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On a certain additive function on the Gaussian integers

by

PIOTR ZARZYCKI (Gdańsk)

1. Introduction. In [1] K. Alladi and P. Erdös showed that if

$$\beta_1(n) = \sum_{p|n} p,$$

then

$$\sum_{n\leqslant x} \beta_1(n) \sim \frac{\pi^2 x^2}{12\log x}.$$

Numerous papers have been written concerning the summatory function of $\beta_1(n)$, see for example [2], [3]. The best result is due to J.-M. De Koninck and A. Ivić [2]; they have given the asymptotic formula

$$(1.1)\quad \sum_{n\leqslant x} \beta_1(n) = x^2\left[(d_1/\log x)+(d_2/\log^2 x)+\ldots+(d_m/\log^m x)+O(1/\log^{m+1} x)\right]$$

with arbitrary fixed $m \geqslant 1$ and $d_1 = \pi^2/12$.

Applying their elementary technique to the function

$$(1.2)\quad \beta_\alpha(n) = \sum_{p|n} p^\alpha$$

with $\alpha > 0$ fixed, one can prove the analogous asymptotic formula

$$(1.3)\quad \sum_{n\leqslant x} \beta_\alpha(n) = x^2\left[(d_1(\alpha)/\log x)+(d_2(\alpha)/\log^2 x)+\ldots+(d_m(\alpha)/\log^m x)+O(1/\log^{m+1} x)\right]$$

with $d_1(\alpha) = \zeta(1+\alpha)/(1+\alpha)$.

The formula (1.3) can be obtained by using the complex integration technique and the following

LEMMA. *Let a Dirichlet series* $\sum_{n=1}^{\infty} a_n/n^s$ *converge for* $\mathrm{Re}\, s > 1$, *where* $|a_n|$

$= O((\log n)^k)$ *with* $k > 0$. *If* $\sum_{n=1}^{\infty} a_n/n^s = G(s)\log\zeta(s)$, *where* $G(s)$ *is a regular function for* $\mathrm{Re}\, s > 1/2$ *and bounded for* $\mathrm{Re}\, s \geqslant 1/2+\varepsilon$ *with* $\varepsilon > 0$, *then*

$$(1.4)\qquad \sum_{n\leqslant x} a_n = \frac{x}{\log x}\left[a_0' + a_1'(1/\log x) + \ldots + a_m'(1/\log^m x) + O(1/\log^{m+1} x)\right].$$

This lemma can be proved in a standard way with the use of the estimation of the zero-free region for $\zeta(s)$.

By applying the main theorem from Ramachandra's paper [10] it is possible to prove that

$$(1.5)\qquad \sum_{x<n\leqslant x+h} \beta_\alpha(n) = \zeta(1+\alpha)\frac{hx^\alpha}{\log x} + O\left(\frac{h^2 x^{\alpha-1}}{\log x}\right) + O\left(\frac{hx^\alpha}{\log^2 x}\right) + O\left(hx^\alpha \exp(-\log^{1/3} x)\right) + O\left(x^{(7/12)+\varepsilon+\alpha}\right)$$

with $x^{(7/12)+\varepsilon} \leqslant h \leqslant o(x)$, and

$$(1.6)\qquad \frac{1}{X}\int_X^{2X}\left|\sum_{x<n\leqslant x+h}\beta_\alpha(n) - \zeta(1+\alpha)\frac{hx^\alpha}{\log x}\right|^2 dx = O\left(h^2 X^{2\alpha}\exp(-\log^{1/3} X)\right) + O\left(X^{2((1/6)+\varepsilon+\alpha)}\right)$$

with $X^{(1/6)+\varepsilon} \leqslant h \leqslant o(X)$ (the exponents 7/12 and 1/6 are related to the estimation of $N_\chi(a, T)$, the number of zeros of $L(s, \chi)$ with real part at least a and imaginary part not exceeding T in absolute value).

The formulae (1.5), (1.6) can be obtained by De Koninck and Ivić's method. It follows from Ivić's result on the number of primes in short intervals (see [5]) that the formula (1.5) holds even for $h \gg x^{7/12}\log^{22} x$.

Let us note that the formulae (1.3), (1.5) remain valid if $\beta_\alpha(n)$ is replaced by the functions

$$B_\alpha(n) = \sum_{p^k \| n} kp^\alpha, \qquad T_\alpha(n) = (P(n))^\alpha,$$

where $P(n)$ is the largest prime factor of $n \geqslant 2$.

In the present paper we study the distribution of values of an additive function $\mathscr{B}_\alpha(\mathfrak{a})$ on the Gaussian integers given by

$$(1.7)\qquad \mathscr{B}_\alpha(\mathfrak{a}) = \sum_{\mathfrak{p}|\mathfrak{a}}{}^{*} N^\alpha(\mathfrak{p})$$

with fixed $\alpha \geqslant 0$; the asterisk means that the summation is over the non-associated prime divisors $\mathfrak{p}$ of a Gaussian integer $\mathfrak{a}$, and $N(\mathfrak{p}) = N(x+iy) = x^2+y^2$ is the norm of $\mathfrak{p}$. This function is a generalization of the function $\beta_\alpha(n)$. In case $\alpha = 0$ we get the function $\omega(\mathfrak{a})$ which has been studied in [6].

In this note we obtain asymptotic formulae for the summatory function $\sum_{\mathfrak{a}\in D}^{*} \mathscr{B}_\alpha(\mathfrak{a})$ $(\alpha > 0)$, where D is a certain set of Gaussian integers which depends on a parameter x.

We shall prove the following theorems:

THEOREM 1. *For* $x \to \infty$

$$(1.8)\qquad \sum_{N(\mathfrak{a})\leqslant x}^{*} \mathscr{B}_\alpha(\mathfrak{a}) = \frac{\zeta(1+\alpha)L(1+\alpha, \chi_4)}{1+\alpha}\frac{x^{1+\alpha}}{\log x}\left[1 + O\left(\frac{1}{\log x}\right)\right].$$

THEOREM 2. *Let* φ_1, φ_2 *be real numbers such that* $0 \leqslant \varphi_1 < \varphi_2 \leqslant \pi/2$. *If* $\varphi_2 - \varphi_1 \gg \exp(-c_1 \log^{(3/5)-\varepsilon} x)$, $\varepsilon > 0$, *and* $x \geqslant x_0(\alpha)$, *then*

$$(1.9)\qquad \sum_{\substack{N(\mathfrak{a})\leqslant x\\ \varphi_1\leqslant \arg\mathfrak{a}\leqslant\varphi_2}}^{*} \mathscr{B}_\alpha(\mathfrak{a}) = \frac{2(\varphi_2-\varphi_1)}{\pi}\frac{\zeta(1+\alpha)L(1+\alpha, \chi_4)}{1+\alpha}\frac{x^{1+\alpha}}{\log x} \times\left[1 + O\left(\frac{1}{\log x}\right)\right] + O\left(x^{1+\alpha}\exp(-c_1\log^{(3/5)-\varepsilon} x)\right).$$

