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On Randers changes of mth root Finsler metrics $L = \sqrt[m]{A}$ without irreducibility of A

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Abstract. We study Randers changes of mth root Finsler metrics, and provide necessary and sufficient conditions for the Finsler metric obtained by a Randers change of an mth root metric to be dually flat. We also prove that if the Finsler metric obtained by a Randers change of an mth root metric is projectively flat, then the mth root metric is locally Minkowskian or Riemannian.

1. Introduction. A change of Finsler metric $L \mapsto \bar{L}$ is called a *Randers change* if $\bar{L}(x,y) = L(x,y) + \beta(x,y)$, where $\beta(x,y) = b_i(x)y^i$ is a 1-form on a smooth manifold M^n. It is easy to see that if $\|\beta\|_L < 1$, then $\bar{L}$ is also a Finsler metric. The notion of Randers change has been proposed by M. Matsumoto [M]. When L is a Riemannian metric, $\bar{L}$ becomes a *Randers metric*. Randers metrics were introduced by the physicist G. Randers in 1941 in the context of general relativity. The name was given by R. S. Ingarden [I], who used it to study the theory of the electron microscope. Randers changes of Finsler metrics have been studied from different viewpoints [HI], [M], [PL].

Let $F^n = (M^n, L)$ be an n-dimensional Finsler space with fundamental metric function $L(x,y)$. The *mth root metrics*, first defined by H. Shimada [Shi], form an important class of Finsler metrics with

$$L(x,y) = \sqrt[m]{A}, \tag{1.1}$$

where $A = a_{i_1\ldots i_m}(x)y^{i_1}\ldots y^{i_m}$ and the coefficients $a_{i_1\ldots i_m}(x)$ are the components of a symmetric tensor field covariant of order m, depending on the position x only. A second root metric is just a Riemannian metric. Thus an mth root metric can be regarded as a generalization of a Riemannian metric. Third and fourth root metrics are called *cubic metrics* and *quartic*

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metrics respectively. The special root metric $L = \sqrt[n]{y^1 \dots y^n}$ is called the *Berwald–Moor metric*. This metric is singular in y and not positive definite. The positive definiteness of mth root Finsler metrics is discussed in Section 3. mth root Finsler metrics are used in the theory of space-time structure, gravitation and unified gauge field theories [A], [P1], [P2]. Moreover, they are also applied in biology as an ecological metric [AIM]. Thus it is important to study the geometric properties of mth root Finsler metrics. Throughout the paper, we only consider positive definite mth root metrics.

Tamássy [T] studied tensorial connections for mth root Finsler metrics. Li–Shen [LS] studied locally projectively flat fourth root metrics and proved that they are locally Minkowskian. Yu–You [YY] showed that an mth root Einstein Finsler metrics is Ricci-flat. Tayebi–Najafi [TN] characterized locally dually flat and Antonelli mth root metrics. A. Tayebi, T. Tabatabaeifar and E. Peyghan [TTP] found a necessary and sufficient condition for Finsler metrics obtained by a Kropina change of mth root Finsler metrics to be locally dually flat. They also proved that the Finsler metric obtained by a Kropina change of an mth root Finsler metric is locally projectively flat if and only if the mth root metric is locally Minkowskian. However, the above results in [LS], [TN] and [TTP] are obtained under the assumption of irreducibility of A. So it is a natural problem to study the mth root metrics without this condition.

The purpose of this paper is to study Randers changes of non-Riemannian mth root metrics. We drop the irreducibility condition on A and get a necessary and sufficient condition under which the Finsler metric obtained by a Randers change of an mth root metric is dually flat. We generalize the results of [TPN2]. For details, see Theorem 1.1.

We denote by $L_{y^i}, L_{y^i y^j}, \dots$ the partial derivative(s) of L with respect to the coordinate y^i, and similarly for the coordinate x^i. Further, denote

$$A_i := A_{y^i}, \quad A_{ij} := A_{y^i y^j}, \quad A_0 := A_{x^i} y^i, \quad A_{0i} := A_{x^l y^i} y^l.$$

THEOREM 1.1. *Let $L(x,y) = \sqrt[m]{A}$ be a non-Riemannian mth root metric on a manifold M^n. Suppose that $\bar{L}(x,y) = L(x,y) + \beta(x,y)$ is a Finsler metric obtained by a Randers change of L, where $\beta(x,y) = b_i(x)y^i$ is a 1-form on M^n. Then $\bar{L}$ is dually flat if and only if*

(1) *there is a 1-form $\theta = \theta_i y^i$ on M^n such that*

$$A_{x^l} = \tfrac{1}{3}\{2\theta A_l + mA\theta_l\}, \tag{1.2}$$

(2) *β satisfies*

$$\frac{\partial b_i}{\partial x^j} = \tfrac{2}{3}\theta_i b_j + \tfrac{1}{3}\theta_j b_i. \tag{1.3}$$

Letting $\beta = 0$ in Theorem 1.1, we easily obtain

COROLLARY 1.2. *Let $L(x,y) = \sqrt[m]{A}$ be a non-Riemannian mth root metric on a manifold M^n. Then L is dually flat if and only if there is a 1-form $\theta = \theta_i y^i$ on M^n such that*

$$A_{x^l} = \tfrac{1}{3}\{2\theta A_l + mA\theta_l\}.$$

REMARK 1.3. The non-Riemannian condition in Theorem 1.1 and Corollary 1.2 cannot be dropped. For more details, see Sections 5 and 6.

