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MINIMAX CONTROL OF A SYSTEM WHICH CANNOT BE OBSERVED WITHOUT ERROR

0. Introduction. The paper is devoted to the problem of minimax control in the sense determined in [6]. It deals with the discrete-time linear stochastic system with additive disturbances the distributions of which depend on an unknown parameter. Various problems of minimax control of such a system are solved under the assumption that the states of the system are exactly observed (see [4]–[6]). In this paper it is assumed that the system cannot be observed without error. This additional difficulty causes that the assumptions about the disturbances are less general than the ones in the quoted papers.

The problem considered in the paper is exactly formulated in Section 1. The Bayes and the minimax control strategies are found in Sections 2 and 3, respectively. The results of both these sections are obtained under the assumption that the distribution of the initial state is given. The problem of minimax control when the initial state has a distribution depending on an unknown parameter is considered in Section 4.

Another approach to minimax control and other systems are considered, e.g., in [1]–[3].

1. Let us consider a discrete-time linear stochastic system defined by the equations

$$X_{n+1} = \alpha_n X_n + \omega_n u_n + \gamma_n V_n, \qquad n = 0, 1, \ldots, \tag{1}$$

where X_n is a state variable, u_n is a control, ω_n is a random actuation error which is introduced into the system by exercising control, $V_0, V_1, \ldots$ are random disturbances, and α_n, γ_n are given constants. We assume that the initial state X_0 is a random variable having a normal distribution with mean m_0 and variance g_0^2. We denote this distribution by $N(m_0, g_0^2)$.

We deal with the problem of control of such a system when we cannot observe it exactly. Specifically, we assume that we observe a process (V_n, Y_n, ω_n), $n = 0, 1, \ldots$, where

$$Y_n = \varepsilon_n X_n + Z_n \tag{2}$$

with ε_n being a given constant, Z_n being a random variable having the distribution $N(0, \sigma_n^2)$, σ_n^2 being known. It may also be true that $\varepsilon_n \neq 0$ and $\sigma_n^2 = 0$ for some value of n. The state X_n at this stage can be observed exactly.

We assume that, for every n, ω_n has the distribution $N(\mu_{n1}, v_n^2)$ and that $V_0, V_1, \ldots$ is a sequence of identically, normally distributed random variables having the density

$$f(v/\lambda) = \frac{1}{\sqrt{2\pi q}} \exp\left[-\frac{(v-\lambda q)^2}{2q}\right], \tag{3}$$

where $q > 0$ is a given constant, whereas λ is an unknown parameter.

Let N denote the horizon of control. The horizon is assumed to be a random variable with the distribution

$$P(N = k) = p_k, \quad k = 0, 1, \ldots, M, \; p_M > 0, \; \sum_{k=0}^{M} p_k = 1.$$

We assume that the random variables $N, X_0, V_n, \omega_n, Z_n (n = 0, 1, \ldots)$ are independent.

The following data are available at the time n:

$$V^{n-1} = (V_0, \ldots, V_{n-1}), \quad Y^{n-1} = (Y_0, \ldots, Y_{n-1}), \quad \omega^{n-1} = (\omega_0, \ldots, \omega_{n-1}),$$
$$U^{n-1} = (u_0, \ldots, u_{n-1}).$$

For convenience we denote the vector $(V^{n-1}, Y^{n-1}, \omega^{n-1}, U^{n-1})$ by W^n.

We assume that the control u_n is a Borel function of W^n. Let $u_0, \ldots, u_M$ be the controls. The vector $U = (u_0, \ldots, u_M)$ is called a *control strategy*. For the given control strategy U we define the risk function as follows:

$$\begin{aligned} R(\lambda, U) &= \mathrm{E}_p \{\mathrm{E}_\lambda \sum_{i=0}^{N} [(X_i, \lambda, 1) A^{(i)} (X_i, \lambda, 1)^T + k_i u_i^2]\} \\ &\overset{\mathrm{df}}{=} \mathrm{E}_p \mathrm{E}_\lambda \sum_{i=0}^{N} L_i(X_i, \lambda, u_i), \end{aligned} \tag{4}$$

where $\mathrm{E}_p(\cdot)$ denotes the expectation with respect to the distribution of the random variable N, $\mathrm{E}_\lambda(\cdot)$ denotes the expectation with respect to the distributions of the random variables X_0, ω_n, V_n $(n = 0, \ldots, M)$ when λ is a fixed parameter of density (3), $A^{(i)} = [a_{kl}^{(i)}]_{3\times 3}$, $i = 0, \ldots, M$, are nonnegative definite matrices, $a_{11}^{(i)} > 0$, $k_i > 0$.

For the prior distribution π of the parameter λ and for the control strategy U we define a *Bayes risk* connected with π and U as follows:

$$r(\pi, U) = \mathrm{E}_\pi R(\lambda, U),$$

where $E_\pi(\cdot)$ denotes the expectation with respect to the distribution π. The Bayes risk is here the cost of control.

A control strategy U^* is called a *Bayes control strategy* (with respect to the distribution π) if

$$r(\pi, U^*) = \inf_{U \in \Delta_\pi} r(\pi, U),$$

where Δ_π is the class of control strategies U for which $r(\pi, U)$ exists, maybe infinite.

We sometimes have the information that $\pi \in \Gamma$, where Γ is a known subclass of the class of all prior distributions of λ. Let us denote the class of control strategies U for which $r(\pi, U)$ exists for each $\pi \in \Gamma$ by Δ_Γ. A strategy $\hat{U}$ is called a *minimax control strategy* with respect to Γ, ΓMCS, if

$$\sup_{\pi \in \Gamma} r(\pi, \hat{U}) = \inf_{U \in \Delta_\Gamma} \sup_{\pi \in \Gamma} r(\pi, U).$$

Our principal aim in this paper is to find ΓMCS under the assumption that Γ is the class of the prior distributions of the parameter λ which have uniformly bounded second moments.

It is well known from decision theory that the minimax decisions are very often found among the Bayes or extended Bayes decisions. Thus, in the next section we shall find the Bayes control strategies with respect to the conjugate prior distributions.

2. Let us suppose that the parameter λ of density (3) has the prior distribution $\pi_{\beta,r}$ with the density

$$g(\lambda, \beta, r) = \sqrt{\frac{\beta}{2\pi}} \exp\left[-\frac{\beta}{2}\left(\lambda - \frac{r}{\beta}\right)^2\right], \tag{5}$$

where β and r are real constants, $\beta > 0$. It is well known that then the posterior distribution of the parameter λ has the density

$$g(\lambda / V^{n-1}) = g(\lambda, \beta_n, r_n), \tag{6}$$

where

$$\beta_n = \beta + nq, \qquad r_n = r + \sum_{i=0}^{n-1} V_i. \tag{7}$$

It is easy to verify that the conditional distribution of the random variable V_n given the vector V^{n-1} has the density

$$f(v / V^{n-1}) = \sqrt{\frac{\beta_n}{2\pi q \beta_{n+1}}} \exp\left[-\frac{\beta_n}{2q\beta_{n+1}}\left(v - q\frac{r_n}{\beta_n}\right)^2\right]. \tag{8}$$

Note that $g(\lambda / W^n) = g(\lambda / V^{n-1})$ and $f(v / W^n) = f(v / V^{n-1})$.