THEOREM 3. *Let* x, X *be sufficiently large and let* $1 \leqslant h \leqslant x$. *Let* φ_1, φ_2 *be real numbers such that*

$$0 \leqslant \varphi_1 < \varphi_2 \leqslant \pi/2, \qquad \varphi_2 - \varphi_1 \gg \exp\left(-c(\log^{1/3} x)(\log\log x)^{-1}\right).$$

Then

$$(1.10)\qquad \sum_{\substack{x<N(\mathfrak{a})\leqslant x+h\\ \varphi_1\leqslant \arg\mathfrak{a}\leqslant\varphi_2}}^{*} \mathscr{B}_\alpha(\mathfrak{a}) = \frac{2(\varphi_2-\varphi_1)}{\pi}\zeta(1+\alpha)L(1+\alpha, \chi_4)\frac{hx^\alpha}{\log x} + O\left(\frac{h^2x^{\alpha-1}}{\log x}\right) + O\left(\frac{hx^\alpha}{\log^2 x}\right) + O\left(x^{\beta+\alpha}\right) + O\left(hx^\alpha \exp\left(-c(\log^{1/3} x)(\log\log x)^{-1}\right)\right),$$

$$(1.11)\qquad \frac{1}{X}\int_X^{2X}\left|\sum_{\substack{x<N(\mathfrak{a})\leqslant x+h\\ \varphi_1\leqslant \arg\mathfrak{a}\leqslant\varphi_2}}^{*} \mathscr{B}_\alpha(\mathfrak{a}) - \frac{2(\varphi_2-\varphi_1)}{\pi}\zeta(1+\alpha)L(1+\alpha, \chi_4)\frac{hx^\alpha}{\log x}\right|^2 dx = O\left(h^2X^{2\alpha}\exp\left(-c(\log^{1/3} X)(\log\log X)^{-1}\right)\right) + O\left(X^{2(\beta'+\alpha)}\right),$$

where the constants β, β' *are defined in Lemma* 5.

(χ_4 in Theorems 1, 2 and 3 denotes the non-principal Dirichlet character modulo 4.)

2. Auxiliary results.

LEMMA 1. *There exists an absolute constant $c > 0$ such that $\zeta(s)$ and $L(s, \chi_4)$ have no zeros whenever*

$$\mathrm{Re}\, s \geqslant 1-(c/\delta), \quad |\mathrm{Im}\, s| \leqslant T, \quad \delta = (\log^{2/3} T)(\log\log T)^{1/3}.$$

The analogous result holds for the Hecke L-function

$$Z(s, m) = \sum_{\mathfrak{a}}{}^{*} \exp(4mi \arg \mathfrak{a})/N^s(\mathfrak{a}),$$

where $m \in \mathbf{Z}$ is fixed and the summation is over all non-zero non-associated Gaussian integers; in this case

$$\delta = (\log^{2/3}(T^2+m^2))\log\log(T^2+m^2).$$

For the proof for $\zeta(s)$ and $L(s, \chi_4)$, see [8], and for $Z(s, m)$, see [4].

LEMMA 2. *There exists an absolute constant $a > 0$ such that if $1/2 \leqslant \mathrm{Re}\, s = \sigma \leqslant 1$, $2 \leqslant |\mathrm{Im}\, s| \leqslant T$, then*

$$\zeta(s),\ L(s, \chi_4) = O(T^{a(1-\sigma)^{3/2}} \log T), \tag{2.1}$$

$$Z(s, m) = O\big((T^2+m^2)^{a(1-\sigma)^{3/2}} \log^4(T^2+m^2)\big). \tag{2.2}$$

For the proof of (2.1), see [11], and for (2.2), see [4].

LEMMA 3. *Let $F(s) = \sum_{n=1}^{\infty} a_n/n^s$ converge absolutely for $\mathrm{Re}\, s > 1$. Let $F(s)$ be regular and non-zero in a rectangle $1-\delta \leqslant \mathrm{Re}\, s \leqslant 2$, $|\mathrm{Im}\, s| \leqslant T$, except possibly at $s = 1$, where $F(s)$ has a pole of k-th order and $F(s)(s-1)^k = a_0 + O(|s-1|)$ with an absolute constant in the O-term. Moreover, let for $1-\delta \leqslant \mathrm{Re}\, s \leqslant 2$, $3 \leqslant |\mathrm{Im}\, s| \leqslant T$*

$$F(s) = O\big((1/\delta)^l \log T\big)$$

and

$$F(1+iT),\ F^{-1}(1+iT),\ F'(1+iT) = O\big((1/\delta)^b + \log^b T\big)$$

with some constants $l > 0$, $b > 0$.

Then

$$\log F(s) = O\big(\log(1/\delta) + \log\log T\big)$$

provided that $1-(\delta/5) \leqslant \mathrm{Re}\, s \leqslant 2$, $|\mathrm{Im}\, s| \leqslant T$, $|s-1| \geqslant \log^{-1} T$.

Proof. We start with $|\mathrm{Im}\, s| \leqslant 3$. For

$$1-(\delta/2) \leqslant \mathrm{Re}\, s \leqslant 2, \qquad |\mathrm{Im}\, s| \leqslant 3$$

we have

$$\mathrm{Re}\log\big(F(s)(s-1)^k\big) = \log|F(s)(s-1)^k| = \log|a_0| + O(|s-1|) = O(1).$$

Therefore, by the Borel–Carathéodory theorem (see [12])

$$|\log F(s)| = O\big((\log(s-1))^l\big) = O(\log\log T).$$

Now, let $|\mathrm{Im}\, s| \geqslant 3$ and $s = \sigma + it$, where $1-(\delta/5) \leqslant \sigma \leqslant 2$, $3 \leqslant |t| \leqslant T$. Let us take $s_0 = 1+it$. Then for every s such that $1-\delta \leqslant \mathrm{Re}\, s \leqslant 2$, $3 \leqslant |\mathrm{Im}\, s| \leqslant T$, we get

$$|F(s)/F(s_0)| = O\big((1/\delta)^{l+b} \log^{1+b} T\big),$$

$$|F'(s_0)/F(s_0)| = O\big((1/\delta)^{2b} + \log^{2b} T\big).$$

Therefore, by Landau's theorem (see [9], supplement), if $1-(\delta/5) \leqslant \mathrm{Re}\, s \leqslant 2$, $3 \leqslant |\mathrm{Im}\, s| \leqslant T$, then

$$|F'(s)/F(s)| \leqslant (C/\delta)\big[\log(1/\delta)^{l+b} + \log(\log T)^{1+b}\big].$$

