.7.4).$$

Examining the assumptions of the lemma, let $(X, [Y^1, Y^2, \alpha, \beta], [Z^1, Z^2, B, C])$ be a representative of $p \in Q \cap S^{\mathrm{spec}}$. Reasoning as in 5.3.3 (see (14) and the definition of det that follows) and 5.6.2, we see that the ideal (26) equals

$$\langle B+\text{sth 3-flat},\ C+\text{sth 3-flat},\ \det,\ F_1,\ F_2,\ F_3\rangle.$$

We know that $F_1 \equiv C \mod \mathfrak{m}^2$, $F_3 \equiv \det \mod \mathfrak{m}^2$. Hence "$p \in Q \cap S^{\mathrm{spec}}$" boils down to

(27) "$B, C, \det, F_2$ are differentially dependent at 0".

Let us derive an equation of this dependence.

Since (X, Y, Z) is, modulo $\mathfrak{m}^3$, a special form (III) triple, $(\partial/\partial w)(0)$ spans $T_0 M_3$ (the curve M_3 is defined by (X, Y, Z)). Consequently, B_w, C_w, $\det_w$ vanish at 0 (cf. (18), valid $\mod \mathfrak{m}^2$ here). But $B, C, \det$ are differentially independent in the directions $(\partial/\partial x, \ \partial/\partial y, \ \partial/\partial z)(0)$ (since $(X, Y, Z) \in S_{(0,2)(0,1)(1,3)}$). Hence the requirement (27) boils down to $F_{2w}(0) = 0$, and further to

$$(\beta_{xw} - \alpha_x C_{xw})(0) = 0 \tag{28}$$

(on account of (6), (12) and the description of F_2 in 5.6.2). The function on the LHS of (28), defined on S^{spec}, is smooth and regular (its regularity at p reveals itself e.g. on the line $p(t) = j^2(X, Y + txw\partial/\partial w, Z)$ lying entirely in S^{spec}). Thus (28) describes $Q \cap S^{\mathrm{spec}}$ as a codimension 1 regular submanifold of S^{spec}. The excision is now completed using Lemma 4.4.

We can thus restrict ourselves generically to the codimension 4 (semialgebraic) manifold

$$\begin{aligned} S^{(\mathrm{IV})} = S_{(0,2)(0,1)(1,3)} \cap S_{(1,3)(1,3)} \\ \cap (S_{(0,2)(0,1)} \cap S_{(1,3)})_{(-1,4)}. \end{aligned} \tag{29}$$

5.7.6. Note. The geometric meaning of the generalized set of type S occurring in (29) is nothing but M_2 being in general position with respect to M_3 (i.e. $T_0 M_2 \cap T_0 M_3 = \{0\}$). This has been guaranteed in the course of 5.7.5.

6. Regular intersection of M_1 and M_2.

6.1. The algebraic lemma. *For every germ (X, Y, Z) in $S^{(\mathrm{I})}$*

$$\begin{aligned} \partial_{(0,2)}(X, Y, Z) + \partial_{(1,3)}(X, Y, Z) = \partial_{(0,2)(0,1)}(X, Y, Z) \\ + \partial_{(1,3)}(X, Y, Z). \end{aligned} \tag{30}$$

Proof. It suffices to deal with special forms. This is due to a general transformation rule valid for ∂-ideals. Namely, in the general situation of $X \in \mathscr{H}^{n,k}$, $(g, F) \in H^{n,k}$, for every admissible sequence I (cf. [M1], (2))

$$\partial_I((g, F)X) = \partial_I(X) \circ g^{-1}. \tag{31}$$

(A proof is included in [M2]. It suffices to prove the inclusion $\subset$. The opposite one follows from this, applied to $(g, F)X$ and $(g, F)^{-1}$.)

If (X, Y, Z) is a special form (I) triple, $\partial_{(0,2)(0,1)}(X, Y, Z)$ has the description as in 5.3.3 (except that (12), (17) need not hold), and $\partial_{(1,3)}(X, Y, Z)$ is as in 5.6 (but (19) need not hold). Obviously LHS $\subset$ RHS. The opposite inclusion now follows from (21). ∎