Since the distribution of the initial state X_0 is assumed to be a normal one and since each of the random variables V_n, ω_n, Z_n $(n = 0, 1, \ldots)$ has a normal distribution, it can be seen that the conditional distribution of X_n given W^n at each stage is again a normal distribution. We consider now the manner in which this distribution (its mean and variance) changes from one stage of the system to another.

We have already assumed that the distribution of X_0 is $N(m_0, g_0^2)$. For $n = 1, \ldots, M$ let us denote the mean and variance of the conditional distribution of X_n, given W^n, by m_n and g_n^2, respectively. For $n = 0, 1, \ldots, M$ it follows from (2) that after the value of Y_n has been observed, the conditional distribution of X_n, given (W^n, Y_n), is a normal distribution for which the mean $\bar{m}_n$ and the variance $\bar{g}_n^2$ are as follows:

(9)
$$\bar{m}_n = \frac{\sigma_n^2 m_n + g_n^2 \varepsilon_n Y_n}{\varepsilon_n^2 g_n^2 + \sigma_n^2} \stackrel{\text{df}}{=} b_n m_n + t_n Y_n, \qquad \bar{g}_n^2 = b_n g_n^2.$$

Suppose now, for a given value of n $(n = 0, \ldots, M)$, that the conditional distribution of X_n, given (W^n, Y_n), is a normal distribution with mean $\bar{m}_n$ and variance $\bar{g}_n^2$. By (1) it can be seen that for any choice of the control value u_n, the next state X_{n+1} has a normal distribution for which the mean m_{n+1} and the variance g_{n+1}^2 are

(10)
$$m_{n+1} = \alpha_n \bar{m}_n + \omega_n u_n + \gamma_n V_n, \qquad g_{n+1}^2 = \alpha_n^2 \bar{g}_n^2.$$

Thus the conditional distribution of X_n, given W^n $(n = 1, \ldots, M)$, has the density

(11)
$$h(x/W^n) = \frac{1}{\sqrt{2\pi g_n^2}} \exp\left[-\frac{1}{2g_n^2}(x - m_n)^2\right],$$

where the values of the mean m_n and the variance g_n^2 can be calculated with the help of (9) and (10) with the initial condition given by the known constants m_0 and g_0^2. Note that the variance g_n^2 changes in a deterministic way from one stage to another because neither the values of the observations nor the values chosen for the controls affect the variance.

Using (5)–(11) the following conditional expectations can be calculated:

$$\mathrm{E}(m_{n+1}/W^n, u_n) = \alpha_n m_n + \mu_{n1} u_n + \gamma_n q \frac{r_n}{\beta_n},$$

$$\mathrm{E}(m_{n+1}^2/W^n, u_n) = \alpha_n^2 m_n^2 + \mu_{n2} u_n^2 + \gamma_n^2 q^2 \frac{r_n^2}{\beta_n^2} + 2\alpha_n \mu_{n1} m_n u_n$$
$$+ 2\gamma_n \mu_{n1} q u_n \frac{r_n}{\beta_n} + 2\alpha_n \gamma_n q m_n \frac{r_n}{\beta_n} + \gamma_n^2 q \frac{\beta_{n+1}}{\beta_n} + \alpha_n^2 g_n^2 \varepsilon_n t_n,$$

(12)
$$\mathrm{E}(r_{n+1}/W^n, u_n) = \beta_{n+1} \frac{r_n}{\beta_n},$$

$$\mathrm{E}(r_{n+1}^2/W^n, u_n) = \beta_{n+1}^2 \frac{r_n^2}{\beta_n^2} + q \frac{\beta_{n+1}}{\beta_n},$$

$$\mathrm{E}(m_{n+1}r_{n+1}/W^n, u_n) = \alpha_n\beta_{n+1}m_n\frac{r_n}{\beta_n} + \mu_{n1}\beta_{n+1}u_n\frac{r_n}{\beta_n}$$
$$+\gamma_n q\frac{\beta_{n+1}}{\beta_n} + \gamma_n q\beta_{n+1}\frac{r_n^2}{\beta_n^2},$$

where $\mathrm{E}(\cdot/\cdot) = \mathrm{E}_{\pi_{\beta,r}}\mathrm{E}_\lambda(\cdot/\cdot)$, $\mu_{n2} = \mu_{n1}^2 + \nu_n^2$, and t_n is given in (9).

Now we shall look for the Bayes control strategy with respect to the distribution given by (5). Let us suppose that we are at the n-th stage and we start to control our system. Let us consider the problem of Bayes control in such a case. At this moment the data W^n are known and the Bayes risk for this problem is the following:

$$r_n(\pi_{\beta,r}, U_n) = \mathrm{E}_p\{\mathrm{E}[\sum_{i=n}^{N} L_i(X_i, \lambda, u_i): W^n]/N \geqslant n\},$$

where $U_n = (u_n, \ldots, u_M)$. This risk can be transformed to the form (see [4])

$$r_n(\pi_{\beta,r}, U_n) = \mathrm{E}\left[\sum_{i=n}^{M}\frac{\pi_i}{\pi_n}L_i(X_i, \lambda, u_i)/W^n\right], \tag{13}$$

where

$$\pi_k = \sum_{i=k}^{M} p_i.$$

It is easy to verify that for $n = M$ we obtain

$$r_M(\pi_{\beta,r}, U_M) = r_M(\pi_{\beta,r}, u_M) = L_M\left(m_M, \frac{r_M}{\beta_M}, u_M\right) + a_{11}^{(M)}g_M^2 + a_{22}^{(M)}\frac{1}{\beta_M}.$$

It can also be seen that $u_M = 0$ minimizes the above expression.

For $n = 0, 1, \ldots, M$ we denote $\inf_{U_n} r_n(\pi_{\beta,r}, U_n)$ by $\varkappa_n(m_n)$. The infimum is taken over all control strategies U_n for which the risk $r_n(\pi_{\beta,r}, U_n)$ exists.

With the help of dynamic programming methods we obtain the following Bellman equation for our problem:

$$\begin{aligned}
\varkappa_n(m_n) &= \inf_{U_n}\mathrm{E}\left[\sum_{i=n}^{M}\frac{\pi_i}{\pi_n}L_i(X_i, \lambda, u_i)/W^n\right] \\
&= \inf_{U_n}\mathrm{E}\left\{\left[L_n(X_n, \lambda, u_n) + \mathrm{E}\left(\sum_{i=n+1}^{M}\frac{\pi_i}{\pi_n}L_i(X_i, \lambda, u_i)/W^{n+1}\right)\right]\Big/W^n\right\} \\
&= \inf_{u_n}\mathrm{E}\left\{\left[L_n(X_n, \lambda, u_n)\right.\right. \\
&\qquad \left.\left. + \frac{\pi_{n+1}}{\pi_n}\inf_{U_{n+1}}\mathrm{E}\left(\sum_{i=n+1}^{M}\frac{\pi_i}{\pi_{n+1}}L_i(X_i, \lambda, u_i)/W^{n+1}\right)\right]\Big/W^n\right\} \\
&= \inf_{u_n}\left\{\mathrm{E}[L_n(X_n, \lambda, u_n)/W^n] + \frac{\pi_{n+1}}{\pi_n}\mathrm{E}[\varkappa_{n+1}(m_{n+1})/W^n]\right\},
\end{aligned} \tag{14}$$

$$n = 0, 1, \ldots, M-1.$$

Using (8), (11) and (14) we obtain

$$\varkappa_n(m_n) = \inf_{u_n} L_n\left(m_n, \frac{r_n}{\beta_n}, u_n\right) + a_{11}^{(n)} g_n^2 + a_{22}^{(n)} \frac{1}{\beta_n} + \frac{\pi_{n+1}}{\pi_n} \mathrm{E}\left[\varkappa_{n+1}(m_{n+1})/W^n\right], \quad n = 0, 1, \ldots, M-1. \tag{15}$$