Since

$$\big|\log|F(\sigma+it)| - \log|F(1+it)|\big| = \left|\int_{\sigma}^{1} \frac{F'(\eta+it)}{F(\eta+it)}\, d\eta\right| \leqslant \int_{\sigma}^{1} \left|\frac{F'(\eta+it)}{F(\eta+it)}\right| d\eta$$

$$\leqslant \delta \frac{C}{\delta}\big[(l+b)\log(1/\delta) + (1+b)\log\log T\big]$$

$$= O\big(\log(1/\delta) + \log\log T\big),$$

we have

$$|\log F(s)| = O\big(\log(1/\delta) + \log\log T\big).$$

Finally, by the Borel–Carathéodory theorem we get

$$\log F(s) = O\big(\log(1/\delta) + \log\log T\big).$$

COROLLARY 1. *If $\mathrm{Re}\, s \geqslant 1 - \dfrac{c}{\log^{(2/3)+\varepsilon} T}$, $|\mathrm{Im}\, s| \leqslant T$, $|s-1| \geqslant \log^{-1} T$ $(T \geqslant 2)$, then*

$$\log \zeta(s) = O(\log\log T),$$

$$\log L(s, \chi_4) = O(\log\log T).$$

Proof. In fact, $\zeta(s)$ and $L(s, \chi_4)$ have no zeros in the considered region, so by (2.1)

$$\zeta(s),\ L(s, \chi_4) = O(T^{a(1-\sigma)^{3/2}} \log^4 T) = O(\log^4 T).$$

Moreover, it is known that

$$\zeta'(1+iT),\ L'(1+iT, \chi_4),\ \zeta^{\pm 1}(1+iT),\ L^{\pm 1}(1+iT, \chi_4) = O(\log^{10} T).$$

Hence, we can use Lemma 3 with $\delta = c/\log^{(2/3)+\varepsilon} T$.

COROLLARY 2. *If* $\operatorname{Re} s \geqslant 1 - \dfrac{c}{\log^{(2/3)+\varepsilon}(T^2+m^2)}$, $|\operatorname{Im} s| \leqslant T$, *then for* $m \neq 0$

$$\log Z(s, m) = O\big(\log\log(T^2+m^2)\big) = O\big(\log\log(T|m|)\big).$$

This corollary follows from Lemma 3 with

$$\delta = \log^{(2/3)+\varepsilon}(T^2+m^2).$$

LEMMA 4. *Let* r *be a positive integer and let* $0 < \Delta < \pi/4$. *Let* φ_1, φ_2 *be real numbers such that* $0 \leqslant \varphi_2 - \varphi_1 \leqslant \pi/2 + 2\Delta$. *There exists a periodic function* $f(\varphi)$ *with period* $\pi/2$ *such that*

1. $f(\varphi) = 1$ *for* $\varphi \in [\varphi_1 + \Delta, \varphi_2 - \Delta]$,
   $0 \leqslant f(\varphi) \leqslant 1$ *for* $\varphi \in [\varphi_1 - \Delta, \varphi_2 + \Delta]$,
   $f(\varphi) = 0$ *for other points from* $[0, \pi/2]$,
2. $f(\varphi)$ *has the following Fourier-series expansion*

$$f(\varphi) = \sum_{m=-\infty}^{+\infty} a_m \exp(4mi\varphi),$$

where for $m \neq 0$

$$|a_m| \leqslant \begin{cases} \dfrac{2}{\pi}(\varphi_2 - \varphi_1 + \Delta), \\ 2(\pi|m|)^{-1}, \\ 2(\pi|m|)^{-1}\left(r\dfrac{\pi}{2}(\pi|m|\Delta)^{-1}\right)^r. \end{cases}$$

This is a modified version of the famous lemma of I. M. Vinogradov ([13], Lemma 2, p. 23, see also [7], Lemma 5).

LEMMA 5 (analogue of the Ramachandra theorem). *Let for* $\operatorname{Re} s > 1$ *each of the series*

$$F_m(s) = \sum_{\mathfrak{a}}{}^{*} \frac{a(\mathfrak{a})}{N^s(\mathfrak{a})} \exp(4mi \arg \mathfrak{a}), \quad m = 0, \pm 1, \pm 2, \ldots,$$

be representable in the form

$$F_m(s) = \big(Z(s, m)\big)^z \mathscr{A}_m(s, z) \log Z(s, m),$$

where $z \in Q(i)$, $|z| < 2$, z *does not depend on* m, *and* $\mathscr{A}_m(s, z)$ *is representable by a Dirichlet series absolutely convergent for* $\operatorname{Re} s > 1/2$. *Let* $N_m(\sigma, T)$ *be the number of zeros of* $Z(s, m)$ *in the rectangle* $\sigma \leqslant \operatorname{Re} s \leqslant 1$, $|\operatorname{Im} s| \leqslant T$, *and let* $\mathscr{B}_0, \mathscr{B}, \mathscr{D}_0, \mathscr{D}$ *be constants independent of* m *such that*

$$N_m(\sigma, T) \ll (TM)^{\mathscr{B}(1-\sigma)}(\log TM)^{\mathscr{D}}, \quad m \neq 0,\ M = |m| + 2,$$

$$N_0(\sigma, T) \ll T^{\mathscr{B}_0(1-\sigma)}(\log T)^{\mathscr{D}_0}.$$

Let

$$\beta_0 = 1 - \mathscr{B}_0^{-1} + \varepsilon, \quad \beta_0' = 1 - 2\mathscr{B}_0^{-1} + \varepsilon,$$
$$\beta = 1 - \mathscr{B}^{-1} + \varepsilon, \quad \beta' = 1 - 2\mathscr{B}^{-1} + \varepsilon,$$

where $\varepsilon > 0$ *is arbitrary.*

If

$$S(x, h; \varphi_1, \varphi_2; z) = \sum_{\substack{x < N(\mathfrak{a}) \leqslant x+h \\ \varphi_1 \leqslant \arg \mathfrak{a} \leqslant \varphi_2}}{}^{*} a(\mathfrak{a}),$$