6.2. *Automatic regularity of the intersection.* For triples of vector fields in $S^{(\mathrm{IV})}$ (cf. (29)) the ideal (26), and via Lemma 6.1 the ideal $\partial_{(0,2)}(\cdot) + \partial_{(1,3)}(\cdot)$ too, satisfy the $(-1,4)$-condition (i.e. are both equal to $\mathfrak{m}$). (We say that an ideal $\mathfrak{I} \subset \mathscr{F}_0^4$ satisfies the *(i, j)-condition* iff its jacobian extension $\mathfrak{I}_{(i,j)}$ is included in $\mathfrak{m}$, and $\mathfrak{I}_{(i,j-1)}$ equals $\mathscr{F}_0^4$.) We get automatically

$$S^{(\mathrm{IV})} \subset (S_{(0,2)} \cap S_{(1,3)})_{(-1,4)}$$

(cf. [M1], (4)). This means (cf. Note 5.7.6) that the 2-dimensional tangent plane $T_0 M_1$ and the tangent line $T_0 M_2$ cross each other "most sparingly" at $0 \in T_0 \boldsymbol{R}^4$. We underline this automatic inference, once the regular crossing of the lines $T_0 M_2$ and $T_0 M_3$ is known, and comment on this second (as announced in Sec. 1) strange nongenericity in geometrical behaviour later (see Sec. 8).

Thus the geometric genericity of 6.2 holds everywhere in $S^{(\mathrm{IV})}$. We are finally restricted to the (semialgebraic) manifold

$$(S^{\mathrm{F}} = S^{(\mathrm{IV})} =) \ S_{(0,2)(0,1)(1,3)} \cap S_{(1,3)(1,3)} \cap (S_{(0,2)} \cap S_{(1,3)})_{(-1,4)}. \ ∎ \tag{32}$$

This is the long awaited focusing stratum.

7. Properties of S^{F}. Proof of Theorem 2 (the first strange nongenericity).

The focusing stratum S^{F}, constituting a generic part of the initial singularity set (1), may be expressed differently, without resorting to generalized sets of type S:

7.1. Lemma.

$$S^{\mathrm{F}} = S_{(0,2)(0,1)(0,2)} \cap S_{(0,2)(-1,2)} \cap S_{(1,3)(0,2)(-1,4)}.$$

Proof. Of course $S^{\mathrm{F}} \subset S_{(0,2)(0,1)(0,2)} \cap S_{(0,2)(-1,2)}$. To prove the inclusion $\subset$, one has to establish the independence at 0 of the functions F_1, F_2, F_3 and F_4, where

$$F_4 := \begin{vmatrix} F_{1x} & F_{1y} + \alpha F_{1z} + \beta F_{1w} & BF_{1z} + CF_{1w} \\ F_{2x} & F_{2y} + \alpha F_{2z} + \beta F_{2w} & BF_{2z} + CF_{2w} \\ F_{3x} & F_{3y} + \alpha F_{3z} + \beta F_{3w} & BF_{3z} + CF_{3w} \end{vmatrix}.$$

(We refer only to special forms (III), cf. 5.7.2, since all sets S_I are $H^{4,3}$-invariant. The inclusion in $S_{(1,3)(0,2)}$, i.e. a certain 2-minor of the determinant defining F_4 not vanishing at 0, follows from the $S_{(0,2)} \cap S_{(1,3)(1,3)}$-assumption.)

As has already been computed in 5.6.1–2, for special form triples ((I), (II) or (III))

$$F_{1x}(0) = F_{1y}(0) = 0. \tag{33}$$

Hence

$$F_4 \equiv (BF_{1z}+CF_{1w})\begin{vmatrix} F_{2x} & F_{2y} \\ F_{3x} & F_{3y}\end{vmatrix} \bmod \mathfrak{m}^2,$$

where

$$G(0) \neq 0 \quad \text{for } G := \begin{vmatrix} F_{2x} & F_{2y} \\ F_{3x} & F_{3y}\end{vmatrix}, \tag{34}$$

because of the $S_{(1,3)(0,2)}$-assumption.

In turn, by (4) (valid all the more for special forms (III)), $BF_{1z}+CF_{1w} \equiv BB_x C_z \bmod \mathfrak{m}^2$. Thus $F_4 \equiv BB_x C_z G \bmod \mathfrak{m}^2$. Since $B_x C_z G$ is invertible at 0 (cf. (12), (17)), the independence at 0 of F_1, F_2, F_3, F_4 is equivalent to that of F_1, F_2, F_3, B. By (12),

$$F_1 \equiv C \bmod \mathfrak{m}^2. \tag{35}$$

The independence mentioned above is then equivalent to the $(-1,4)$-condition being fulfilled by the ideal $\langle B, C, F_1, F_2, F_3\rangle$, i.e. by

$$\partial_{(0,2)}(X, Y, Z) + \partial_{(1,3)}(X, Y, Z).$$

This is nothing but the condition for

$$(X, Y, Z) \in (S_{(0,2)} \cap S_{(1,3)})_{(-1,4)}.$$

Proof of $\supset$. We do not know for a while whether we have the $S_{(0,2)(1,2)}$-condition fulfilled. This leads us to an (irrelevantly) weaker special form (II′) construction than in 5.1.