It follows from the above considerations, in view of Bellman's optimality principle, that the control strategy $U^* = (u_0^*, \ldots, u_M^*)$ is the Bayes one if $u_M^* = 0$ and for the controls u_n^*, $n = 0, 1, \ldots, M-1$, the right-hand side of (15) reaches its minimum. Hence it results that for $n = 0, 1, \ldots, M-1$ the following equation determines the Bayes controls:

$$2k_n u_n^* + \frac{\partial}{\partial u_n} \frac{\pi_{n+1}}{\pi_n} \mathrm{E}\left[\varkappa_{n+1}(m_{n+1})/W^n\right] \Big|_{u_n = u_n^*} = 0. \tag{16}$$

We shall show that $\varkappa_n(m_n)$ is of the following form:

$$\varkappa_n(m_n) = A_n m_m^2 + 2B_n m_n \frac{r_n}{\beta_n} + C_n \frac{r_n^2}{\beta_n^2} + 2D_n m_n + 2E_n \frac{r_n}{\beta_n} + F_n. \tag{17}$$

For $n = M$ the formula (17) holds with

$$\begin{gathered} A_M = a_{11}^{(M)}, \quad B_M = a_{12}^{(M)}, \quad C_M = a_{22}^{(M)}, \quad D_M = a_{13}^{(M)}, \quad E_M = a_{23}^{(M)}, \\ F_M = a_{11}^{(M)} g_M^2 + \frac{1}{\beta_M} a_{22}^{(M)} + a_{33}^{(M)}. \end{gathered} \tag{18}$$

Assuming (17) to be true for $n+1$ and using (12) we obtain

$$\frac{\partial}{\partial u_n} \mathrm{E}\left[\varkappa_{n+1}(m_{n+1})/W^n\right] = 2A_{n+1}\mu_{n2} u_n + 2A_{n+1}\alpha_n \mu_{n1} m_n + 2A_{n+1}\gamma_n \mu_{n1} q \frac{r_n}{\beta_n} + 2B_{n+1}\mu_{n1} \frac{r_n}{\beta_n} + 2D_{n+1}\mu_{n1}. \tag{19}$$

Note that we can use the formulae (12) in order to obtain (19), because $\mathrm{E}(\cdot/W^n) = \mathrm{E}(\cdot/W^n, f(W^n))$ for every Borel function $f(\cdot)$. This fact will also be used in further considerations.

From (16) and (19) we infer that the optimal value u_n^* of the control is given by

$$u_n^* = -\frac{\pi_{n+1} A_{n+1} \alpha_n \mu_{n1}}{\pi_n k_n + \pi_{n+1} A_{n+1} \mu_{n2}} m_n - \frac{\pi_{n+1}(A_{n+1}\gamma_n q + B_{n+1})\mu_{n1}}{\pi_n k_n + \pi_{n+1} A_{n+1} \mu_{n2}} \frac{r_n}{\beta_n} - \frac{\pi_{n+1} D_{n+1} \mu_{n1}}{\pi_n k_n + \pi_{n+1} A_{n+1} \mu_{n2}} \stackrel{\mathrm{df}}{=} -P_n m_n - H_n \frac{r_n}{\beta_n} - T_n. \tag{20}$$

Note that the constants P_n, H_n and T_n are independent of β.

If in the formula (15) we set $u_n = u_n^*$, then, using the inductive assumption and (12), we obtain

$$
\begin{aligned}
(21)\quad \varkappa_n(m_n) = {} & L_n\left(m_n, \frac{r_n}{\beta_n}, u_n^*\right) + a_{11}^{(n)} g_n^2 + \frac{1}{\beta_n} a_{22}^{(n)} + \frac{\pi_{n+1}}{\pi_n}\Bigg[A_{n+1}\Bigg(\alpha_n^2 m_m^2 \\
& + \mu_{n2} u_n^{*2} + \gamma_n^2 q^2 \frac{r_n^2}{\beta_n^2} + 2\alpha_n \mu_{n1} m_n u_n^* + 2\gamma_n \mu_{n1} q u_n^* \frac{r_n}{\beta_n} + 2\alpha_n \gamma_n q m_n \frac{r_n}{\beta_n} \\
& + \alpha_n^2 g_n^2 \varepsilon_n t_n + \gamma_n^2 q \frac{\beta_{n+1}}{\beta_n}\Bigg) + 2B_{n+1}\Bigg(\alpha_n m_n \frac{r_n}{\beta_n} + \mu_{n1} u_n^* \frac{r_n}{\beta_n} + \gamma_n q \frac{r_n^2}{\beta_n^2} \\
& + \gamma_n \frac{q}{\beta_n}\Bigg) + C_{n+1}\left(\frac{r_n^2}{\beta_n^2} + \frac{q}{\beta_n \beta_{n+1}}\right) + 2D_{n+1}\left(\alpha_n m_n + \mu_{n1} u_n^* + \gamma_n q \frac{r_n}{\beta_n}\right) \\
& + 2\mathrm{E}_{n+1} \frac{r_n}{\beta_n} + F_{n+1}\Bigg].
\end{aligned}
$$

Substituting the right-hand side of (20) for u_n^* in (21) we see that $\varkappa_n(m_n)$ takes the form (17) with coefficients given by the following recursive formulae:

$$
(22)\quad
\begin{aligned}
A_n &= a_{11}^{(n)} + d_n A_{n+1} \alpha_n^2 G_n, \\
B_n &= a_{12}^{(n)} + d_n \alpha_n Q_n G_n, \\
C_n &= a_{22}^{(n)} + d_n (C_{n+1} + Q_n K_n + \gamma_n q B_{n+1}), \\
D_n &= a_{13}^{(n)} + d_n D_{n+1} \alpha_n G_n, \\
E_n &= a_{23}^{(n)} + d_n (D_{n+1} K_n + E_{n+1}), \\
F_n &= a_{11}^{(n)} g_n^2 + a_{22}^{(n)} \frac{1}{\beta_n} + a_{33}^{(n)} + d_n \Bigg[\frac{q}{\beta_n}\Bigg(A_{n+1} \gamma_n^2 \beta_{n+1} + 2B_{n+1} \gamma_n \\
&\quad + C_{n+1} \frac{1}{\beta_{n+1}}\Bigg) + A_{n+1} \alpha_n^2 g_n^2 \varepsilon_n t_n - D_{n+1} T_n \mu_{n1} + F_{n+1}\Bigg],
\end{aligned}
$$

where

$$
d_n = \frac{\pi_{n+1}}{\pi_n}, \quad G_n = \frac{k_n + d_n A_{n+1} v_n^2}{k_n + d_n A_{n+1} \mu_{n2}}, \quad K_n = \gamma_n q - \mu_{n1} H_n,
$$
$$
Q_n = A_{n+1} \gamma_n q + B_{n+1}.
$$

The proof of the following theorem is now completed.