$$I(x, h; z) = \frac{1}{2\pi i}\int_0^h \Big(\int_{C_0(r)} F_0(s)(x+v)^{s-1}\, ds\Big)\, dv$$

with

$$r = c(\log^{-2/5} x)(\log\log x)^{-1},$$

then for $0 \leqslant \varphi_1 < \varphi_2 \leqslant \pi/2$, $\varphi_2 - \varphi_1 \gg \exp\big(-c(\log^{1/3} x)(\log\log x)^{-1}\big)$,

$$\text{(2.3)} \quad S(x, h; \varphi_1, \varphi_2; z) = \frac{2(\varphi_2 - \varphi_1)}{\pi} I(x, h; z) + O\Big(h\exp\big(-c(\log^{1/3} x)(\log\log x)^{-1}\big)\Big) + O(x^{\beta}),$$

$$\text{(2.4)} \quad \frac{1}{X}\int_X^{2X} \left|S(x, h; \varphi_1, \varphi_2; z) - \frac{2(\varphi_2 - \varphi_1)}{\pi} I(x, h; z)\right|^2 dx = O\Big(h^2\exp\big(-c(\log^{1/3} X)(\log\log X)^{-1}\big)\Big) + O(X^{2\beta'}),$$

where $C_0(r)$ *is a positively oriented circle of radius* r *centred at* $s = 1$, *with* $s = 1 - r$ *removed.*

For the proof see [4], and for the Ramachandra theorem, see [10].

3. Proof of Theorem 1. Notice that

$$\mathscr{B}_\alpha(\mathfrak{a}) = \sum_{\mathfrak{p}|\mathfrak{a}}{}^{*} N^\alpha(\mathfrak{p}) = \sum_{\mathfrak{d}|\mathfrak{a}}{}^{*} b(\mathfrak{d}) \cdot c(\mathfrak{a}/\mathfrak{d}),$$

where

$$b(\mathfrak{d}) = \begin{cases} N^\alpha(\mathfrak{d}) & \text{if } \mathfrak{d} \text{ is prime,} \\ 0 & \text{otherwise,} \end{cases} \qquad c(\mathfrak{d}) \equiv 1.$$

Hence

$$\sum_{\mathfrak{a}}{}^{*} \mathscr{B}_\alpha(\mathfrak{a})/N^s(\mathfrak{a}) = \Big(\sum_{\mathfrak{a}}{}^{*} b(\mathfrak{a})/N^s(\mathfrak{a})\Big)\Big(\sum_{\mathfrak{a}}{}^{*} c(\mathfrak{a})/N^s(\mathfrak{a})\Big) = \Big(\sum_{\mathfrak{p}}{}^{*} 1/N^{s-\alpha}(\mathfrak{p})\Big)\Big(\sum_{\mathfrak{a}}{}^{*} 1/N^s(\mathfrak{a})\Big).$$

It follows from the Euler identity for $Z(s, m)$,

$$Z(s, m) = \sum_{\mathfrak{a}}^{*} \frac{\exp(4mi \arg \mathfrak{a})}{N^s(\mathfrak{a})} = \prod_{\mathfrak{p}}^{*} \left(1 - \frac{\exp(4mi \arg \mathfrak{p})}{N^s(\mathfrak{p})}\right)^{-1}$$

that

$$\log Z(s, m) = \sum_{\mathfrak{p}}^{*} \frac{\exp(4mi \arg \mathfrak{p})}{N^s(\mathfrak{p})} + G(s, m),$$

where $G(s, m)$ is a regular function for $\operatorname{Re} s > 1/2$.

Hence, by $Z(s, 0) = \zeta(s) L(s, \chi_4)$ we have

$$\sum_{\mathfrak{a}}^{*} \mathscr{B}_\alpha(\mathfrak{a})/N^s(\mathfrak{a}) = \zeta(s) L(s, \chi_4) [\log \zeta(s-\alpha) + \log L(s-\alpha, \chi_4) + G(s-\alpha)],$$

$$\sum_{\mathfrak{a}}^{*} \mathscr{B}_\alpha(\mathfrak{a})/N^{s+\alpha}(\mathfrak{a}) = \zeta(s+\alpha) L(s+\alpha, \chi_4) [\log \zeta(s) + \log L(s, \chi_4) + G(s)].$$

If we put

$$G_1(s) = \zeta(s+\alpha) L(s+\alpha, \chi_4) \log \zeta(s),$$

$$G_2(s) = \zeta(s+\alpha) L(s+\alpha, \chi_4) \log L(s, \chi_4),$$

$$G_3(s) = \zeta(s+\alpha) L(s+\alpha, \chi_4) G(s),$$

it follows that

(3.1) $$\sum_{\mathfrak{a}}^{*} \frac{\mathscr{B}_\alpha(\mathfrak{a})/N^\alpha(\mathfrak{a})}{N^s(\mathfrak{a})} = G_1(s) + G_2(s) + G_3(s) = F(s).$$

We shall use

$$\frac{1}{2\pi i} \int_{b-iT}^{b+iT} (y^s/s)\, ds = \begin{cases} 1 + O(y^b/T \log y) & \text{if } y > 1, \\ \frac{1}{2} + O(1/T) & \text{if } y = 1, \\ O(y^b/T|\log y|) & \text{if } 0 < y < 1, \end{cases}$$

where $b > 1$, $T > 1$.

By the uniform convergence of the series (3.1) in the half-plane $\operatorname{Re} s > b$ we get

$$\frac{1}{2\pi i} \int_{b-iT}^{b+iT} \sum_{\mathfrak{a}}^{*} \frac{\mathscr{B}_\alpha(\mathfrak{a})/N^\alpha(\mathfrak{a})}{N^s(\mathfrak{a})} \frac{x^s}{s} ds$$

$$= \sum_{N(\mathfrak{a}) \leqslant x}^{*} \mathscr{B}_\alpha(\mathfrak{a})/N^\alpha(\mathfrak{a}) + \sum_{N(\mathfrak{a}) \neq x}^{*} \frac{\left(\mathscr{B}_\alpha(\mathfrak{a})/N^\alpha(\mathfrak{a})\right)\left(x/N(\mathfrak{a})\right)^b}{T\left|\log\left(x/N(\mathfrak{a})\right)\right|}$$