First we get a special form (I) triple of vector fields (as in 4.1). In particular, according to (4), $B_w^{(I)}(0) = C_w^{(I)}(0) = 0$ for a while. But after an additional step 5°″ (which cannot coincide in phrasing with 5° from 5.1, since we do not know that (10) is *not* valid) we will not have such information.

5°″ Change coordinates in the (z, w)-plane so that $[X, Z](0)$ is the new versor $(\partial/\partial z)(0)$ and the new versor $(\partial/\partial w)(0)$ makes with it an angle of $+\pi/2$ in the old coordinates in this plane.

A *special form* (II′) triple coming out after steps 1°–4° and 5°″ yields less information than a special form (II). We do not know if $(\partial/\partial w)(0) \in T_0 M_1$, i.e. if $B_w(0) = C_w(0) = 0$; however, some properties of special forms (II) are acquired, namely the property (12). By (12) and the $S_{(0,2)(-1,2)}$-assumption we only know that $C_z(0) \neq 0$ *or* $C_w(0) \neq 0$.

7.1.1. Note. (12) alone does not guarantee here the inclusion in $S_{(0,2)(1,2)}$ as it did for special forms (I) in Sec. 4. We do not know (5), which rests on (4), and the latter is temporarily unknown.

Therefore, more carefully than in the "$\supset$" part of the proof, we calculate

$$BF_{1z}+CF_{1w} \equiv BB_x C_z + CB_x C_w \bmod \mathfrak{m}^2.$$

Hence

$$F_4 \equiv (BB_x C_z + CB_x C_w) G \bmod \mathfrak{m}^2,$$

and the assumption that F_1, F_2, F_3, F_4 are independent at 0 implies that $F_1, F_2, F_3, BB_x C_z + CB_x C_w$ are independent at 0. But (35) is still valid, implying that $F_1, F_2, F_3, BB_x C_z$ are independent at 0. We have $B(0) = 0$, so (17) must hold, and F_1, F_2, F_3, B are independent at 0. All the more, the ideal $\langle B, C, F_1, F_2, F_3\rangle$ fulfils the $(-1, 4)$-condition.

7.1.2. Note. One could argue more briefly: $F_1, F_2, F_3, BB_x C_z + CB_x C_w$ independent $\Rightarrow \langle B, C, F_1, F_2, F_3\rangle$ fulfils the $(-1, 4)$-condition. Yet the preservation of (17) is important for an argument below.

On the other hand, $F_{1z}(0) = (B_x C_z)(0) \neq 0$, and (33), (34) are still valid (the latter because of the former and the $S_{(1,3)(0,2)}$-assumption). Therefore F_1, F_2, F_3 are independent in the directions $(\partial/\partial x, \partial/\partial y, \partial/\partial z)(0) = (X, Y, [X, Z])(0)$, i.e. the germ of the triple belongs to $S_{(1,3)(1,3)}$. Once (17) is established, the inclusion in $S_{(0,2)(1,2)}$ holds automatically (cf. (12)). Weaker properties of special forms (II′) in comparison with special forms (II) have turned out to be irrelevant, because they have paved the way back to the situation in 5.1. Since $S_{(0,2)(0,1)(0,2)}$ is assumed, Corollary 5.5.2 yields the inclusion in $S_{(0,2)(0,1)(1,3)}$. ∎

7.2. *Algebraic formulation of the strange nongenericity occurring in* S^{F}. The strange nongenericity described in Th. 2 (see 2.3) has a short phrasing with the help of a generalized set of type S:

THEOREM 2′. $S^{\mathrm{F}} \subset (S_{(0,2)} \sqcup S_{(1,3)})_{(0,0)}$.

Proof. Set

$$\mathfrak{I} := \partial_{(0,2)}(X, Y, Z) \cap \partial_{(1,3)}(X, Y, Z) = \langle B, C\rangle \cap \langle F_1, F_2, F_3\rangle. \tag{36}$$

(We work only with special form (III) triples.) For triples of vector fields in $\mathrm{Spec}^{\mathrm{III}} \cap S^{\mathrm{F}}$

$$dB(0) \notin \operatorname{span}(dF_1, dF_2, dF_3)(0)$$

(because the ideals equated by (30) are equal to $\mathfrak{m}$). On account of (35), for every $fB+gC \in \mathfrak{I}$, $f(0) = 0$. Both $X(0)$ and $Y(0)$ differentiate to zero every function in $\mathfrak{J}$, because

$$XC(0) = YC(0) = 0 \tag{37}$$

(see (4), (12)). This yields that $L_0(X, Y, Z)(0)$ differentiates to 0 the ideal $\mathfrak{J}$.