THEOREM 1. *Let us consider the linear stochastic system described by* (1) *and* (2). *Assume that*

(i) *the initial state X_0 is a random variable having a known distribution $N(m_0, g_0^2)$, where m_0 and g_0^2 are given constants;*

(ii) *$V_0, V_1, \ldots$ are identically distributed random variables having density* (3) *dependent on an unknown parameter λ;*

(iii) *the prior distribution $\pi_{\beta,r}$ of the parameter is given by* (5);

(iv) *the control strategy* $U^*(\beta, r) = (u_0^*, \ldots, u_M^*)$ *is defined by*

$$u_M^* = 0, \quad u_n^* = -P_n m_n - H_n \frac{r_n}{\beta_n} - T_n, \; n = 0, 1, \ldots, M-1,$$

where m_n *can be obtained from* (9) *and* (10) *with the initial condition given by the known constant* m_0, β_n *and* r_n *can be obtained from* (7), *the constants* P_n, H_n, T_n *independent of* β *can be obtained from* (20), (22) *with the boundary conditions* (18).

Then the strategy $U^*(\beta, r)$ *is the Bayes control strategy.*

3. In this section we deal with the problem of minimax control of the stochastic system described in Section 1.

Let the class Γ_1 of the prior distributions of the parameter λ be defined as follows: the distribution π belongs to Γ_1 iff $\mathrm{E}_\pi \lambda^2 \leqslant \varkappa$, where $\varkappa$ is a given positive constant. We look for the minimax control strategy with respect to such a class Γ_1.

First, we consider the case where the parameter λ and the control strategy U are given. Then the distribution of the random variables $V_0, V_1, \ldots$ is known and the risk $R(\lambda, U)$ can be calculated. For this purpose we define the truncated risk $R_n(\lambda, U)$ as follows:

$$R_n(\lambda, U) = \mathrm{E}_p\{\mathrm{E}_\lambda [\sum_{i=n}^{N} L_i(X_i, \lambda, u_i)/W^n]\}. \tag{23}$$

Note that $R(\lambda, U) = R_0(\lambda, U)$. The right-hand side of (23) can be transformed to the form

$$R_n(\lambda, U) = \mathrm{E}_\lambda \left[\sum_{i=n}^{M} \frac{\pi_i}{\pi_n} L_i(X_i, \lambda, u_i)/W^n \right].$$

Calculations similar to those performed in (14) allow us to show that $R_n(\lambda, U)$ fulfils the following recursive relation:

$$\begin{aligned} & R_M(\lambda, U) = L_M(m_M, \lambda, u_M) + a_{11}^{(M)} g_M^2, \\ & R_n(\lambda, U) = L_n(m_n, \lambda, u_n) + a_{11}^{(n)} g_n^2 + \frac{\pi_{n+1}}{\pi_n} \mathrm{E}_\lambda [R_{n+1}(\lambda, U)/W^n], \\ & \qquad\qquad n = 0, 1, \ldots, M-1. \end{aligned} \tag{24}$$

Let $U^*(\beta, r) = (u_0^*, \ldots, u_M^*)$ be the Bayes control strategy with respect to the distribution $\pi_{\beta,r}$. For convenience we denote $R_n(\lambda, U^*(\beta, r))$ by R_n. Now we look for the form of the risk R_n. We apply the following equations which can be obtained using (5)–(7) and (10):

$$\begin{aligned} & \mathrm{E}_\lambda (m_{n+1}/W^n, u_n) = \alpha_n m_n + \mu_{n1} u_n + \gamma_n q \lambda, \\ & \mathrm{E}_\lambda (m_{n+1}^2/W^n, u_n) = \alpha_n^2 m_n^2 + \mu_{n2} u_n^2 + \gamma_n^2 q^2 \lambda^2 + 2\alpha_n \mu_{n1} m_n u_n \\ & \qquad + 2\gamma_n \mu_{n1} q \lambda u_n + 2\alpha_n \gamma_n q \lambda m_n + \alpha_n^2 g_n^2 \varepsilon_n t_n + \gamma_n^2 q, \end{aligned} \tag{25}$$

$$E_\lambda(r_{n+1}/W^n, u_n) = r_n + q\lambda,$$

$$E_\lambda(r_{n+1}^2/W^n, u_n) = r_n^2 + 2q\lambda r_n + q^2\lambda^2 + q.$$

We shall show that R_n is of the following form:

$$R_n = c_0^{(n)} m_n^2 + c_1^{(n)} r_n^2 + c_2^{(n)} \lambda^2 + 2c_3^{(n)} m_n \lambda + 2c_4^{(n)} r_n \lambda + 2c_5^{(n)} \lambda + 2c_6^{(n)} m_n + c_7^{(n)}, \quad n = 0, 1, \ldots, M. \tag{26}$$

For $n = M$ this holds with

$$\begin{gathered} c_0^{(M)} = a_{11}^{(M)}, \quad c_2^{(M)} = a_{22}^{(M)}, \quad c_3^{(M)} = a_{12}^{(M)}, \quad c_5^{(M)} = a_{23}^{(M)}, \quad c_6^{(M)} = a_{13}^{(M)}, \\ c_7^{(M)} = a_{11}^{(M)} g_M^2 + a_{33}^{(M)}, \quad c_1^{(M)} = c_4^{(M)} = 0. \end{gathered} \tag{27}$$

Let us suppose that (26) is true for $n+1$. Then, using (24) and (25), we obtain the following expression for R_n:

$$\begin{aligned} R_n = L_n(m_n, \lambda, u_n^*) + a_{11}^{(n)} g_n^2 + \frac{\pi_{n+1}}{\pi_n} \big[& c_0^{(n+1)} (\alpha_n^2 m_n^2 + \mu_{n2} u_n^{*2} + \gamma_n^2 q^2 \lambda^2 \\ & + 2\alpha_n \mu_{n1} m_n u_n^* + 2\gamma_n \mu_{n1} q \lambda u_n^* + 2\alpha_n \gamma_n q \lambda m_n + \alpha_n^2 g_n^2 \varepsilon_n t_n + \gamma_n^2 q) \\ & + c_1^{(n+1)} (r_n^2 + 2q\lambda r_n + q^2 \lambda^2 + q) + c_2^{(n+1)} \lambda^2 + 2c_3^{(n+1)} (\alpha_n m_n + \mu_{n1} u_n^* + \gamma_n q \lambda) \lambda \\ & + 2c_4^{(n+1)} (r_n + q\lambda) \lambda + 2c_5^{(n+1)} \lambda + 2c_6^{(n+1)} (\alpha_n m_n + \mu_{n1} u_n^* + \gamma_n q) + c_7^{(n+1)} \big]. \end{aligned}$$