A Finsler manifold is called *projectively flat* if its geodesics can be mapped into straight lines of the Euclidean space. We prove

THEOREM 1.4. *Let $L(x,y) = \sqrt[m]{A}$ be an mth root metric on a manifold M^n. Suppose that $\bar{L}(x,y) = L(x,y)+\beta(x,y)$ is a Finsler metric obtained by a Randers change of L, where $\beta(x,y) = b_i(x)y^i$ is a 1-form on M^n. Then $\bar{L}$ is projectively flat if and only if L is locally Minkowskian or Riemannian with constant sectional curvature and β is closed.*

Letting $\beta = 0$ in Theorem 1.4, we immediately get

COROLLARY 1.5. *Let $L(x,y) = \sqrt[m]{A}$ be an mth root metric on a manifold M^n. Then L is projectively flat if and only if it is locally Minkowskian or Riemannian with constant sectional curvature.*

REMARK 1.6. If we add the irreducibility condition of A and take $\bar{\theta} = m\theta$ in Corollary 1.2, it is just the main theorem of [TN]. Thus Theorem 1.1 generalizes the main result of [TN]. In all of the above results, we do not assume the irreducibility of A. We conjecture that the irreducibility of A is not necessary when one characterises mth root metrics or generalized mth root metrics [TPN1].

2. Preliminaries. The local coordinates on a manifold M^n are denoted as (x^i) and those of TM^n in the induced chart as (x^i, y^i). A *Finsler metric* on M^n is a function $L : TM^n \to [0,\infty)$ which has the following properties [C]:

(a) *Regularity*: L is C^∞ on $TM^n \setminus \{0\}$,
(b) *Homogeneity*: $L(x,\lambda y) = \lambda L(x,y)$ for all $\lambda > 0$,
(c) *Strong convexity*: For any tangent vector $y \in T_xM^n \setminus \{0\}$, the following bilinear symmetric form $g_y : T_xM^n \otimes T_xM^n \to \mathbb{R}$ is positive definite:

$$g_y(u,v) := \frac{1}{2}\,\frac{\partial^2}{\partial s\partial t}[L^2(x,y+su+tv)]\Big|_{s=t=0}.$$

Let

$$g_{ij}(x,y) := \frac{1}{2}\,\frac{\partial^2 L^2(x,y)}{\partial y^i\partial y^j}.$$

By the homogeneity of L, we have

$$L(x,y) = \sqrt{g_{ij}(x,y)y^iy^j}.$$

For a given Finsler metric $L = L(x, y)$, the geodesics of L are characterized locally by a system of second order ODEs as follows [C]:

$$\frac{d^2x^i}{dt^2} + 2G^i\left(x, \frac{dx}{dt}\right) = 0,$$

where

$$G^i = \tfrac{1}{4}g^{il}\{[L^2]_{x^m y^l}y^m - [L^2]_{x^l}\}$$

and $(g^{ij}) := (g_{ij})^{-1}$. The G^i are called the *geodesic coefficients* of L. We call L a *Berwald metric* if the G^i are quadratic in y. Clearly, every Riemannian metric is a Berwald metric. L is called a *Landsberg metric* if

$$L_{y^i}\frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l} = 0.$$

Thus Landsberg metrics are Berwald metrics.

For any $x \in M$ and $y \in T_xM \setminus \{0\}$, the *Riemann curvature* $\mathbf{R}_y = R^i{}_k \frac{\partial}{\partial x^i} \otimes dx^k$ is defined by [C]

$$R^i{}_k = 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^m \partial y^k}y^m + 2G^m\frac{\partial^2 G^i}{\partial y^m \partial y^k} - \frac{\partial G^i}{\partial y^m}\frac{\partial G^m}{\partial y^k}. \tag{2.1}$$

For a tangent plane $P \subset T_xM$ containing y, let

$$\mathbf{K}(P, y) := \frac{\mathbf{g}_y(\mathbf{R}_y(u), u)}{\mathbf{g}_y(y, y)\mathbf{g}_y(u, u) - [\mathbf{g}_y(y, u)]^2},$$

where $u \in P$ such that $P = \mathrm{span}\{y, u\}$. $\mathbf{K} = \mathbf{K}(P, y)$ is called the *flag curvature*. A Finsler metric L is said to be of *scalar flag curvature* if the flag curvature $\mathbf{K}(P, y) = K(x, y)$ is a scalar function on the slit tangent bundle $TM \setminus \{0\}$. In 1975, S. Numata proved the following important theorem.

THEOREM 2.1 ([N]). *Let L be a Landsberg metric of scalar flag curvature on a manifold M^n of dimension $n \geq 3$. If $\mathbf{K} \neq 0$, then L is Riemannian.*

The condition $n \geq 3$ in the above theorem is necessary. In 1928, L. Berwald proved that

THEOREM 2.2 ([B1]). *Every Berwald metric with $\mathbf{K} = 0$ is locally Minkowskian.*

REMARK 2.3. A Finsler metric $L = L(x, y)$ is said to be *locally Minkowskian* if at every point there is a local coordinate domain in which the metric $L = L(y)$ is independent of x.