$$= \sum_{N(\mathfrak{a}) \leqslant x}^{*} \mathscr{B}_\alpha(\mathfrak{a})/N^\alpha(\mathfrak{a}) + \sum_{N(\mathfrak{a}) < x/2}^{*} + \sum_{N(\mathfrak{a}) > 2x}^{*} + \sum_{\substack{x/2 \leqslant N(\mathfrak{a}) \leqslant 2x \\ N(\mathfrak{a}) \neq x}}^{*}$$

$$= \sum_{N(\mathfrak{a}) \leqslant x}^{*} \mathscr{B}_\alpha(\mathfrak{a})/N^\alpha(\mathfrak{a}) + O\left(\frac{x^b}{T} \sum_{\mathfrak{a}}^{*} \frac{\mathscr{B}_\alpha(\mathfrak{a})/N^\alpha(\mathfrak{a})}{N^b(\mathfrak{a})}\right)$$

$$+ O\left(\frac{1}{T} \sum_{\substack{x/2 \leqslant N(\mathfrak{a}) \leqslant 2x \\ N(\mathfrak{a}) \neq x}}^{*} \frac{\mathscr{B}_\alpha(\mathfrak{a})/N^\alpha(\mathfrak{a})}{\left|\log\left(x/N(\mathfrak{a})\right)\right|}\right).$$

Let

$$\sum_{\mathfrak{a}}^{*} \frac{\mathscr{B}_\alpha(\mathfrak{a})/N^\alpha(\mathfrak{a})}{N^b(\mathfrak{a})} = \sum_{n=1}^{\infty} e(n)/n^b,$$

where

$$e(n) = \sum_{\substack{\mathfrak{a} \\ N(\mathfrak{a}) = n}}^{*} 1/n^\alpha \sum_{\mathfrak{p} | \mathfrak{a}}^{*} N^\alpha(\mathfrak{p}) = (1/n^\alpha) \sum_{\substack{\mathfrak{a} \\ N(\mathfrak{a}) = n}}^{*} \sum_{\mathfrak{p} | \mathfrak{a}}^{*} N^\alpha(\mathfrak{p}).$$

To estimate $e(n)$ notice that if

$$n = 2^{a_0} p_1^{a_1} \ldots p_k^{a_k} q_1^{2b_1} \ldots q_m^{2b_m} = n_1 2^{a_0} q_1^{2b_1} \ldots q_m^{2b_m},$$

where p_i is a prime number of the form $4l+1$ and q_j is a prime number of the form $4l+3$, then

$$e(n) = \frac{1}{n^\alpha} \sum_{\substack{\mathfrak{a} \\ N(\mathfrak{a}) = n}}^{*} \left(\sum_{i=1}^{k} p_i^\alpha + \sum_{j=1}^{m} q_j^{2\alpha} + 2^\alpha\right).$$

Since the number of n for which $N(\mathfrak{a}) = n$ does not exceed $\tau(n_1) = (a_1+1) \ldots (a_k+1)$ we get

$$e(n) \leqslant \frac{1}{n^\alpha}(a_1+1) \ldots (a_k+1)\left(\sum_{i=1}^{k} p_i^\alpha + \sum_{j=1}^{m} q_j^{2\alpha}\right)$$

$$\leqslant \left(2^k/(p_1 \ldots p_k q_1^2 \ldots q_m^2)^\alpha\right)\left(\sum_{i=1}^{k} p_i^\alpha + \sum_{j=1}^{m} q_j^{2\alpha}\right)$$

$$\leqslant 2^k \sum_{i=1}^{k} 1/(p_1 \ldots p_{i-1} p_{i+1} \ldots p_k)^\alpha \leqslant 2^k k/[(k-1)!]^\alpha = c(\alpha).$$

Thus

$$\sum_{\mathfrak{a}}^{*} \frac{\mathscr{B}_\alpha(\mathfrak{a})/N^\alpha(\mathfrak{a})}{N^b(\mathfrak{a})} \leqslant c(\alpha) \sum_{n=1}^{\infty} 1/n^b = O_\alpha(1/(b-1)),$$

$$\sum_{\substack{x/2 \leqslant N(\mathfrak{a}) \leqslant 2x \\ N(\mathfrak{a}) \neq x}}^{*} \frac{\mathscr{B}_\alpha(\mathfrak{a})/N^\alpha(\mathfrak{a})}{\left|\log\left(x/N(\mathfrak{a})\right)\right|} = \sum_{\substack{x/2 \leqslant n \leqslant 2x \\ n \neq x}} e(n)/|\log(x/n)|$$

$$\leqslant c(\alpha) \sum_{\substack{x/2 \leqslant n \leqslant 2x \\ n \neq x}} 1/|\log(x/n)|$$

$$= c(\alpha)\Big[\sum_{x/2 \leqslant n < x} 1/\log(x/n) + \sum_{x < n \leqslant 2x} 1/|\log(x/n)|\Big]$$

$$= O_\alpha(x \log 2x) + O_\alpha(x/\|x\|) = O_\alpha(x \log 2x).$$

($\|x\|$ is the distance from x to the nearest integer, we can assume without loss of generality that x is an integer plus one-half.)

Therefore

$$\sum_{N(\mathfrak{a}) \leqslant x}^{*} \mathscr{B}_\alpha(\mathfrak{a})/N^\alpha(\mathfrak{a})$$

$$= \frac{1}{2\pi i} \int_{b-iT}^{b+iT} F(s) \frac{x^s}{s} ds + O_\alpha\big(x^b/T(b-1)\big) + O_\alpha(x \log 2x/T).$$

To estimate the integral

$$\frac{1}{2\pi i} \int_{b-iT}^{b+iT} G_2(s) \frac{x^s}{s} ds$$

we use Lemma 3. If we move the segment of integration to $\mathrm{Re}\, s = 1-\delta$ we get

$$(3.2)\qquad \frac{1}{2\pi i} \int_{b-iT}^{b+iT} G_2(s) \frac{x^s}{s} ds = O\big(x^{1-\delta}(\log(1/\delta) + \log\log T)\log T\big)$$

$$+ O\left(x^b \left(\frac{b-1+\delta}{T}\right)(\log(1/\delta) + \log\log T)\right).$$

By moving the segment of integration in

$$\frac{1}{2\pi i} \int_{b-iT}^{b+iT} G_3(s) \frac{x^s}{s} ds$$

to $\mathrm{Re}\, s = 1-(\alpha/2)$ we obtain

$$(3.3)\qquad \frac{1}{2\pi i} \int_{b-iT}^{b+iT} G_3(s) \frac{x^s}{s} ds = O_\alpha(x^{1-(\alpha/2)} \log T) + O_\alpha(x^b/T).$$

To deal with the integral

$$\frac{1}{2\pi i} \int_{b-iT}^{b+iT} G_1(s) \frac{x^s}{s} ds$$

we move the integral to the contour L consisting of:

L_1: the line segment $[1-\delta, 1-\delta+iT]$,
L_2: the line segment $[1-\delta+iT, b+iT]$,
L_3: the line segment $[b-iT, 1-\delta-iT]$,

L_4: the line segment $[1-\delta-iT, 1-\delta]$,
L_5, L_6: the lower and upper edges of the cut in the complex plane along the line segment $[1-\delta, 1-\varrho]$,
C_ϱ: the positively oriented circle of radius ϱ centred at $s=1$, with $s = 1-\varrho$ removed.