Theorem 2′ is proved (cf. [M1], (4)). ∎

7.3. Corollary. $S^F \subset (S_{(0,2)} \cup S_{(1,3)})_{(1,1)}$.

Proof. The ideal $\mathfrak{J}$ (see (36)), by Theorem 2′, satisfies the (0, 0)-condition (for every point of S^F). Thus it can satisfy the $(1, j)$-condition with $j = 0$ or $j = 1$ only.

But $F_1 \in \mathfrak{J}$ by the definition, and

$$[X, Z] F_1(0) = dF_1(0)([X, Z](0)) = dC\big((\partial/\partial z)(0)\big) = C_z(0) \neq 0$$

(cf. (12), (17), (35)). Thus, for every point of $\mathrm{Spec}^{\mathrm{III}} \cap S^F$, we have $j = 1$. Hence $j = 1$ for the whole S^F, since generalized sets of type S are $H^{4,3}$-invariant. ∎

7.4. Claim.

$$S^F \subset (S_{(0,2)(0,1)} \cup S_{(1,3)})_{(0,1)}, \qquad S^F \subset (S_{(0,2)(0,1)} \cup S_{(1,3)})_{(1,2)}.$$

Proof. Set

$$\begin{aligned} \mathfrak{N} := \partial_{(0,2)(0,1)}(X, Y, Z) \cap \partial_{(1,3)}(X, Y, Z) \\ = \langle B, C, \det\rangle \cap \langle F_1, F_2, F_3\rangle. \end{aligned} \tag{38}$$

(Without loss of generality we restrict the argument to special form (III) triples of vector fields.) By the definition (cf. especially (21)) we know that $F_1, F_3 \in \mathfrak{N}$. Notice that

$$fF_2 \in \mathfrak{N} \Rightarrow f(0) = 0 \tag{39}$$

($F_2 \in \mathfrak{N}$ would result in $\langle F_1, F_2, F_3\rangle \subset \langle B, C, \det\rangle$, which is impossible within $\mathrm{Spec}^{\mathrm{III}} \cap S^F$, where the sum of these two ideals is $\mathfrak{m}$, see 5.7).

The matrix of partial derivatives of F_1, F_3 in the directions $(\partial/\partial x, \partial/\partial y, \partial/\partial z)(0)$ is the following (cf. 5.6):

$$\begin{pmatrix} 0 & 0 & C_z(0) \\ * & C_{yy}(0) & * \end{pmatrix}.$$

Recalling now the geometric positions of $L_0(X, Y, Z)(0)$ and $L_1(X, Y, Z)(0)$, and knowing (39), we have both inclusions of the claim justified. ∎

7.5. Remark. Recall that $(S_I \cup S_J)_{(i,j)}$ consists of certain points of $S_I \cap S_J$ only, namely those where the ideal $\partial_I(\cdot) \cap \partial_J(\cdot)$ satisfies the (i, j)-condition.

7.6. *Discussion of the definition of* $(S_I \cup S_J)_{(i,j)}$. The definition of $(S_I \cup S_J)_{(i,j)}$, as stated in [M1], (4), has been applied here always in situations

when the traces M_2 (of $S_{(1,3)}$) and either M_1 (of $S_{(0,2)}$) or M_3 (of $S_{(0,2)(0,1)}$) intersect regularly. When traces like these become tangent in some directions, we need a finer definition of $(S_I \cup S_J)_{(i,j)}$ which coincides with the previous one in regular cases as in this section. In tangent cases the common part of ∂-ideals may not reflect adequately the geometric contents. Take, for instance, $k = 2$ in (24). For that triple, $\partial_{(0,2)(0,1)}(X, Y, Z) = \langle x, z+y^2, 2y\rangle = \langle x, y, z\rangle$, while

$$\begin{aligned} \partial_{(1,3)}(X, Y, Z) &= \langle 2x+w^2, z+y^2, 2y-(z+y^2)2xw\rangle \\ &= \langle 2x+w^2, z+y^2, 2y\rangle = \langle 2x+w^2, y, z\rangle. \end{aligned}$$

Therefore $\partial_{(0,2)(0,1)}(X, Y, Z) \cap \partial_{(1,3)}(X, Y, Z) = \langle x(2x+w^2), y, z\rangle$, and this ideal fulfils only the (0, 1)-condition; the triple formally belongs to $(S_{(0,2)(0,1)} \cup S_{(1,3)})_{(0,1)}$. But, as is explained in 5.7.1, for this triple, the curves M_2 and M_3 are tangent at 0, their common tangent direction being $(\partial/\partial w)(0)$. So geometrically $L_0(X, Y, Z)(0) = \mathrm{span}(\partial/\partial x, \partial/\partial y)(0)$ has codimension 2 relative to $M_2 \cup M_3$. Therefore one needs another definition making use of a certain ideal satisfying the (0, 2)-condition in this case. Such an ideal should have the span of derivatives of its generators equal to the intersection of analogous spans for $\partial_I(\cdot)$, $\partial_J(\cdot)$. In the present section, the intersections of ∂-ideals are already sufficiently thick so that there is no need to replace $\partial_I(\cdot) \cap \partial_J(\cdot)$ by any other ideal.