Finding the terms which contain m_n^2, r_n^2, λ^2, $m_n\lambda$, $r_n\lambda$, λ, $m_n r_n$, m_n, r_n, 1, respectively, we see that (26) is true for n and the coefficients can be calculated from the following equations:

$$\begin{aligned} & c_0^{(n)} = A_n, \quad c_3^{(n)} = B_n, \quad c_5^{(n)} = E_n, \quad c_6^{(n)} = D_n, \\ & c_1^{(n)} = \frac{1}{\pi_n} S_n \frac{1}{\beta_n^2} + d_n c_1^{(n+1)}, \\ & c_2^{(n)} = a_{22}^{(n)} + d_n (A_{n+1} \gamma_n^2 q^2 + 2B_{n+1} \gamma_n q + q^2 c_1^{(n+1)} + c_2^{(n+1)} + 2q c_4^{(n+1)}), \\ & c_4^{(n)} = -\frac{1}{\pi_n} S_n \frac{1}{\beta_n} + d_n (c_4^{(n+1)} + q c_1^{(n+1)}), \\ & c_7^{(n)} = a_{11}^{(n)} g_n^2 + a_{33}^{(n)} + d_n \big[A_{n+1} (\alpha_n^2 g_n^2 \varepsilon_n t_n + \gamma_n^2 q) + q c_1^{(n+1)} \\ & \qquad - \mu_{n1} D_{n+1} T_n + c_7^{(n+1)} \big], \end{aligned} \tag{28}$$

where the new constant S_n independent of β is given by

$$S_n = H_n^2 (\pi_n k_n + \pi_{n+1} A_{n+1} \mu_{n2}).$$

Hence we have just proved the following lemma:

LEMMA 1. *The risk $R(\lambda, U^*(\beta,r))$ connected with the Bayes control strategy $U^*(\beta,r)$ takes the form*

$$R(\lambda, U^*(\beta,r)) = c_0^{(0)}m_0^2 + c_1^{(0)}r^2 + c_2^{(0)}\lambda^2 + 2c_3^{(0)}m\lambda + 2c_4^{(0)}r\lambda + 2c_5^{(0)}\lambda + 2c_6^{(0)}m_0 + c_7^{(0)},$$

where the coefficients $c_i^{(0)}$, $i = 0, 1, \ldots, 7$, can be obtained from the system of equations (28) *with the boundary conditions* (27).

Note that the coefficients $c_1^{(n)}$, $c_2^{(n)}$, $c_4^{(n)}$, $c_7^{(n)}$ of the risk R_n, in contradistinction to the remaining ones, are dependent on β. The dependence can be expressed as follows:

$$\begin{aligned}
c_1^{(n)} &= \frac{1}{\pi_n}\sum_{i=n}^{M-1}\frac{S_i}{\beta_i^2},\\
c_2^{(n)} &= \frac{1}{\pi_n}\left[h_n + q^2\sum_{i=n+1}^{M-1}(i-n)^2\frac{S_i}{\beta_i^2} - 2q\sum_{i=n+1}^{M-1}(i-n)\frac{S_i}{\beta_i}\right],\\
c_4^{(n)} &= \frac{1}{\pi_n}\left[q\sum_{i=n+1}^{M-1}(i-n)\frac{S_i}{\beta_i^2} - \sum_{i=n}^{M-1}\frac{S_i}{\beta_i}\right],\\
c_7^{(n)} &= \frac{1}{\pi_n}\left[e_n + q\sum_{i=n}^{M-1}(i-n)\frac{S_i}{\beta_i^2}\right],
\end{aligned} \tag{29}$$

where the constants h_n and e_n are independent of β and take the form

$$h_n = \pi_n a_{22}^{(n)} + \sum_{i=n}^{M-1}\pi_{i+1}(A_{i+1}\gamma_i^2 q^2 + 2B_{i+1}\gamma_i q + a_{22}^{(i+1)}),$$

$$\begin{aligned}
e_n = \pi_n(a_{11}^{(n)}g_n^2 + a_{33}^{(n)}) + \sum_{i=n}^{M-1}\pi_{i+1}\,[&A_{i+1}(\alpha_i^2 g_i^2\varepsilon_i t_i + \gamma_i^2 q)\\
&-\mu_{i1}D_{i+1}T_i + a_{11}^{(i+1)}g_{i+1}^2 + a_{33}^{(i+1)}].
\end{aligned}$$

Let us define the control strategy $\hat{U}(\varrho) = (\hat{u}_0, \ldots, \hat{u}_M)$ by

$$\hat{u}_M = 0, \quad \hat{u}_n = -P_n m_n - H_n\varrho - T_n, \quad n = 0, 1, \ldots, M-1, \tag{30}$$

where ϱ is a fixed constant, m_n can be obtained from (9) and (10) with the initial condition given by the known value m_0, and P_n, H_n, T_n are given in (20).

By the methods used in the proof of Lemma 1 the risk $R(\lambda, \hat{U}(\varrho))$ can be calculated. Finding the formula for the risk, using Lemma 1 and (29) one can prove that the following equation is valid:

$$R(\lambda, \hat{U}(\varrho)) = \lim_{\substack{\beta\to\infty\\ r/\beta\to\varrho}} R(\lambda, U^*(\beta,r)). \tag{31}$$

It follows from Lemma 1 that for every prior distribution π of the parameter λ the Bayes risk $r(\pi, U^*(\beta,r))$ takes the form

$$\begin{aligned}
r(\pi, U^*(\beta,r)) &= c_2^{(0)}\mathrm{E}_\pi\lambda^2 + (2c_3^{(0)}m_0 + 2c_4^{(0)}r + 2c_5^{(0)})\,\mathrm{E}_\pi\lambda + c_0^{(0)}m_0^2 +\\
&+ c_1^{(0)}r^2 + 2c_6^{(0)}m_0 + c_7^{(0)} \overset{\mathrm{df}}{=} c_2^{(0)}(\beta)\,\mathrm{E}_\pi\lambda^2 + Z_1(\beta,r)\,\mathrm{E}_\pi\lambda + Z_2(\beta,r),
\end{aligned} \tag{32}$$

where the coefficients $c_2^{(0)}$, Z_1 and Z_2 depend on the variables which are indicated in the parentheses. In view of the equations (31) and (32) it is clear that for every prior distribution π of the parameter λ the following equation is true:

$$r(\pi, \hat{U}(\varrho)) = \lim_{\substack{\beta\to\infty \\ r/\beta\to\varrho}} r(\pi, U^*(\beta, r)). \tag{33}$$

Elementary analysis of the form of the coefficient $Z_1(\beta, r)$ shows that for every $\varkappa > 0$ the functions

$$f_1(\beta) = Z_1(\beta, \sqrt{\varkappa\beta^2 - \beta}) \quad \text{and} \quad f_2(\beta) = Z_1(\beta, -\sqrt{\varkappa\beta^2 - \beta}), \quad \beta > 1/\varkappa,$$

are continuous and satisfy the condition $f_1(1/\varkappa) = f_2(1/\varkappa)$. Using these facts, it can be shown that one of the following three conditions is fulfilled:

A. For each $\beta \geqslant 1/\varkappa$, $f_1(\beta) > 0$.

B. For each $\beta \geqslant 1/\varkappa$, $f_2(\beta) < 0$.

C. There exist $\hat{\beta} > 0$ and $\hat{r}$ for which

$$\hat{r}^2/\hat{\beta}^2 + 1/\hat{\beta} = \varkappa \quad \text{and} \quad Z_1(\hat{\beta}, \hat{r}) = 0.$$

It can also be seen that $c_2^{(0)} \geqslant 0$. This follows from Lemma 1 and from the obvious inequality $R(\lambda, U) \geqslant 0$, which holds for each λ and for each control strategy U for which the risk exists.

Now we are ready to solve the problem formulated at the beginning of the present section.