The contour L is shown in Fig. 1.

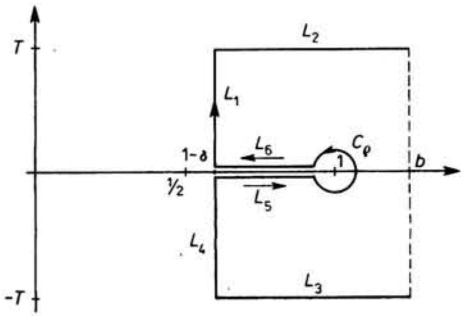

Fig. 1

We first note that

$$\int_{C_\varrho} G_1(s) \frac{x^s}{s} ds \to 0 \quad \text{if } \varrho \to 0.$$

Further, on the contour $L_5 \cup L_6$

$$G_1(s)\frac{x^s}{s} = \zeta(s+\alpha) L(s+\alpha, \chi_4)\big(\log\zeta(s)\big)\frac{x^s}{s}$$

$$= \zeta(s+\alpha) L(s+\alpha, \chi_4)\frac{x^s}{s}\big[\log\zeta(s)(s-1) - \log(s-1)\big]$$

$$= \zeta(s+\alpha) L(s+\alpha, \chi_4)\frac{x^s}{s}\log(s-1)$$

$$+ O\big(\zeta(s+\alpha) L(s+\alpha, \chi_4) x^s (s-1)/s\big),$$

where

$$\log(s-1) = \begin{cases} \log|s-1| + i\pi & \text{if } s \in L_6, \\ \log|s-1| - i\pi & \text{if } s \in L_5. \end{cases}$$

Let $b = 1 + (1/\log x)$, $T = \exp\big(c(\log^{3/5} x)\big)$, $\delta = c\log^{-(2/3)-\varepsilon} T$. We use the bounds of $\zeta(s)$, $L(s, \chi_4)$ (Lemma 2), $\log\zeta(s)$ (Corollary 1). By Cauchy's residue theorem we get

$$(3.4)\qquad \frac{1}{2\pi i} \int_{b-iT}^{b+iT} G_1(s) \frac{x^s}{s} ds = \zeta(1+\alpha) L(1+\alpha, \chi_4) \frac{x}{\log x}\left[1 + O\left(\frac{1}{\log x}\right)\right]$$

(note that computing $\int_{L_5 \cup L_6} G_1(s)\frac{x^s}{s}ds$ can be reduced to computing $\int_{1-\delta}^{1} x^s(1-s)^k ds$; but we have

$$\int_{1-\delta}^{1} x^s(1-s)^k ds = \frac{x}{\log x}[b_0 + b_1(1/\log x) + \ldots + b_m(1/\log^m x) + O(1/\log^{m+1} x)]$$

with arbitrary fixed $m \geq 1$ and with computable $b_0, b_i, b_i', b_0 \neq 0$).

It follows from (3.2), (3.3), (3.4) that

$$\sum_{N(\mathfrak{a}) \leq x}^{*} \mathcal{B}_\alpha(\mathfrak{a})/N^\alpha(\mathfrak{a}) = \zeta(1+\alpha)L(1+\alpha, \chi_4)\frac{x}{\log x}\left[1 + O\left(\frac{1}{\log x}\right)\right]$$

where the constant in the O-term depends only on α.

By the Abel lemma on partial summation we obtain

$$\sum_{N(\mathfrak{a}) \leq x}^{*} \mathcal{B}_\alpha(\mathfrak{a}) = \frac{\zeta(1+\alpha)L(1+\alpha, \chi_4)}{1+\alpha}\frac{x^{1+\alpha}}{\log x}[1 + O(1/\log x)].$$

This completes the proof of Theorem 1.

4. Proof of Theorem 2. Notice that

$$S = \sum_{\substack{N(\mathfrak{a}) \leq x \\ \varphi_1 \leq \arg \mathfrak{a} \leq \varphi_2}}^{*} \mathcal{B}_\alpha(\mathfrak{a}) = \sum_{N(\mathfrak{a}) \leq x}^{*} \mathcal{B}_\alpha(\mathfrak{a})\chi_{[\varphi_1, \varphi_2]}(\arg \mathfrak{a}),$$

where $\chi_{[\varphi_1, \varphi_2]}$ is the characteristic function of $[\varphi_1, \varphi_2]$ in $[0, \pi/2]$. Let $f_1(\varphi)$, $f_2(\varphi)$ be functions from Lemma 4 constructed for $[\varphi_1, \varphi_2]$ and $[\varphi_1 - \Delta_1, \varphi_2 + \Delta_1]$ respectively ($\Delta_1 = 2\Delta$). If we put

$$S_i = \sum_{N(\mathfrak{a}) \leq x}^{*} \mathcal{B}_\alpha(\mathfrak{a}) f_i(\arg \mathfrak{a}) \quad (i = 1, 2)$$

we get

$$S_1 \leq S \leq S_2.$$

It is sufficient to prove that S_1, S_2 have the same asymptotic representation. We shall estimate S_1 (the case of S_2 is similar). It follows from Lemma 4 that

$$S_1 = \sum_{N(\mathfrak{a}) \leq x}^{*} \mathcal{B}_\alpha(\mathfrak{a}) f_1(\arg \mathfrak{a}) = \sum_{N(\mathfrak{a}) \leq x}^{*} \mathcal{B}_\alpha(\mathfrak{a}) \sum_{m=-\infty}^{+\infty} a_m \exp(4mi \arg \mathfrak{a})$$

$$= \sum_{m=-\infty}^{+\infty} a_m \sum_{N(\mathfrak{a}) \leq x}^{*} \mathcal{B}_\alpha(\mathfrak{a}) \exp(4mi \arg \mathfrak{a})$$
$$= a_0 \sum_{N(\mathfrak{a}) \leq x}^{*} \mathcal{B}_\alpha(\mathfrak{a}) + \sum_{m \neq 0} a_m \sum_{N(\mathfrak{a}) \leq x}^{*} \mathcal{B}_\alpha(\mathfrak{a}) \exp(4mi \arg \mathfrak{a}).$$