7.7. Proof of Theorem 2. An algebraic proof is already given in 7.2: Theorem 2′ turns out to be equivalent to Theorem 2. This is so because within S^F, with the thickness mentioned in 7.6!, we have

$$\begin{aligned} d\big(\partial_{(0,2)}(\cdot) \cap \partial_{(1,3)}(\cdot)\big)(0) = d\big(\partial_{(0,2)}(\cdot)\big)(0) \\ \cap d\big(\partial_{(1,3)}(\cdot)\big)(0). \end{aligned} \tag{40}$$

(For special form (III) germs both sides are equal to $\mathrm{span}(dF_1(0)) = \mathrm{span}(dC(0))$ by (35) and the inference $fB + gC \in \mathfrak{J} \Rightarrow f(0) = 0$ in the proof of Theorem 2′ in 7.2. This identity extends to the whole S^F via (31).) Hence

$$\begin{aligned} T_0 M_1 + T_0 M_2 &= \ker d\big(\partial_{(0,2)}(\cdot)\big)(0) + \ker d\big(\partial_{(1,3)}(\cdot)\big)(0) \\ &= \ker\big[d\big(\partial_{(0,2)}(\cdot)\big)(0) \cap d\big(\partial_{(1,3)}(\cdot)\big)(0)\big] \\ &= \ker d\big(\partial_{(0,2)}(\cdot) \cap \partial_{(1,3)}(\cdot)\big)(0), \end{aligned}$$

the last equality being due to (40). Theorem 2′ asserts that the last space contains the subspace $L_0(X, Y, Z)(0)$. In other words, in the focusing stratum there are no pathologies mentioned in 7.6, so that Theorem 2 is indeed equivalent to Theorem 2′. ∎

Yet it is Th. 2 that highlights the geometric context of the strange nongeneric behaviour of germs in the focusing stratum. We comment on this below.

7.8. *Comments on the strange nongenericity*. A straightforward geometrical argument for Theorem 2 is possible. Let v be a vector spanning $T_0 M_2$. For special form (II) (and all the more (III)) triples, M_2 has the description $F_1 = F_2 = F_3 = 0$ (cf. 5.6), so that $vF_1(0) = 0$ and hence, by (35), $vC(0) = 0$. Because also $XC(0) = 0$ (see (37)), the vector $X(0) - \frac{1}{vB(0)} v$ differentiates both B and C to 0. Thus

$$X(0) \in \operatorname{span}(v, T_0 M_1) = T_0 M_1 + T_0 M_2.$$

A supplementary direction of $L_0(X, Y, Z)(0)$ is $Y(0)$, which for special forms (III) belongs to $T_0 M_1$. The strange nongenericity—confinement of the vectors $(X, Y, Z)(0)$ to a certain 3-dimensional space depending on the *second* jets of the vector fields, as pictured in Fig. 2—is established for special form (III) triples of vector fields, and hence for the whole S^F. The similarity (35) of F_1 and C (for special forms!) is crucial. We can describe this nongenericity by saying that in the focusing stratum the vector fields in a given triple cannot behave entirely independently of the sets materializing their singularities ("traces of singularity sets" throughout this paper). To the contrary, they are bound together to some extent at $0 \in \boldsymbol{R}^4$, the point focusing the whole complex singularity of the considered generic triple of vector fields. The specimen triple (25) yields an example of this strange behaviour, as computed in [M1], (12).

The nongenericity arising from Th. 2 may be considered similar, in some aspects, to the strange nongenericity (in codimension 2) in $\mathscr{H}^{3,2}$ discovered in [JP2] (Introduction, 1.12 and 5.1. (5)); however, the former (in $\mathscr{H}^{4,3}$) appears in higher codimension.

8. The second strange nongenericity related to S^F. As was observed in 6.2, the property of regular intersection of the curves M_2 and M_3 (established for the whole focusing stratum S^F in 5.7) translates automatically into the property of regular intersection of the surface M_1 and the curve M_2. The immediate translation is obtained via Lemma 6.1 (the strange identity (30), which itself could be named nongeneric). It seems fitting to attribute the adjective "nongeneric" to the resulting geometric restriction limiting the arbitrariness of mutual positions of M_1 and M_2. Indeed, 6.2 implies immediately (we omit "germ of" in the sequel):

8.1. *Observation on the geometry of* M_1, M_2 *and* M_3. The curve M_2 cannot cross regularly the curve M_3 and simultaneously be tangent to the surface M_1. The only direction of possible tangency of M_2 to M_1 is the M_3-direction. The last tangency may occur only outside the focusing stratum. Yet the described kind of tangency is inherited from the neighbouring focusing stratum, because this stratum renders identity (30), which forces geometric restrictions.