THEOREM 2. *Let us consider the system described in Section* 1. *If the assumptions* (i) *and* (ii) *from Theorem* 1 *are satisfied and if Γ_1 is the class of the prior distributions π of the parameter λ which hold under the condition* $E_\pi\lambda^2 \leqslant \varkappa$, *then there exists $\Gamma_1 MCS$ and*

(i) *if the condition* A *is fulfilled, then $\hat{U}(\sqrt{\varkappa})$ is $\Gamma_1 MCS$;*

(ii) *if the condition* B *is fulfilled, then $\hat{U}(-\sqrt{\varkappa})$ is $\Gamma_1 MCS$;*

(iii) *if the condition* C *is fulfilled, then $U^*(\hat{\beta}, \hat{r})$ is $\Gamma_1 MCS$.*

In order to prove the theorem we use the following lemma:

LEMMA 2. *Let $\{\pi_k\}_1^\infty$ be a sequence of the prior distributions of the parameter λ, $\pi_k \in \Gamma$ and let $\{U_k\}_1^\infty$ and $\{r(\pi_k, U_k)\}_1^\infty$ be the corresponding sequences of Bayes strategies (with respect to π_k) and Bayes risks. If $\hat{U}$ is a strategy for which the Bayes risk satisfies the condition*

$$\sup_{\pi\in\Gamma} r(\pi, \hat{U}) \leqslant \lim_{k\to\infty} \sup r(\pi_k, U_k),$$

then the strategy $\hat{U}$ is ΓMCS.

The above lemma is well known from decision theory (see [7]). We quote it in the version given in [5].

P r o o f o f T h e o r e m 2. Let us suppose that the condition A is fulfilled. Then for each distribution $\pi \in \Gamma_1$ we obtain

$$
\begin{aligned}
r(\pi, \hat{U}(\sqrt{\varkappa})) &= \lim_{\beta\to\infty} r(\pi,\ U^*(\beta, \sqrt{\varkappa\beta^2-\beta})) \\
&= \lim_{\beta\to\infty} [c_2^{(0)}(\beta) \mathrm{E}_\pi \lambda^2 + f_1(\beta) \mathrm{E}_\pi \lambda + Z_2(\beta, \sqrt{\varkappa\beta^2-\beta})] \\
&\leqslant \lim_{\beta\to\infty} [c_2^{(0)}(\beta)\varkappa + f_1(\beta)\sqrt{\varkappa} + Z_2(\beta, \sqrt{\varkappa\beta^2-\beta})] \\
&= \lim_{\beta\to\infty} \left[c_2^{(0)}(\beta)\varkappa + f_1(\beta)\frac{1}{\beta}\sqrt{\varkappa\beta^2-\beta} + Z_2(\beta, \sqrt{\varkappa\beta^2-\beta}) \right] \\
&= \lim_{\beta\to\infty} r(\pi_{\beta,\sqrt{\varkappa\beta^2-\beta}}, U^*(\beta, \sqrt{\varkappa\beta^2-\beta})).
\end{aligned}
$$

This inequality, in view of Lemma 2, implies that the control strategy $\hat{U}(\sqrt{\varkappa})$ is Γ_1MCS. Similarly, it can be proved that the strategy $\hat{U}(-\sqrt{\varkappa})$ is Γ_1MCS when the condition B is fulfilled.

Now we consider the case where the condition C holds. Then for each $\pi \in \Gamma_1$ we have

$$
\begin{aligned}
r(\pi, U^*(\hat{\beta}, \hat{r})) &= c_2^{(0)}(\hat{\beta}) \mathrm{E}_\pi \lambda^2 + Z_1(\hat{\beta}, \hat{r}) \mathrm{E}_\pi \lambda + Z_2(\hat{\beta}, \hat{r}) \\
&= c_2^{(0)}(\hat{\beta}) \mathrm{E}_\pi \lambda^2 + Z_2(\hat{\beta}, \hat{r}) \leqslant c_2^{(0)}(\hat{\beta})\varkappa + Z_2(\hat{\beta}, \hat{r}) = r(\pi_{\hat{\beta},\hat{r}},\ U^*(\hat{\beta}, \hat{r})).
\end{aligned}
$$

If in Lemma 2 for each k we set $\pi_k = \pi_{\hat{\beta},\hat{r}}$ and $U_k = U^*(\hat{\beta}, \hat{r})$, then we see that $U^*(\hat{\beta}, \hat{r})$ is Γ_1MCS. Thus the theorem is proved.

Remark. The above result is very intuitive. Comparing it with the results obtained in [4] or [5] it can be seen that the formulae giving the minimax controls in our problem are similar to those obtained when the states of the system are observed exactly. The only difference is the replacement of the exact value X_n by its expected value m_n. This phenomenon occurs in various decision problems involving quadratic loss functions.

4. In this section we deal with the problem of minimax control of the system described in Section 1, but now we omit the assumption that both parameters of the distribution of the initial state X_0 are known. Now we assume that X_0 has the distribution $N(m, g_0^2)$, where g_0^2 is a given constant, whereas m is an unknown parameter. The parameter is assumed to be a random variable independent of the random variables λ, V_n, ω_n, Z_n $(n = 0, 1, \ldots)$.

Let now $R(m, \lambda, U)$ denote the risk $R(\lambda, U)$ (given by (4)) in the case where the parameter of the distribution of X_0 is known and equal to m. Let $\mathscr{D}$ be a prior distribution of the random vector (m, λ) and let Γ be a subclass of the class of all such distributions. For such a risk, distributions and the class Γ we define the Bayes risk, the Bayes and the minimax control strategies by the same formulae as those given in Section 1.

Let $U_\eta^*(\beta, r)$ denote the Bayes control strategy defined in Theorem 2 in the case where the mean of X_0 is equal to η (i.e., $m_0 = \eta$; see Theorem 2). Similarly, let $\hat{U}_\eta(\varrho)$ denote the strategy given by the formula (30) in the case where we compute the control values under the assumption that $m_0 = \eta$ (see (30) and (9), (10)).

Let Γ_2 be the class of the prior distributions $\mathscr{D}$ of the parameter (m, λ) for which the conditions

(34) $$\mathrm{E}_{\mathscr{D}} m^2 \leqslant \vartheta, \quad \mathrm{E}_{\mathscr{D}} \lambda^2 \leqslant \varkappa$$

hold, where ϑ and $\varkappa$ are given positive constants. Our aim is to find the minimax control strategy with respect to such a class Γ_2.

Let us suppose that, although the state X_0 has the distribution with mean equal to m, we use the strategy $U_\eta^*(\beta, r)$. The methods used in the proof of Lemma 1 and the formulae (24), (25) allow us to show that the risk takes the following form in such a case:

(35) $$\begin{aligned} R\big(m, \lambda, U_\eta^*(\beta, r)\big) = {} & c_{01}^{(0)} m^2 + c_{02}^{(0)} \eta^2 + c_{03}^{(0)} m\eta + c_{11}^{(0)} r^2 + c_{22}^{(0)} \lambda^2 \\ & + 2c_{31}^{(0)} m\lambda + 2c_{32}^{(0)} \eta\lambda + 2c_{44}^{(0)} r\lambda + 2c_{55}^{(0)} \lambda + 2c_{61}^{(0)} m \\ & + 2c_{62}^{(0)} \eta + 2c_{81}^{(0)} mr + 2c_{82}^{(0)} \eta r + c_{77}^{(0)}, \end{aligned}$$

where $c_{ii}^{(0)} = c_i^{(0)}$ for $i = 1, 2, 4, 5, 7$, whereas the remaining coefficients in this expression can be obtained from the following equations:

$$\begin{aligned} c_{01}^{(n)} &= a_{11}^{(n)} + d_n \alpha_n^2 \big(c_{01}^{(n+1)} + \varepsilon_n t_n c_{03}^{(n+1)}\big), \\ c_{02}^{(n)} &= k_n P_n^2 + d_n \big[P_n^2 A_{n+1} \mu_{n2} + \alpha_n^2 c_{02}^{(n+1)} - P_n \alpha_n \mu_{n1} \big(2c_{02}^{(n+1)} + b_n c_{03}^{(n+1)}\big)\big], \\ c_{03}^{(n)} &= d_n \big(\alpha_n^2 b_n c_{03}^{(n+1)} - P_n \alpha_n \mu_{n1} R_{13}^{(n)}\big), \\ c_{31}^{(n)} &= a_{12}^{(n)} + d_n \alpha_n \big(\tfrac{1}{2}\gamma_n q R_{13}^{(n)} + \gamma_n q c_{81}^{(n+1)} + c_{31}^{(n+1)}\big), \\ c_{32}^{(n)} &= c_3^{(n)} - c_{31}^{(n)}, \\ c_{61}^{(n)} &= a_{13}^{(n)} + d_n \alpha_n \big(c_{61}^{(n+1)} - \tfrac{1}{2} T_n \mu_{n1} R_{12}^{(n)}\big), \\ c_{62}^{(n)} &= c_6^{(n)} - c_{61}^{(n)}, \\ c_{81}^{(n)} &= d_n \alpha_n \big(c_{81}^{(n+1)} - \tfrac{1}{2} H_n \mu_{n1} R_{13}^{(n)}\big) \frac{1}{\beta_n}, \\ c_{82}^{(n)} &= -c_{81}^{(n)}. \end{aligned}$$

In these equations we have

$$R_{kl}^{(n)} = 2c_{0k}^{(n+1)} + (1 + \varepsilon_n t_n) c_{0l}^{(n+1)},$$

whereas the remaining constants were given earlier. The boundary conditions are the following:

$$c_{01}^{(M)} = a_{11}^{(M)}, \quad c_{31}^{(M)} = a_{12}^{(M)}, \quad c_{61}^{(M)} = a_{13}^{(M)},$$
$$c_{02}^{(M)} = c_{03}^{(M)} = c_{32}^{(M)} = c_{62}^{(M)} = c_{81}^{(M)} = c_{82}^{(M)} = 0.$$

Similarly, one can prove that

$$(36)\quad R(m,\lambda,\hat{U}_\eta(\varrho)) = c_{01}^{(0)}m^2+c_{02}^{(0)}\eta^2+c_{03}^{(0)}m\eta+c_{12}^{(0)}\varrho^2+c_{21}^{(0)}\lambda^2 \\ +2c_{34}^{(0)}m\lambda+2c_{35}^{(0)}\eta\lambda+2c_{41}^{(0)}\lambda\varrho+2c_{55}^{(0)}\lambda+2c_{61}^{(0)}m+2c_{62}^{(0)}\eta \\ +2c_{81}^{(0)}m\varrho+2c_{82}^{(0)}\eta\varrho+c_{71}^{(0)},$$

where the coefficients which have not been defined previously can be obtained from the equations

$$\begin{aligned}
c_{12}^{(n)} &= \frac{S_n}{\pi_n}+d_n c_{12}^{(n+1)},\\
c_{21}^{(n)} &= a_{22}^{(n)}+d_n(A_{n+1}\gamma_n^2 q^2+c_{21}^{(n+1)}+2B_{n+1}\gamma_n q),\\
c_{34}^{(n)} &= a_{12}^{(n)}+d_n\alpha_n(\gamma_n q R_{13}^{(n)}+c_{34}^{(n+1)}),\\
c_{35}^{(n)} &= c_3^{(n)}-c_{34}^{(n)},\\
c_{41}^{(n)} &= -\frac{S_n}{\pi_n}+d_n c_{41}^{(n+1)},\\
c_{71}^{(n)} &= a_{11}^{(n)}g_n^2+a_{33}^{(n)}+k_n T_n^2+d_n\big[A_{n+1}(\gamma_n^2 q+\alpha_n^2 g_n^2\varepsilon_n t_n)\\
&\quad -2c_6^{(n+1)}\mu_{n1}T_n+c_{71}^{(n+1)}\big]
\end{aligned}$$

with the boundary conditions

$$c_{34}^{(M)} = a_{12}^{(M)},\quad c_{21}^{(M)} = a_{22}^{(M)},\quad c_{71}^{(M)} = a_{11}^{(M)}g_n^2+a_{33}^{(M)},\quad c_{12}^{(M)} = c_{35}^{(M)} = c_{41}^{(M)} = 0.$$

In the case where the expected value of X_0 is given and equal to η we denote Γ_1MCS by U_η. It follows from Theorem 2 that the strategy U_η exists and it takes one of two different forms:

$$U_\eta = U_\eta^*(\beta, r) \text{ for some } \beta, r \quad\text{or}\quad U_\eta = \hat{U}_\eta(\varrho).$$

In the case which we consider ϱ may be equal to $\sqrt{\varkappa}$ or $-\sqrt{\varkappa}$. Using the formulae (35) and (36) it is easy to verify that the risk $R(m,\lambda,U_\eta)$ is a quadratic polynomial with respect to m (independently of the form of U_η). It can be expressed as

$$(37)\qquad R(m,\lambda,U_\eta) = c_{01}^{(0)}m^2+\theta_1(\eta,\lambda)m+\theta_2(\eta,\lambda,\lambda^2),$$

where the functions $\theta_1(\cdot,\cdot)$ and $\theta_2(\cdot,\cdot,\cdot)$ satisfy the following condition: for every distribution π of the parameter λ,

$$\mathrm{E}_\pi\theta_1(\eta,\lambda) = \theta_1(\eta,\mathrm{E}_\pi\lambda),\quad \mathrm{E}_\pi\theta_2(\eta,\lambda,\lambda^2) = \theta_2(\eta,\mathrm{E}_\pi\lambda,\mathrm{E}_\pi\lambda^2).$$

The forms of the functions depend on the form of U_η and they can be easily obtained from the formula (35) or (36). We note also that the coefficient $c_{01}^{(0)}$ is a positive constant (see the note before Theorem 2).

The following theorem is the main result of the present section.

THEOREM 3. *Let us consider the stochastic system described in Section* 1. *Assume that*

(i) *the random variable* X_0 *has the normal distribution* $N(m, g_0^2)$, *where m is an unknown parameter*;

(ii) $V_0, V_1, \ldots$ *are identically distributed random variables having density* (3) *dependent on an unknown parameter* λ;

(iii) *the parameters m and* λ *are independent random variables*;

(iv) Γ_2 *is the class of prior distributions of the random vector* (m, λ) *defined by* (34).

Then

(i) *if* $\theta_1(\sqrt{\vartheta}, \lambda) \geqslant 0$ *for each* $\lambda \in \langle -\sqrt{\varkappa}, \sqrt{\varkappa} \rangle$, *then the control strategy* $U_{\sqrt{\vartheta}}$ *is* $\Gamma_2 MCS$;

(ii) *if* $\theta_1(-\sqrt{\vartheta}, \lambda) \leqslant 0$ *for each* $\lambda \in \langle -\sqrt{\varkappa}, \sqrt{\varkappa} \rangle$, *then the control strategy* $U_{-\sqrt{\vartheta}}$ *is* $\Gamma_2 MCS$.

Proof. Let us consider the case where the mean of X_0 is given and equal to η. Note that the control strategy U_η is the Bayes one. This is obvious when $U_\eta = U_\eta^*(\beta, r)$ for some β and r. If $U_\eta = \hat{U}_\eta(\varrho)$ for any ϱ, then, similarly as in the proof of Theorem 1, one can easily verify that U_η is the Bayes control strategy with respect to the distribution concentrated at the point ϱ. We denote such a distribution by δ_ϱ. For convenience, we denote by π_η the distribution with respect to which the control strategy U_η is the Bayes one.

It follows from the independence of the parameters m and λ that for each distribution $\mathscr{D} \in \Gamma_2$ we have

$$r(\mathscr{D}, U) = \mathrm{E}_{\mathscr{P}} \mathrm{E}_\pi R(m, \lambda, U), \tag{38}$$

where $\mathscr{P}$ and π are the marginal distributions of the distribution $\mathscr{D}$, $\mathscr{P}$ is the distribution of the parameter m, π is the distribution of the parameter λ. We introduce the following notation:

$$r(\mathscr{P} \times \pi, U) = r(\mathscr{P}, \pi, U) \quad \text{and} \quad r(\delta_\varrho, \pi, U) = r(\varrho, \pi, U).$$

Let Γ_1 be the class defined in Theorem 2. We shall show that

$$\sup_{\pi \in \Gamma_1} r(\eta, \pi, U_\eta) = r(\eta, \pi_\eta, U_\eta). \tag{39}$$

First, let us consider the case where $U_\eta = U_\eta^*(\beta, r)$. Then for each $\pi \in \Gamma_1$ we obtain (see Theorem 2 and its proof)

$$\begin{aligned} r(\eta, \pi, U_\eta^*(\beta, r)) &= c_2^{(0)} \mathrm{E}_\pi \lambda^2 + Z_1(\beta, r) \mathrm{E}_\pi \lambda + Z_2(\beta, r) \\ &= c_2^{(0)} \mathrm{E}_\pi \lambda^2 + Z_2(\beta, r) \leqslant c_2^{(0)} \varkappa + Z_2(\beta, r) \\ &= r(\eta, \pi_{\beta, r}, U_\eta^*(\beta, r)) = r(\eta, \pi_\eta, U_\eta). \end{aligned}$$

Now, let us suppose that $U_\eta = \hat{U}_\eta(\sqrt{\varkappa})$. Then for each $\pi \in \Gamma_1$ we obtain

$$
\begin{aligned}
r(\eta,\pi,\hat{U}_\eta(\sqrt{\varkappa})) &= \lim_{\substack{\beta\to\infty\\ r/\beta\to\sqrt{\varkappa}}} r(\eta,\pi,U_\eta^*(\beta,r))\\
&= \lim_{\substack{\beta\to\infty\\ r/\beta\to\sqrt{\varkappa}}} \left[c_2^{(0)}(\beta)\mathrm{E}_\pi\lambda^2+Z_1(\beta,r)\,\mathrm{E}_\pi\lambda+Z_2(\beta,r)\right]\\
&\leqslant \lim_{\substack{\beta\to\infty\\ r/\beta\to\sqrt{\varkappa}}} \left[c_2^{(0)}(\beta)\varkappa+Z_1(\beta,r)\sqrt{\varkappa}+Z_2(\beta,r)\right]\\
&= \lim_{\beta\to\infty}\left[c_2^{(0)}(\beta)\varkappa+Z_1(\beta,\sqrt{\varkappa\beta^2-\beta})\frac{1}{\beta}\sqrt{\varkappa\beta^2-\beta}+Z_2(\beta,\sqrt{\varkappa\beta^2-\beta})\right]\\
&= \lim_{\beta\to\infty} r(\eta,\pi_{\beta,\sqrt{\varkappa\beta^2-\beta}},U_\eta^*(\beta,\sqrt{\varkappa\beta^2-\beta}))\\
&= r(\eta,\delta_{\sqrt{\varkappa}},\hat{U}_\eta(\sqrt{\varkappa})) = r(\eta,\pi_\eta,U_\eta).
\end{aligned}
$$

Similarly, one can verify that an analogous inequality holds when $U_\eta = \hat{U}_\eta(-\sqrt{\varkappa})$. Since the above inequalities hold for each $\pi\in\Gamma_1$, the equation (39) is valid.

Let us suppose now that $\theta_1(\sqrt{\vartheta},\lambda)\geqslant 0$ for each $\lambda\in\langle-\sqrt{\varkappa},\sqrt{\varkappa}\rangle$. We shall show that

$$
\sup_{\mathscr{P}\times\pi\in\Gamma_2} r(\mathscr{P},\pi,U_{\sqrt{\vartheta}}) = \sup_{\pi\in\Gamma_1} r(\sqrt{\vartheta},\pi,U_{\sqrt{\vartheta}}). \tag{40}
$$

It follows from the above assumption and (37) that for each distribution $\mathscr{P}\times\pi\in\Gamma_2$ the following relations are valid:

$$
\begin{aligned}
r(\mathscr{P},\pi,U_{\sqrt{\vartheta}}) &= c_{01}^{(0)}\mathrm{E}_{\mathscr{P}}m^2+\theta_1(\sqrt{\vartheta},\mathrm{E}_\pi\lambda)\mathrm{E}_{\mathscr{P}}m+\theta_2(\sqrt{\vartheta},\mathrm{E}_\pi\lambda,\mathrm{E}_\pi\lambda^2)\\
&\leqslant c_{01}^{(0)}\vartheta+\theta_1(\sqrt{\vartheta},\mathrm{E}_\pi\lambda)\sqrt{\vartheta}+\theta_2(\sqrt{\vartheta},\mathrm{E}_\pi\lambda,\mathrm{E}_\pi\lambda^2) = r(\delta_{\sqrt{\vartheta}},\pi,U_{\sqrt{\vartheta}}).
\end{aligned}
$$

Hence the equation (40) is true. Using the formulae (40) and (39) we obtain

$$
\sup_{\mathscr{D}\in\Gamma_2} r(\mathscr{D},U_{\sqrt{\vartheta}}) = \sup_{\mathscr{P}\times\pi\in\Gamma_2} r(\mathscr{P},\pi,U_{\sqrt{\vartheta}}) = \sup_{\pi\in\Gamma_1} r(\sqrt{\vartheta},\pi,U_{\sqrt{\vartheta}}) = r(\sqrt{\vartheta},\pi_{\sqrt{\vartheta}},U_{\sqrt{\vartheta}}).
$$

On the other hand, we know that the control strategy $U_{\sqrt{\vartheta}}$ is the Bayes one with respect to the distribution $\delta_{\sqrt{\vartheta}}\times\pi_{\sqrt{\vartheta}}$. Therefore, if in Lemma 2 we set $\pi_k = \delta_{\sqrt{\vartheta}}\times\pi_{\sqrt{\vartheta}}$ and $U_k = U_{\sqrt{\vartheta}}$, $k = 1,2,\ldots$, we see that $U_{\sqrt{\vartheta}}$ is Γ_2MCS. Thus the part (i) of our assertion is proved.

In the same manner one can prove that (ii) is valid.

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