Let us estimate the sum

$$\sum_{N(\mathfrak{a}) \leq x}^{*} \mathcal{B}_\alpha(\mathfrak{a}) \exp(4mi \arg \mathfrak{a}) \quad (m \neq 0).$$

Similarly to the case $m = 0$ it is easy to verify that for $m \neq 0$

$$\sum_{\mathfrak{a}}^{*} \frac{\mathcal{B}_\alpha(\mathfrak{a}) \exp(4mi \arg \mathfrak{a})/N^\alpha(\mathfrak{a})}{N^s(\mathfrak{a})} = \zeta(s+\alpha)L(s+\alpha, \chi_4)[\log Z(s, m) + G'(s, m)],$$

where $G'(s, m)$ is a regular function for $\operatorname{Re} s > 1/2$.

If we put

$$G_1'(s) = \zeta(s+\alpha)L(s+\alpha, \chi_4)\log Z(s, m),$$
$$G_2'(s) = \zeta(s+\alpha)L(s+\alpha, \chi_4)G'(s, m),$$

we get

$$\sum_{N(\mathfrak{a}) \leq x}^{*} \mathcal{B}_\alpha(\mathfrak{a}) \exp(4mi \arg \mathfrak{a})/N^\alpha(\mathfrak{a})$$
$$= \frac{1}{2\pi i}\int_{b-iT}^{b+iT} G_1'(s)\frac{x^s}{s}ds + \frac{1}{2\pi i}\int_{b-iT}^{b+iT} G_2'(s)\frac{x^s}{s}ds + O(x^b/T(b-1)) + O(x \log 2x/T).$$

Let us take $T = \exp(c(\log^{3/5} x))$.

We move the contour of integration to $\operatorname{Re} s = 1-(\delta/2)$. By the bound for $\log Z(s, m)$ (Corollary 2) and $\zeta(s+\alpha)$, $L(s+\alpha, \chi_4) = O(1)$ on the line $\operatorname{Re} s = 1-(\delta/2)$ we have

$$(4.1) \qquad \frac{1}{2\pi i}\int_{b-iT}^{b+iT} G_1'(s)\frac{x^s}{s}ds = O(x^{1-(\delta/2)}(\log\log T|m|)\log T).$$

Moving the contour of integration in $\frac{1}{2\pi i}\int_{b-iT}^{b+iT} G_2'(s)\frac{x^s}{s}ds$ to $\operatorname{Re} s = 1-(\alpha/2)$, we get

$$(4.2) \qquad \frac{1}{2\pi i}\int_{b-iT}^{b+iT} G_2'(s)\frac{x^s}{s}ds = O(x^{1-(\alpha/2)}\log T) + O(x^b/T)$$

(since $\delta = \log^{-(2/3)-\varepsilon} T$, for given $\alpha > 0$ we can find $x_0(\alpha)$ such that $\alpha > \delta$ if $x \geqslant x_0(\alpha)$).

It follows from (4.1), (4.2) with $b = 1+(1/\log x)$ that

$$\sum_{N(\mathfrak{a})\leqslant x}^{*} \mathscr{B}_\alpha(\mathfrak{a})\exp(4mi\arg\mathfrak{a})/N^\alpha(\mathfrak{a}) = O\big(x\exp(-(c/3)(\log^{(3/5)-\varepsilon} x))\big).$$

Hence, by the Abel lemma

$$\sum_{N(\mathfrak{a})\leqslant x}^{*} \mathscr{B}_\alpha(\mathfrak{a})\exp(4mi\arg\mathfrak{a}) = O(x^{1+\alpha}\exp(-c_1\log^{(3/5)-\varepsilon} x)). \tag{4.3}$$

Then by Lemma 4 and Theorem 1 we have

$$S_1 = \frac{2(\varphi_2-\varphi_1+\Delta)}{\pi}\frac{\zeta(1+\alpha)L(1+\alpha,\chi_4)}{1+\alpha}\frac{x^{1+\alpha}}{\log x}\left[1+O\left(\frac{1}{\log x}\right)\right]$$
$$+\sum_{1\leqslant|m|\leqslant 1/\Delta} a_m \sum_{N(\mathfrak{a})\leqslant x}^{*} B_\alpha(\mathfrak{a})\exp(4mi\arg\mathfrak{a})$$
$$+\sum_{|m|>1/\Delta} a_m \sum_{N(\mathfrak{a})\leqslant x}^{*} B_\alpha(\mathfrak{a})\exp(4mi\arg\mathfrak{a})$$
$$=\frac{2(\varphi_2-\varphi_1+\Delta)}{\pi}\frac{\zeta(1+\alpha)L(1+\alpha,\chi_4)}{1+\alpha}\frac{x^{1+\alpha}}{\log x}[1+O(1/\log x)]$$
$$+O\Big(\sum_{1\leqslant|m|\leqslant 1/\Delta}\frac{1}{|m|}x^{1+\alpha}\exp(-c_1\log^{(3/5)-\varepsilon} x)\Big)$$
$$+O\Big(\sum_{|m|>1/\Delta}\frac{1}{|m|^{r+1}}\left(\frac{1}{\Delta}\right)^r\exp(-c_1\log^{(3/5)-\varepsilon} x)(\log T)(\log\log T|m|)\Big)$$
$$=\frac{2(\varphi_2-\varphi_1)}{\pi}\frac{\zeta(1+\alpha)L(1+\alpha,\chi_4)}{1+\alpha}\frac{x^{1+\alpha}}{\log x}[1+O(1/\log x)]$$
$$+O(x^{1+\alpha}\Delta/\log x)+O\big(x^{1+\alpha}(\exp(-c_1\log^{(3/5)-\varepsilon} x))\log(1/\Delta)\big)$$
$$+O\big(x^{1+\alpha}(1/\Delta)^{-r}(1/\Delta)^r\exp(-c_1\log^{(3/5)-\varepsilon} x)(\log\log(T/\Delta))(\log T)\big)$$
$$=\frac{2(\varphi_2-\varphi_1)}{\pi}\frac{\zeta(1+\alpha)L(1+\alpha,\chi_4)}{1+\alpha}\frac{x^{1+\alpha}}{\log x}[1+O(1/\log x)]$$
$$+O\big(x^{1+\alpha}\exp(-c_1\log^{(3/5)-\varepsilon} x)\big).$$

The obtained formula is non-trivial if

$$\varphi_2-\varphi_1 \gg \Delta = \exp(-c_1\log^{(3/5)-\varepsilon} x).$$

This completes the proof of Theorem 2.