8.2. *Consequences for certain nongeneric triples of vector fields on a 4-manifold.* The restriction in nontypical behaviour mentioned in 8.1 has consequences for certain triples of smooth vector fields on 4-dimensional manifolds. We mean the triples which are nongeneric in the sense of [M1], Sec. 6, yet transversal to $S_{(0,2)(0,1)}(M)$ and to $S_{(1,3)}(M)$, where M stands for a given smooth 4-manifold. Their nongenericity consists in the nontransversality to the manifold $(S_{(0,2)} \cap S_{(1,3)})(M)$, i.e. in nonregular intersection of M_1 and M_2 for a given triple. For these triples of vector fields on M the direction of possible tangent intersection of the corresponding curves M_2 with the surfaces M_1 is uniquely determined. M_2 may cross M_1 only at points belonging to the curve M_3 (cf. also 3.3) and the mentioned direction is the direction of the curve M_3 at the point of intersection.

9. Completion of the proof of Theorem 1. Section 6 ends with establishing that S^F is a codimension 4 semialgebraic manifold. In order to complete the proof of Theorem 1 we want to know how many different functions ∂_I: $S^F \to \{\text{ideals of } \mathscr{F}_0^4\}$ there exist on S^F. We shall call such a function ∂_I *unessential* iff for every $(X, Y, Z) \in S^F$

$$\partial_I(X, Y, Z) = \mathscr{F}_0^4.$$

We shall call ∂_I *proper* iff for every $(X, Y, Z) \in S^F$

$$\partial_I(X, Y, Z) \subset \mathfrak{m}.$$

We introduce the same notions for ideals occurring in the definition of generalized sets of type S (we mean the definition in terms of local ideals, equivalent to the one in [M1], (4); cf. the comment on this in the Introduction).

The assertion of Theorem 1 means that each of the ∂_I, $(\partial_I + \partial_J)_{(I,J)}$ and $(\partial_I \cap \partial_J)_{(I,J)}$ is either unessential or proper.

9.1. Lemma. *Every ∂_I is either unessential or proper. More precisely, the following proper nonzero ∂_I functions exist*:

$$\partial_{(0,2)}, \quad \partial_{(0,2)(0,1)}, \quad \partial_{(1,3)}, \quad \partial_{(1,3)(0,2)}.$$

This means that a further jacobian extension of ideals which are values of the listed functions, as well as performing another sequence of such extensions from the beginning, gives values of, unless unessential, ideal-valued functions already present on this list.

Proof. For germs in S^F, $L_2(X, Y, Z)(0)$ has maximal dimension 4 (see Remark 5.4). Thus, for constructing the ideals $\partial_I(X, Y, Z)$, one may use at the first step only the distributions $L_0(X, Y, Z)$ and $L_1(X, Y, Z)$. And generally, since the distributions $L_i(\cdot)$, $i \geqslant 2$, are full-dimensional, in the remainder of the proof of Th. 1 we shall make use of $L_{-1}(\cdot) = TR^4$ to replace and eliminate them everywhere.

So we first consider $\partial_{(0,2)}(X, Y, Z)$ and $\partial_{(1,3)}(X, Y, Z)$. ($\partial_{(0,j)}$ for $j = 0, 1$ and $\partial_{(1,j)}$ for $j = 0, 1, 2$ are unessential since $S^F \subset S_{(0,2)} \cap S_{(1,3)}$; $\partial_{(0,3)}$ and $\partial_{(1,4)}$ are zero functions.)

9.1.1. What can jacobian extensions of $\partial_{(0,2)}(X, Y, Z)$ be for $(X, Y, Z) \in S^F$? This ideal has 2 generators which have nonproportional derivatives in the directions of $L_1(X, Y, Z)(0)$ (cf. (11) in 4.5). Hence $\partial_{(0,2)(1,0)}$ and $\partial_{(0,2)(1,1)}$ are unessential, while $\partial_{(0,2)(1,j)}$ for $j = 2, 3, 4$ does not differ from $\partial_{(0,2)}$ (the jacobian extension procedure adds minors of size $\geqslant 3$, using only 2 generators that span $\partial_{(0,2)}(\cdot)$ over $\mathscr{F}_0^4$).

No extension is due to $L_{-1}(\cdot)$ either—the two generators have nonproportional differentials. But an interesting extension with the help of $L_0(\cdot)$ is possible. The function $\partial_{(0,2)(0,1)}$ is proper and different from $\partial_{(0,2)}$ (S^F omits $S_{(0,2)(0,2)} \subset S_{(1,4)}$, cf. 3.3). As above, $\partial_{(0,2)(0,0)}$ is unessential and $\partial_{(0,2)(0,j)}$ for $j = 2, 3, 4$ does not differ from $\partial_{(0,2)}$.

Note now that $\partial_{(0,2)(0,1)}(\cdot)$ has 3 generators independent in the directions of $L_1(\cdot)(0)$ ($S^F \subset S_{(0,2)(0,1)(1,3)}$, cf. Corollary 5.5.2). So neither $L_1(\cdot)$ nor $L_{-1}(\cdot)$ gives a new jacobian extension of it different from $\mathscr{F}_0^4$. The only possibility is the L_0-extension $\partial_{(0,2)(0,1)(0,2)}(\cdot)$, since $S^F \subset S_{(0,2)(0,1)(0,2)}$ (cf. (16); $\partial_{(0,2)(0,1)(0,j)}$ is unessential for $j = 0, 1$ and equal to $\partial_{(0,2)(0,1)}$ for $j = 3, 4$). But this also gives nothing more than the function $\partial_{(0,2)(0,1)}$. This is so because (we argue for special forms (III) only) the additional generator that extends $\partial_{(0,2)(0,1)}(X, Y, Z)$ to $\partial_{(0,2)(0,1)(0,2)}(X, Y, Z)$ is a 3-minor with a column of derivatives in the Z-direction. Hence this generator is a combination of the coordinates of Z, i.e. of B and C, and belongs to $\langle B, C\rangle = \partial_{(0,2)}(X, Y, Z)$.

Therefore the jacobian extensions of $\partial_{(0,2)(0,1)}(\cdot)$ yield values of no new proper ideal-valued function.

9.1.2. What can jacobian extensions of $\partial_{(1,3)}(X, Y, Z)$ be for $(X, Y, Z) \in S^F$? This ideal is generated by 3 functions independent in the directions of $L_1(X, Y, Z)(0)$ (cf. 5.6), thus having no new proper L_1- or L_{-1}-jacobian extension (by an argument similar to 9.1.1). There exists only one nontrivial L_0-jacobian extension of $\partial_{(1,3)}(\cdot)$, namely $\partial_{(1,3)(0,2)}(\cdot)$.

Analogously to the previous cases, $\partial_{(1,3)(0,j)}$ is unessential for $j = 0, 1$ and equal to $\partial_{(1,3)}$ for $j = 3, 4$. Lemma 7.1 yields that for $(X, Y, Z) \in S^F$

$$\partial_{(1,3)(0,2)}(X, Y, Z) = \mathfrak{m}. \tag{41}$$

(Four independent functions on R^4 vanishing at 0 may be considered as new coordinate functions on R^4, spanning then the unique maximal ideal $\mathfrak{m} \subset \mathscr{F}_0^4$.)

By (41) the jacobian extensions of $\partial_{(1,3)(0,2)}(\cdot)$ are evaluations of no new proper ideal-valued function. Lemma 9.1 is proved. ∎

9.2. *Continuation of the proof.* We can now show that S^F is not subdivided by any set S_I. The inclusion $S^F \subset S_I$, $I = (i_1, j_1)\ldots(i_m, j_m)$, is equivalent to the condition: $\partial_{(i_1,j_1)\ldots(i_m,j_m)}$ is proper and for $r = 1, \ldots, m$, $\partial_{(i_1,j_1)\ldots(i_r,j_r-1)}$ is unessential (the latter condition is void if $j_r = 0$).

It follows from Lemma 9.1 that every function ∂_J is proper or unessential. Hence $S^F \not\subset S_I$ is equivalent to

$$\partial_{(i_1,j_1)\ldots(i_m,j_m)} \text{ is unessential or}$$

$$\exists\, 1 \leqslant r \leqslant m: \quad j_r \geqslant 1 \text{ and } \partial_{(i_1,j_1)\ldots(i_r,j_r-1)} \text{ is proper,}$$

which implies the disjointness of S^F and S_I.

9.3. *Completing the proof.* We should finally examine the mutual positions of S^F and generalized sets of type S (cf. 2.2 and 7.6). The inclusion $S^F \subset (S_I \cap S_J)_{(i,j)}$ is equivalent to the conditions:

$$S^F \subset S_I \cap S_J,$$

$$(\partial_I + \partial_J)_{(i,j)} \text{ is proper,}$$

$$(\partial_I + \partial_J)_{(i,j-1)} \text{ is unessential (no condition if } j = 0).$$

Under these conditions ∂_I and ∂_J are proper too. Thus, unless being 0, these functions are equal to certain ones included in the list in Lemma 9.1. After evaluating the latter at $(X, Y, Z) \in S^F$, we get the pairs of ideals

$$\partial_{(0,2)}(X, Y, Z) \subset \partial_{(0,2)(0,1)}(X, Y, Z),$$

$$\partial_{(1,3)}(X, Y, Z) \subset \partial_{(1,3)(0,2)}(X, Y, Z).$$

Hence $\partial_I(X, Y, Z) + \partial_J(X, Y, Z)$ can only be either 0 or the evaluation of one of the listed functions (the option depends on I, J and not on (X, Y, Z), cf. Lemma 6.1, 6.2, (41)). So, by Lemma 9.1 again, for every admissible j', $(\partial_I + \partial_J)_{(i,j')}$ is either proper or unessential.

Analogously, the inclusion in $(S_I \cup S_J)_{(i,j)}$ boils down to: the inclusion in $S_I \cap S_J$, $(\partial_I \cap \partial_J)_{(i,j)}$ being proper, and $(\partial_I \cap \partial_J)_{(i,j-1)}$ being unessential (no condition if $j = 0$).

The ideal $\partial_I(\cdot) \cap \partial_J(\cdot)$ can be 0, $\mathfrak{J}$, $\mathfrak{N}$ or the evaluation of one of the functions listed in 9.1 (cf. (36), (38)). Which is the case depends on I, J only.

It follows from 7.3, 7.4 and the description of $\mathfrak{J}$, $\mathfrak{N}$ for special form triples that for $i = -1, 2, 3, \ldots$

(a) $\mathfrak{J}$ satisfies the $(i, 1)$-condition,

(b) $\mathfrak{N}$ satisfies the $(i, 2)$-condition,

for any triple in S^{F}. All this together with Lemma 9.1 yields that for every admissible pair (i', j'), $(\partial_I \cap \partial_J)_{(i',j')}$ is either proper or unessential.

Thus S^{F} is either included in, or disjoint from every generalized set of type S.

At long last, Theorem 1 is proved. ∎

9.4. *Concluding remarks on a possible strengthening of Theorem* 1. Remark 7.6 warns that the definition of generalized sets of type S, of the form $(S_I \cup S_J)_{(i,j)}$, as appeared in [M1], (4), is too rough (yet right in the context of the focusing stratum S^{F} discussed in the present paper) and ought to be refined — possibly as 7.6 suggests.

But if, regardless of 7.6, we considered all possible generalizations in the sense suggested in [JP2], 2.7, a far from clear picture of available general ∂-type ideals would appear for points in S^{F}. Consider e.g. the specimen triple (25) again. Looking back to Th. 2′ in 7.2 one could ask about $(\mathfrak{I})_{(0,0)}$. For the triple (25) this ideal is equal to $\mathfrak{m}$, and the same can be proved everywhere in S^{F}.

Similarly, after Corollary 7.3, what about $(\mathfrak{I})_{(1,1)}$? The triple (25) yields it equal to $\mathfrak{m}$, which also turns out to be valid in the whole S^{F}.

Analogously, after Claim 7.4, one can compute $(\mathfrak{N})_{(0,1)} = \mathfrak{m}$ for this specimen; yet I cannot prove it everywhere in S^{F}. Again after Claim 7.4, let us ask about $(\mathfrak{N})_{(1,2)}$ for the specimen triple. Curiously enough, the answer is

$$(\mathfrak{N})_{(1,2)} = \langle 4x+w,\ y,\ z,\ x^2 \rangle.$$

Still for this triple, $\big((\mathfrak{N})_{(1,2)}\big)_{(0,2)} = \mathfrak{m}$. But strangely enough,

$$\big((\mathfrak{N})_{(1,2)}\big)_{(1,3)} = \langle x^2,\ xw,\ 4x+w,\ y,\ z \rangle = (\mathfrak{N})_{(1,2)},$$

so that one has here

$$\mathfrak{N} \subsetneqq (\mathfrak{N})_{(1,2)} = \big((\mathfrak{N})_{(1,2)}\big)_{(1,3)} \subsetneqq \big((\mathfrak{N})_{(1,2)}\big)_{(0,2)} = \mathfrak{m}. \tag{42}$$

Such irregular behaviour of ∂-type ideals hampers us in the further investigation of interrelations between the focusing stratum S^{F} and possible generalizations of the sets S_I.

The natural question arises whether that more abundant family of ∂-type ideals distinguishes some triples in S^{F} (forgetting about the possible lack of geometrical interpretation discussed in 7.6), i.e. whether our classification of singularities is full with respect to the operations "+", "$\cap$" and the jacobian extensions of ideals.

I suppose, aiming at strengthening Theorem 1, that S^{F} cannot be dissected this way, yet I am unable to prove it.

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