Locally dually flat Finsler metrics are studied in Finsler information geometry and naturally arise from the investigation of the so-called flat information structure [Shen]. A Finsler metric $L = L(x, y)$ is said to be *locally dually flat* if at every point there is a coordinate system (x^i) such that

$$[L^2]_{x^k y^l}y^k = 2[L^2]_{x^l}. \tag{2.2}$$

Such a coordinate system is called an *adapted coordinate system.*

A Finsler metric $L = L(x, y)$ is projectively flat if and only if its spray coefficients satisfy $G^i = P(x, y)y^i$. This is equivalent to the following *Hamel equation*:

$$L_{x^k y^l} y^k = L_{x^l}. \tag{2.3}$$

In this case, the projective factor P is $L_{x^k} y^k/(2L)$ and the flag curvature is given by

$$\mathbf{K} = \frac{P^2 - P_{x^k} y^k}{L^2}. \tag{2.4}$$

Thus locally projectively flat Finsler metrics are of scalar flag curvature. L. Berwald [B2] proved

THEOREM 2.4 ([B2]). *Let L be a locally projectively flat Landsberg metric on a surface M. Then L is either Riemannian with non-zero constant Gauss curvature or locally Minkowskian.*

3. The proof of Theorem 1.1. Let $L(x, y) = \sqrt[m]{A}$ be an mth root metric on a manifold M^n, where $A = a_{i_1\dots i_m}(x) y^{i_1} \dots y^{i_m}$ and the coefficients $a_{i_1\dots i_m}(x)$ depend on the position alone, with symmetric indices $i_1, \dots, i_m$. The fundamental tensor is given by

$$g_{ij} = \frac{A^{2/m-2}}{m^2}[mAA_{ij} + (2-m)A_i A_j]. \tag{3.1}$$

For further computation, we need the following lemma from linear algebra.

LEMMA 3.1 ([C]). *Let $K = (g_{ij})$ and $H = (h_{ij})$ be symmetric $n \times n$ matrices and $C = (c_i)$ be an n-vector. Assume that H is invertible with $H^{-1} = (h^{ij})$, and*

$$g_{ij} = h_{ij} + \delta c_i c_j.$$

Then

$$\det(g_{ij}) = (1 + \delta c^2)\det(h_{ij}),$$

where $c := \sqrt{h^{ij} c_i c_j}$. If $1 + \delta c^2 \neq 0$, then K is invertible. The inverse matrix $K^{-1} = (g^{ij})$ is given by

$$g^{ij} = h^{ij} - \frac{\delta c^i c^j}{1 + \delta c^2}, \quad \textit{where} \quad c^i := h^{ij} c_j.$$

By the above lemma and (3.1), it is obvious that (g_{ij}) is positive definite if and only if (A_{ij}) is positive definite. Let $(A_{ij})^{-1} = (A^{ij})$. Then the inverse matrix (g^{ij}) is given by

$$g^{ij} = A^{-2/m}\left[mAA^{ij} + \frac{m-2}{m-1} y^i y^j\right].$$

Now we consider the dually flat Finsler metric $\bar{L} = \sqrt[m]{A} + \beta(x, y)$, where $\beta := b_i(x) y^i$ is a 1-form on M^n. It is easy to prove

Proposition 3.2. *Let* $L(x,y) = \sqrt[m]{A}$ *be a non-Riemannian mth root metric on a manifold* M^n*. Suppose that* $\bar{L}(x,y) = L(x,y) + \beta(x,y)$ *is a Finsler metric obtained by a Randers change of* L*, where* $\beta(x,y) = b_i(x)y^i$ *is a* 1*-form on* M^n*. Then* $\bar{L}$ *is dually flat if and only if*

(1) *there is a* 1*-form* $\theta = \theta_i y^i$ *on* M^n *such that*

$$\frac{\partial b_i}{\partial x^j} = \tfrac{2}{3}\theta_i b_j + \tfrac{1}{3}\theta_j b_i, \tag{3.2}$$

(2) A *satisfies*

$$(2-m)A_0 A_l = mA(2A_{x^l} - A_{0l}), \tag{3.3}$$

$$(mA\theta - A_0)A_l\beta = (m^2 A\theta - mA_0)Ab_l. \tag{3.4}$$

Proof. By assumption and (2.2), we have

$$[\bar{L}^2]_{x^k y^l} y^k = 2[\bar{L}^2]_{x^l}. \tag{3.5}$$

By a direct computation, we get

$$\bar{L}^2 = A^{2/m} + 2\beta A^{1/m} + \beta^2, \tag{3.6}$$

$$(\bar{L}^2)_{x^l} = \frac{2}{m}A^{2/m-1}A_{x^l} + 2\beta_{x^l}A^{1/m} + \frac{2}{m}\beta A^{1/m-1}A_{x^l} + 2\beta\beta_{x^l}, \tag{3.7}$$

$$\begin{aligned}(\bar{L}^2)_{x^k y^l} y^k = {}& \frac{2}{m}\left(\frac{2}{m}-1\right)A^{2/m-2}A_0A_l + \frac{2}{m}A^{2/m-1}A_{0l} + 2\beta_{0l}A^{1/m} \\ &+ \frac{2}{m}\beta_0 A^{1/m-1}A_l + \frac{2}{m}A^{1/m-1}A_0 b_l + \frac{2}{m}\beta A^{1/m-1}A_{0l} \\ &+ \frac{2}{m}\left(\frac{1}{m}-1\right)\beta A^{1/m-2}A_0A_l + 2\beta_0 b_l + 2\beta\beta_{0l},\end{aligned} \tag{3.8}$$

where

$$\beta_0 := \beta_{x^i}y^i, \qquad \beta_{0l} := \beta_{x^i y^l}y^i.$$

Plugging (3.6)–(3.8) into (3.5), we obtain

$$R_l A^{2/m} + S_l A^{1/m} + T_l = 0, \tag{3.9}$$

where

$$R_l = AA_{0l} + \left(\frac{2}{m}-1\right)A_0A_l - 2AA_{x^l}, \tag{3.10}$$

$$T_l = A^2(\beta\beta_{0l} + \beta_0 b_l - 2\beta\beta_{x^l}), \tag{3.11}$$

$$\begin{aligned}S_l = {}& \frac{1}{m}A\beta A_{0l} + \frac{1}{m}AA_0 b_l + \frac{1}{m}\left(\frac{1}{m}-1\right)\beta A_0 A_l \\ &+ A^2\beta_{0l} + \frac{1}{m}A\beta_0 A_l - \frac{2}{m}\beta AA_{x^l} - 2A^2\beta_{x^l}.\end{aligned} \tag{3.12}$$

Noting that R_l, S_l, T_l are polynomials in (y^i), we find that $R_l = S_l = T_l = 0$

by (3.9), i.e.,

$$0 = AA_{0l} + \left(\frac{2}{m} - 1\right)A_0A_l - 2AA_{x^l}, \tag{3.13}$$

$$0 = \beta\beta_{0l} + \beta_0 b_l - 2\beta\beta_{x^l}, \tag{3.14}$$

$$\begin{aligned} 0 = {} & A\beta A_{0l} + AA_0 b_l + \left(\frac{1}{m} - 1\right)\beta A_0 A_l \\ & + mA^2\beta_{0l} + A\beta_0 A_l - 2\beta AA_{x^l} - 2mA^2\beta_{x^l}. \end{aligned} \tag{3.15}$$

It follows from (3.13) that (3.3) holds.

By (3.14), there is a 1-form $\theta = \theta_i y^i$ on M^n such that $\beta_0 = \theta\beta$. Differentiating this with respect to y^l, we obtain

$$\beta_{x^l} + \frac{\partial b_l}{\partial x^k} y^k = \theta_l \beta + \theta b_l. \tag{3.16}$$

Differentiating (3.16) with respect to y^k, we get

$$\frac{\partial b_k}{\partial x^l} + \frac{\partial b_l}{\partial x^k} = \theta_l b_k + \theta_k b_l. \tag{3.17}$$

Plugging $\beta_0 = \theta\beta$ into (3.14) yields

$$2\beta_{x^l} = \beta_{0l} + \theta b_l. \tag{3.18}$$

Combining (3.16) and (3.18), one obtains

$$\beta_{x^l} = \tfrac{2}{3}\theta b_l + \tfrac{1}{3}\theta_l \beta, \tag{3.19}$$

$$\beta_{0l} = \tfrac{1}{3}\theta b_l + \tfrac{2}{3}\theta_l \beta. \tag{3.20}$$

Then (3.2) follows from (3.19).

Now, we prove that (3.4) holds. (3.13)$\times\beta$ – (3.15) yields

$$-\frac{1}{m} A_0 A_l \beta + AA_0 b_l + A\beta_0 A_l = mA^2(2\beta_{x^l} - \beta_{0l}). \tag{3.21}$$

Substituting (3.19), (3.20) and $\beta_0 = \theta\beta$ into (3.21) yields (3.4). The converse is a direct computation. This completes the proof. ∎

To prove the main theorem, the following lemma is necessary.

LEMMA 3.3. *Let $L = \sqrt[m]{A}$ be a non-Riemannian mth root metric on a manifold M^n, where $A = a_{i_1 \dots i_m}(x) y^{i_1} \dots y^{i_m}$ and the coefficients $a_{i_1 \dots i_m}(x)$ are components of a symmetric tensor field covariant of order m. If A satisfies* (3.3) *and* $m \geq 3$, *there is a* 1*-form* η *on* M^n *such that* $A_0 = m\eta A$.

Proof. When A is irreducible, as $\deg(A_l) = m - 1$ and $\deg(A) = m$, it is clear that the conclusion of Lemma 3.3 follows from (3.3).

When A is reducible, we can always express $A = B_1^{r_1} \dots B_p^{r_p}$, where $B_1, \dots, B_p$ are irreducible polynomials in (y^i) and relatively prime. Let $\deg(B_q) = d_q$. It is obvious that $\sum_{q=1}^{p} d_q r_q = m$.

Case 1: $r_q \geq 2$ *for all* q. Denote $(B_s)_0 := (B_s)_{x^l} y^l$, $(B_s)_l := (B_s)_{y^l}$ and $(B_s)_{0l} := (B_s)_{x^k y^l} y^k$. By a direct computation, we have

$$A_{x^l} = \sum_{s=1}^{p} r_s B_1^{r_1} B_2^{r_2} \dots B_s^{r_s-1} \dots B_p^{r_p} (B_s)_{x^l}, \tag{3.22}$$

$$A_l = \sum_{s=1}^{p} r_s B_1^{r_1} B_2^{r_2} \dots B_s^{r_s-1} \dots B_p^{r_p} (B_s)_l, \tag{3.23}$$

$$A_0 = \sum_{s=1}^{p} r_s B_1^{r_1} B_2^{r_2} \dots B_s^{r_s-1} \dots B_p^{r_p} (B_s)_0. \tag{3.24}$$

Further, we have

$$\begin{aligned} A_{0l} = {} & \sum_{s=1}^{p} r_s(r_s-1) B_1^{r_1} B_2^{r_2} \dots B_s^{r_s-2} \dots B_p^{r_p} (B_s)_l (B_s)_0 \\ & + \sum_{s=1}^{p} r_s B_1^{r_1} B_2^{r_2} \dots B_s^{r_s-1} \dots B_p^{r_p} (B_s)_{0l} \\ & + \sum_{s,t=1,\, s\neq t}^{p} r_s r_t B_1^{r_1} B_2^{r_2} \dots B_s^{r_s-1} \dots B_t^{r_t-1} \dots B_p^{r_p} (B_s)_0 (B_t)_l. \end{aligned} \tag{3.25}$$

By (3.22)–(3.24), one can get

$$\begin{aligned} A_0 A_l = {} & \sum_{s,t=1,\, s\neq t}^{p} r_s r_t B_1^{2r_1} B_2^{2r_2} \dots B_s^{2r_s-1} \dots B_t^{2r_t-1} \dots B_p^{2r_p} (B_s)_0 (B_t)_l \\ & + \sum_{s=1}^{p} r_s^2 B_1^{2r_1} B_2^{2r_2} \dots B_s^{2r_s-2} \dots B_p^{2r_p} (B_s)_0 (B_s)_l \end{aligned} \tag{3.26}$$

and

$$\begin{aligned} & (2A_{x^l} - A_{0l})A \\ & = 2\sum_{s=1}^{p} r_s B_1^{2r_1} B_2^{2r_2} \dots B_s^{2r_s-1} \dots B_p^{2r_p} (B_s)_{x^l} \\ & \quad - \sum_{s,t=1,\, s\neq t}^{p} r_s r_t B_1^{2r_1} B_2^{2r_2} \dots B_s^{2r_s-1} \dots B_t^{2r_t-1} \dots B_p^{2r_p} (B_s)_0 (B_t)_l \\ & \quad - \sum_{s=1}^{p} r_s B_1^{2r_1} B_2^{2r_2} \dots B_s^{2r_s-1} \dots B_p^{2r_p} (B_s)_{0l} \\ & \quad - \sum_{s=1}^{p} r_s(r_s-1) B_1^{2r_1} B_2^{2r_2} \dots B_s^{2r_s-2} \dots B_p^{2r_p} (B_s)_0 (B_s)_l. \end{aligned} \tag{3.27}$$

Plugging (3.26) and (3.27) into (3.3) yields

$$
\begin{aligned}
(3.28)\quad 0 = {} & 2m\sum_{s=1}^{p} r_s B_1^{2r_1}B_2^{2r_2}\dots B_s^{2r_s-1}\dots B_p^{2r_p}(B_s)_{x^l} \\
& - 2\sum_{s,t=1,\, s\neq t}^{p} r_s r_t B_1^{2r_1}B_2^{2r_2}\dots B_s^{2r_s-1}\dots B_t^{2r_t-1}\dots B_p^{2r_p}(B_s)_0(B_t)_l \\
& - m\sum_{s=1}^{p} r_s B_1^{2r_1}B_2^{2r_2}\dots B_s^{2r_s-1}\dots B_p^{2r_p}(B_s)_{0l} \\
& - \sum_{s=1}^{p} r_s(2r_s-m)B_1^{2r_1}B_2^{2r_2}\dots B_s^{2r_s-2}\dots B_p^{2r_p}(B_s)_0(B_s)_l.
\end{aligned}
$$

We claim that $2r_s - m \neq 0$ for all s. Suppose that $r_{u_1} = \dots = r_{u_k} = m/2$. Because $\sum_{q=1}^{p} d_q r_q = m$, where $\deg(B_q) = d_q \geq 1$, we have $k \leq 2$. If $k = 2$, then $L = \sqrt[m]{A} = \sqrt[m]{(B_{u_1}B_{u_2})^{m/2}} = \sqrt{B_{u_1}B_{u_2}}$. Hence $\deg(B_{u_1}) = \deg(B_{u_2}) = 1$, i.e., $B_{u_1} = a_i(x)y^i$ and $B_{u_2} = \hat{a}_i(x)y^i$ are 1-forms on M^n. Since L is positive definite, this is impossible. If $k = 1$, then $\deg(B_{u_1}) = 1$ or 2. If $\deg(B_{u_1}) = 2$, then $L = \sqrt{B_{u_1}}$ is a Riemannian metric, a contradiction. If $\deg(B_{u_1}) = 1$, i.e., $B_{u_1} = a_i(x)y^i$, we can always choose $y \neq 0$ such that $B_{u_1} = a_i(x)y^i = 0$, so $L = 0$, contrary to L being positive definite. Thus $2r_s - m \neq 0$ for all s.

Because $B_1, \dots, B_p$ are irreducible polynomials in (y^i) and relatively prime, from (3.28) and $\deg((B_s)_l) = r_s - 1$ we have $B_s \mid (B_s)_0$, i.e., there is a 1-form ξ_s on M^n such that $(B_s)_0 = \xi_s B_s$. Plugging this into (3.24) yields

$$
A_0 = \sum_{s=1}^{p} r_s \xi_s B_1^{r_1}\dots B_s^{r_s}\dots B_p^{r_p} = A\sum_{s=1}^{p} r_s\xi_s.
$$

Taking $\eta = m^{-1}\sum_{s=1}^{p} r_s\xi_s$, we obtain $A_0 = m\eta A$.

Case 2: $r_{u_1} = \dots = r_{u_k} = 1$ for some $1 \leq k \leq p$. Without loss of generality, we suppose that $r_1 = \dots = r_k = 1$. Then $A = B_1\dots B_k B_{k+1}^{r_{k+1}}\dots B_p^{r_p}$, where $r_{k+1}, \dots, r_p \geq 2$. Denote $B := B_{k+1}^{r_{k+1}}\dots B_p^{r_p}$. By a direct computation, we have

$$
(3.29)\qquad A_l = B\sum_{s=1}^{p} B_1\dots B_{s-1}B_{s+1}\dots B_k(B_s)_l + B_1\dots B_k(B)_{y^l},
$$

$$
(3.30)\qquad A_0 = B\sum_{s=1}^{p} B_1\dots B_{s-1}B_{s+1}\dots B_k(B_s)_0 + B_1\dots B_k(B)_{x^l}y^l.
$$

Because $B_1, \dots, B_p$ are irreducible polynomials in (y^i) and relatively prime, from (3.29) and $\deg((B_s)_l) < \deg(B_s)$, we have $B_i \nmid A_l$ for all $1 \leq i \leq k$.

It follows from (3.3) that $B_i \,|\, A_0$ for all $1 \le i \le k$. Then $B_i \,|\, (B_i)_0$ for all $1 \le i \le k$ by (3.30). We also obtain $B_i \,|\, (B_i)_0$ for all $k+1 \le i \le p$ as in Case 1. Thus $B_i \,|\, (B_i)_0$ for all $1 \le i \le p$, i.e., there is a 1-form ξ_s on M^n such that $(B_s)_0 = \xi_s B_s$. Plugging this into (3.24) and taking $\eta = m^{-1}\sum_{s=1}^{p} r_s\xi_s$, we still have $A_0 = m\eta A$. ■

Now we can prove Theorem 1.1. If $m = 2$, then $L = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric. So we always assume $m \ge 3$ in the following.

Proof of Theorem 1.1. By Proposition 3.2 and Lemma 3.3, there is a 1-form η on M^n such that $A_0 = m\eta A$. Plugging this into (3.4) yields

$$(\theta - \eta)(A_l\beta - mAb_l) = 0. \tag{3.31}$$

CASE 1: $\beta \neq 0$. We claim $\eta = \theta$. If $\eta \neq \theta$, by (3.31) we get

$$A_l\beta = mAb_l. \tag{3.32}$$

Because $\beta \neq 0$, we have $\beta \,|\, A$, i.e., there is a polynomial C_1 in (y^i) with $\deg(C_1) = m-1$ such that $A = \beta C_1$. Plugging this into (3.32) yields

$$(C_1)_l\beta = (m-1)C_1 b_l. \tag{3.33}$$

Then (3.33) implies $\beta \,|\, C_1$, i.e., there is a polynomial C_2 in (y^i) with $\deg(C_2) = m-2$ such that $C_1 = \beta C_2$. Denote $C_0 := A$. Thus there is a family of polynomials C_k in (y^i) satisfying $C_{k-1} = \beta C_k$ and $\deg(C_k) = m-k$, where $1 \le k \le m$. Then $A = C_m\beta^m$, where $\deg(C_m) = 0$, i.e., $C_m =$ constant. In this case, $L = \beta\sqrt[m]{C_m}$ is not a Finsler metric, a contradiction.

So $\eta = \theta$, and then

$$A_0 = m\theta A. \tag{3.34}$$

Plugging (3.34) into (3.3) yields

$$2A_{x^l} - A_{0l} = (2-m)\theta A_l. \tag{3.35}$$

Differentiating (3.34) with respect to y^l gives

$$A_{x^l} + A_{0l} = m\theta_l A + m\theta A_l. \tag{3.36}$$

Combining (3.35) and (3.36), we get (1.2). Moreover, (3.2) is just (1.3). Conversely, it follows from (1.2) that (3.3) and (3.4) hold. By Proposition 3.2, $\bar{L}$ is dually flat.

CASE 2: $\beta = 0$. It is obvious that (3.2) and (3.4) hold for any 1-form $\theta := \theta_i y^i$ on M^n. Letting $\eta = \theta$, we have $A_0 = m\theta A$. Then we get (1.2) from (3.35) and (3.36). Conversely, by $\beta = 0$, (3.2) and (3.4) hold. We can also easily obtain (3.3) from (1.2). Thus $\bar{L}$ is dually flat by Proposition 3.2. ■

Using (3.19), Theorem 1.1 and Corollary 1.2, one can deduce

COROLLARY 3.4. *Let $L(x,y) = \sqrt[m]{A}$ be a non-Riemannian mth root metric on a manifold M^n. Suppose that $\bar{L}(x,y) = L(x,y) + \beta(x,y)$ is a Finsler* *metric obtained by a Randers change of L, where $\beta(x,y) = b_i(x)y^i$ is a 1-form on M^n. Then $\bar{L}$ is dually flat if and only if L is dually flat and there is a 1-form $\theta = \theta_i y^i$ on M^n such that*

$$A_0\beta = m\beta_0 A, \qquad \frac{\partial b_i}{\partial x^j} = \tfrac{2}{3}\theta_i b_j + \tfrac{1}{3}\theta_j b_i.$$

4. The proof of Theorem 1.4. Observe that

$$d\beta = db_i \wedge dx^i = \frac{\partial b_i}{\partial x^j} dx^j \wedge dx^i = \frac{1}{2}\left\{\frac{\partial b_j}{\partial x^i} - \frac{\partial b_i}{\partial x^j}\right\} dx^i \wedge dx^j.$$

One has the following

THEOREM 4.1 ([HI]). *Let $\bar{L} = L + \beta(x,y)$ and L be a Finsler metric on manifold M^n, where $\beta := b_i(x)y^i$ is a 1-form. Then $\bar{L}$ is projectively equivalent to L if and only if β is closed.*

Proof of Theorem 1.4. Let $\bar{L} = L + \beta(x,y)$ be a projectively flat Finsler metric on M^n, where $L(x,y) = \sqrt[m]{A}$ is an mth root metric, $A = a_{i_1\dots i_m}(x)y^{i_1}\dots y^{i_m}$ and $\beta := b_i(x)y^i$ is a 1-form. By a direct computation,

$$\bar{L}_{x^l} = \frac{1}{m}A^{1/m-1}A_{x^l} + \beta_{x^l}, \tag{4.1}$$

$$\bar{L}_{x^k y^l}y^k = \frac{1}{m}\left(\frac{1}{m} - 1\right)A^{1/m-2}A_0 A_l + \frac{1}{m}A^{1/m-1}A_{0l} + \beta_{0l}. \tag{4.2}$$

By (2.3), (4.1) and (4.2), we have

$$\beta_{x^l} = \beta_{0l}, \tag{4.3}$$

$$(1-m)A_0 A_l = mA(A_{x^l} - A_{0l}). \tag{4.4}$$

Similarly, there is a 1-form η on M^n such that $A_0 = m\eta A$, as in the proof of Lemma 3.3. Then $L_{x^k}y^k = \eta L$.

It follows from (4.3) that

$$\frac{\partial b_j}{\partial x^i} - \frac{\partial b_i}{\partial x^j} = 0,$$

which implies that β is closed. By Theorem 4.1, L is projectively equivalent to $\bar{L}$. Thus L is projectively flat. Plugging $L_{x^k}y^k = \eta L$ into $P = L_{x^k}y^k/(2L)$ yields $P = \eta/2$. In this case, the spry coefficients $G^i = (\eta/2)y^i$ are quadratic in y, i.e., L is a Berwald metric. When $n = 2$, by Theorem 2.3, L is Riemannian with non-zero constant Gauss curvature or locally Minkowskian. When $n \geq 3$, by Theorems 2.1 and 2.2, L is Riemannian or locally Minkowskian. If L is a Riemannian metric, by the Beltrami theorem in Riemannian geometry, L is locally projectively flat if and only if it is of constant sectional curvature. To sum up, L is Riemannian with constant sectional curvature or locally Minkowskian.

The converse is obvious. ∎

5. Some results on Riemannian metrics. In this section, we provide some results on dually flat Riemannian metrics. When $m = 2$, $L = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric. By the definition of dually flat metrics, we have

THEOREM 5.1. *Let $L = \sqrt{A}$ be a Riemannian metric on M^n, where $A = a_{ij}(x)y^i y^j$. Then L is dually flat if and only if*

$$2A_{x^l} = A_{0l}. \tag{5.1}$$

Now we construct a dually flat Riemannian metric which does not satisfy (1.2).

EXAMPLE 5.2. Let $L = \sqrt{A}$ be a function on the tangent bundle TM^n, where $A = a_{ij}(x)y^i y^j = e^{\frac{1}{2}|x|^2}(|y|^2 + \langle x, y\rangle^2)$. Then

$$a_{ij}(x) = e^{\frac{1}{2}|x|^2}(\delta_{ij} + x^i x^j).$$

We have $\det(a_{ij}(x)) = e^{\frac{n}{2}|x|^2}(1 + |x|^2)\det(\delta_{ij}) = e^{\frac{n}{2}|x|^2}(1 + |x|^2)$. Thus L is a Riemannian metric on M^n. By a direct computation,

$$A_{x^l} = e^{\frac{1}{2}|x|^2}[(|y|^2 + \langle x, y\rangle^2)x^l + 2\langle x, y\rangle y^l],$$
$$A_0 = e^{\frac{1}{2}|x|^2}[(|y|^2 + \langle x, y\rangle^2) + 2|y|^2]\langle x, y\rangle,$$
$$A_{0l} = 2e^{\frac{1}{2}|x|^2}[(|y|^2 + \langle x, y\rangle^2)x^l + 2\langle x, y\rangle y^l].$$

Then $2A_{x^l} = A_{0l}$, and by Theorem 5.1, L is dually flat.

Combining the expressions of A and A_0, we have

$$A_0 = [A + 2|y|^2 e^{\frac{1}{2}|x|^2}]\langle x, y\rangle. \tag{5.2}$$

Noting that $|y|^2$ and $\langle x, y\rangle$ are relatively prime, by (5.2), one has $A \nmid A_0$. We claim L does not satisfy (1.2). If it does, contracting (1.2) with y^l, we get $A_0 = 2\theta A$, a contradiction. Thus the non-Riemannian condition in Corollary 1.2 cannot be dropped.

For any Riemannian metric $L = \sqrt{A}$ on M^n, where $A = a_{ij}(x)y^i y^j$, (1.2) is not a necessary and sufficient condition for L to be dually flat. If we suppose that (1.2) holds, contracting it with y^l, we get $A_0 = 2\theta A$. Differentiating this with respect to y^l yields

$$A_{x^l} + A_{0l} = 2(\theta_l A + \theta A_l). \tag{5.3}$$

Combining (1.2) and (5.3), one obtains

$$A_{0l} = \tfrac{4}{3}(\theta_l A + \theta A_l) = 2A_{x^l}.$$

By Theorem 5.1, L is dually flat. Thus we also have

COROLLARY 5.3. *Let $L = \sqrt{A}$ be a Riemannian metric on M^n, where $A = a_{ij}(x)y^i y^j$. If*

$$A_{x^l} = \tfrac{2}{3}(\theta_l A + \theta A_l), \tag{5.4}$$

then L is dually flat.

6. Some examples. The first example given in [Shen] of a non-Riemannian dually flat metric is the Funk metric on $B^n(1)$.

EXAMPLE 6.1. Let $\bar{L} = L + \beta$, where

$$L := \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y\rangle^2)}}{1 - |x|^2}, \qquad \beta := \frac{\langle x, y\rangle}{1 - |x|^2}.$$

The metric $\bar{L}$ is the *Funk metric* on the unit ball $B^n(1) \subset \mathbb{R}^n$. It is obvious that L is a Riemannian metric and

$$A = \frac{|y|^2 - (|x|^2|y|^2 - \langle x, y\rangle^2)}{(1 - |x|^2)^2}, \qquad b_i(x) = \frac{x_i}{1 - |x|^2}.$$

It is easy to see that β is closed and $\bar{L}\bar{L}_{y^l} = \bar{L}_{x^l}$. Then

$$(\bar{L}^2)_{x^l} = 2\bar{L}\bar{L}_{x^l} = 2\bar{L}^2\bar{L}_{y^l}, \tag{6.1}$$

$$(\bar{L}^2)_{x^k y^l} y^k = [2\bar{L}^2\bar{L}_{y^k}]_{y^l} y^k = 4\bar{L}^2\bar{L}_{y^l}. \tag{6.2}$$

Thus $\bar{L}$ is dually flat. By Theorem 5.1, L is not dually flat. In fact, by a direct computation,

$$2A_{x^l} - A_{0l} = 2\frac{|y|^2 x^l - \langle x, y\rangle y^l}{(1 - |x|^2)^2} \neq 0.$$

By Corollary 5.3, A does not satisfy (5.4), i.e., it does not satisfy (1.2). This implies that in the Riemannian case the conclusion of Theorem 1.1 does not hold.

It is well-known that L in Example 6.1 is a Riemannian metric with constant sectional curvature $\mathbf{K} = -1$. Note that β is closed, thus the Funk metric is also projectively flat.

EXAMPLE 6.2. Let $\bar{L} = L + \beta$, where

$$L := \frac{\sqrt{|y|^2 + \mu(|x|^2|y|^2 - \langle x, y\rangle^2)}}{1 + \mu|x|^2}, \qquad \beta := \frac{\sqrt{-\mu}\,\langle x, y\rangle}{1 + \mu|x|^2}$$

and $\mu < 0$ is a constant. The metric L is a Riemannian metric with constant sectional curvature $\mathbf{K} = \mu$. Further,

$$b_i(x) = \frac{\sqrt{-\mu}\,x_i}{1 + \mu|x|^2}.$$

Since β is closed, $\bar{L}$ is projectively flat. This example satisfies the conditions of Theorem 1.4.

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