5. Proof of Theorem 3. Let

$$F_0(s) = \zeta(s+\alpha)L(s+\alpha,\chi_4)\log\zeta(s)$$
$$+\zeta(s+\alpha)L(s+\alpha,\chi_4)\log L(s,\chi_4)+\zeta(s+\alpha)L(s+\alpha,\chi_4)G(s)$$

(see the proof of Theorem 1).

The function $\zeta(s+\alpha)L(s+\alpha,\chi_4)G(s)$ has the same values at points of the lower and upper "edges" of $[1-\varrho, 1]$, so by Lemma 5 we get

$$I(x,h;z) = \frac{1}{2\pi i}\int_0^h\Big(\int_{C(\varrho)}\zeta(s+\alpha)L(s+\alpha,\chi_4)(\log\zeta(s))(x+v)^{s-1}ds\Big)dv$$
$$+\frac{1}{2\pi i}\int_0^h\Big(\int_{C(\varrho)}\zeta(s+\alpha)L(s+\alpha,\chi_4)(\log L(s,\chi_4))(x+v)^{s-1}ds\Big)dv$$
$$=\frac{1}{2\pi i}\int_{1-\varrho}^{1}\zeta(s+\alpha)L(s+\alpha,\chi_4)(\log\zeta(s))\frac{(x+h)^s-x^s}{s}ds$$
$$+\frac{1}{2\pi i}\int_{1-\varrho}^{1}\zeta(s+\alpha)L(s+\alpha,\chi_4)(\log L(s,\chi_4))\frac{(x+h)^s-x^s}{s}ds$$
$$=\zeta(1+\alpha)L(1+\alpha,\chi_4)\frac{h}{\log x}[1+O(1/\log x)].$$

Therefore

$$\sum_{\substack{x<N(\mathfrak{a})\leqslant x+h\\ \varphi_1\leqslant\arg\mathfrak{a}\leqslant\varphi_2}}^{*}\mathscr{B}_\alpha(\mathfrak{a})/N^\alpha(\mathfrak{a})$$
$$=\frac{2(\varphi_2-\varphi_1)}{\pi}\zeta(1+\alpha)L(1+\alpha,\chi_4)\frac{h}{\log x}[1+O(1/\log x)]$$
$$+O\big(h\exp(-c(\log^{1/3}x)(\log\log x)^{-1})\big)+O(x^\beta).$$

Hence

$$\sum_{\substack{x<N(\mathfrak{a})\leqslant x+h\\ \varphi_1\leqslant\arg\mathfrak{a}\leqslant\varphi_2}}^{*}\mathscr{B}_\alpha(\mathfrak{a}) = \frac{2(\varphi_2-\varphi_1)}{\pi}\zeta(1+\alpha)L(1+\alpha,\chi_4)\frac{hx^\alpha}{\log x}$$
$$+O(h^2x^{\alpha-1}/\log x)+O(hx^\alpha/\log^2 x)$$
$$+O\big(hx^\alpha\exp(-c(\log^{1/3}x)(\log\log x)^{-1})\big)+O(x^{\beta+\alpha}).$$

Similarly, by (2.4) we obtain

$$\frac{1}{X}\int_X^{2X}\Big|\sum_{\substack{x<N(\mathfrak{a})\leqslant x+h\\ \varphi_1\leqslant\arg\mathfrak{a}\leqslant\varphi_2}}^{*}\mathscr{B}_\alpha(\mathfrak{a})-\frac{2(\varphi_2-\varphi_1)}{\pi}\zeta(1+\alpha)L(1+\alpha,\chi_4)\frac{hx^\alpha}{\log x}\Big|^2dx$$
$$=O\big(h^2X^{2\alpha}\exp(-c(\log^{1/3}X)(\log\log X)^{-1})\big)+O(X^{2(\beta'+\alpha)}).$$

This completes the proof of Theorem 3.

6. Remarks. Let us note that the factor $1+O(1/\log x)$ in Theorems 1 and 2 can be improved, namely it can be replaced by

$$1+b_1(1/\log x)+\ldots+b_m(1/\log^m x)+O(1/\log^{m+1} x)$$

It would be interesting to prove Theorem 1 by the elementary methods from [2]. However, it seems that the elementary approach cannot be used for Theorems 2 and 3, because for problems of distribution of values of arithmetical functions in sectorial regions the elementary techniques do not give satisfactory accuracy.

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INSTITUTE OF MATHEMATICS
UNIVERSITY OF GDAŃSK
ul. Wita Stwosza 57
80-952 Gdańsk, Poland

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Sub-bases of pleasant h-bases

by

ERNST S. SELMER (Bergen)

Given an integral basis

$$A_k = \{a_1, a_2, \ldots, a_k\}, \quad 1 = a_1 < a_2 < \ldots < a_k$$

for a positive integer h, we form all the combinations

$$\sum_{i=1}^{k} x_i a_i, \quad x_i \geqslant 0, \quad \sum_{i=1}^{k} x_i \leqslant h,$$

and ask for the smallest integer $N_h(A_k)$ which is not represented by such a combination. The number $n_h(A_k) = N_h(A_k)-1$ is called the *h-range* of A_k. In this connection, A_k is often denoted as *h-basis*.

A popular interpretation arises if we consider the integers a_i as *stamp* denominations, and h as the "size of the envelope". More information on the postage stamp problem can be found for instance in [4]. A comprehensive treatment of this problem is contained in the author's research monograph [5] (freely available on request). We only give here some more definitions which will be needed below.

A representation $n = \sum_{i=1}^{k} x_i a_i$ is called *regular* if we first use a_k as often as possible, then a_{k-1} as often as possible, etc. This means to impose the additional condition

$$\sum_{i=1}^{j} x_i a_i < a_{j+1}, \quad j = 1, 2, \ldots, k-1.$$

If only such representations are allowed, still restricted to at most h addends, we speak of the *regular h-range* $g_h(A_k)$. Clearly $n_h(A_k) \geqslant g_h(A_k)$ for all A_k and h. In contrast to $n_h(A_k)$, the general determination of $g_h(A_k)$ is fairly simple, see for instance [3].

A given integer may have several representations by a basis A_k. A *minimal* representation (not necessarily unique) is one with the smallest number of addends from the basis. Djawadi [1] called a basis *pleasant* (German: