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A problem on semicubical powers

by

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1. Introduction. Newton observed that an unordered set $\{h_1, \ldots, h_N\}$ of real numbers is determined by the first N power sums: $h_1^K + \ldots + h_N^K$, for $K = 1$ to N ([7]). If we are given that $h_1, \ldots, h_N$ are integers, then the set $\{h_1, \ldots, h_N\}$ is often determined by a shorter sequence of power sums. This fact is important in I. M. Vinogradov's treatment of Waring's Problem. In their recent work on exponential sums ([1]) Bombieri and Iwaniec require a result of this type with $N = 4$, when the power sums with $K = 1$ and 2 are given and the power sum with $K = 3/2$ is given approximately. They require an upper bound for $I_8(H, \Delta)$, the number of integer solutions of the equations,

$$\sum_1^4 (h_j^2 - k_j^2) = \sum_1^4 (h_j - k_j) = 0, \tag{1.1}$$

$$\Big|\sum_1^4 (h_j^{3/2} - k_j^{3/2})\Big| \leqslant \Delta H^{3/2}, \tag{1.2}$$

$$H \leqslant h_j, k_j < 2H, \quad j = 1, \ldots, 4, \tag{1.3}$$

where H is a positive integer and $0 \leqslant \Delta \leqslant 1$.

In the technical paper, [2], they obtain the bound $O(H^{4+\varepsilon})$ for any $\varepsilon > 0$, when $\Delta = 1/H$. They also obtain the same bound if the exponents in (1.2), which are all 3/2, are replaced by any real number other than 0, 1 or 2. In this paper we show, by elementary means, that the number of integer solutions of (1.1), (1.2) and

$$0 \leqslant h_j, k_j < H \tag{1.4}$$

is $\ll H^4 + \Delta H^5 \log^3 H$. We use Vinogradov's order of magnitude notation "$F \ll G$" for $F = O(G)$. Replacing H by $2H$, we see that $I_8(H, \Delta) \ll H^4 + \Delta H^5 \log^3 H$.

If we impose the extra condition

$$\Big|\sum_1^4 (h_j^{1/2} - k_j^{1/2})\Big| \leqslant \Delta_2 H^{1/2}, \tag{1.5}$$

where Δ_2 can be chosen from $[\Delta, \Delta H \log H]$, then the number of solutions of (1.1), (1.2), (1.5) and (1.4) is $A(H)+O(\Delta H^5 \log^3 H)$, where

(1.6) $A(H) = 24H^4 - 72H^3 + 82H^2 - 33H$ is the number of "trivial solutions" of (1.1) and (1.4) for which $(k_1, \ldots, k_4)$ is a permutation of $(h_1, \ldots, h_4)$.

Let $S = \sum_{H \leqslant h < 2H} \exp\big(2\pi i(\alpha h^2 + \beta h + \gamma (h/H)^{3/2})\big)$. By Lemma 2.3 of [1],

$$I_8(H, \Delta) \gg \Delta \int_{-1}^{1}\int_{-1}^{1}\int_{-1/\Delta}^{1/\Delta} |S|^8\, d\alpha\, d\beta\, d\gamma \gg \Delta \int_{-1/96H^2}^{1/96H^2}\int_{-1/32H}^{1/32H}\int_{-1/64}^{1/64} |S|^8\, d\alpha\, d\beta\, d\gamma$$

and expanding the product and integrating term by term gives $I_8(H, \Delta) \gg \Delta H^5$. A plausible conjecture would be that $\Delta H^5 \ll |I_8(H, \Delta) - A(H)| \ll \Delta H^5$, for $H^{-5} \ll \Delta \ll 1$. This conjecture is related to the Problem of Prouhet and Tarry, described in Section 21.9 of [4].

Applications to exponential sums will be given in [6]. I would like to thank Dr. M. N. Huxley, my supervisor, for several useful suggestions. I am also indebted to the Carnegie Trust for the Universities of Scotland, who have supported me financially. This work is part of my doctoral thesis for the University of Wales.

2. Linear families. If $h_1, \ldots, k_4$ is a solution of (1.1), then so is $h_1+t, \ldots, k_4+t$, for any real t. We call such a set of solutions a *family*. In this section we show that the integer solutions of (1.1) and (1.4) fall into $O(H^4)$ families.

LEMMA 1. *For integers A, B the number of integer solutions of*

$$x_1^2 + x_2^2 + x_3^2 = A,$$

$$x_1 + x_2 + x_3 = B$$

is

$$\omega(F) = \begin{cases} 1 & \text{if } F = 0, \\ 6\sum_{d|F}\left(\frac{d}{3}\right) & \text{if } F/3 \text{ is a positive integer}, \\ 3\sum_{d|F}\left(\frac{d}{3}\right) & \text{if } F \text{ is a positive integer and } (F, 3) = 1, \\ 0 & \text{in all other cases}, \end{cases}$$

where $\left(\frac{a}{b}\right)$ is Jacobi's symbol and $F = (3A - B^2)/2$.

Furthermore, if $F = 0$, then $x_1 = x_2 = x_3 = B/3$ and $B^2 = 3A$.

Proof. Writing $u = x_2 - x_1$, $v = x_3 - x_1$ we have

$$u^2 - uv + v^2 = x_1^2 + x_2^2 + x_3^2 - x_1 x_2 - x_2 x_3 - x_3 x_1 = (3A - B^2)/2 = F.$$

As $u^2 - uv + v^2$ is positive definite, solutions only exist if F is a non-negative integer. If $F = 0$, then $u = v = 0$ and $x_2 = x_1 = x_3$. If F is a positive integer, then, by a similar argument to Sections 16.9 and 16.10 of [4], there are $6\sum_{d|F}\left(\frac{d}{3}\right)$ integer solutions for (u, v). Then $3x_1 = B - (u+v)$, $x_2 = u + x_1$, $x_3 = v + x_1$ and

$$(u+v)^2 = 3(u^2+v^2) - 2(u^2 - uv + v^2) \equiv -2F \pmod 3 \equiv B^2 \pmod 3.$$

If $3|F$, then $3|(u+v)$, $3|B$ and x_1 is an integer. If $(F, 3) = 1$, then $(B, 3) = (u+v, 3) = 1$, $u+v \equiv \pm B \pmod 3$ and from each pair of solutions (u, v), $(-u, -v)$ (which are distinct, as $u = v = 0$ implies $F = 0$) only one will correspond to an integral value for x_1. This completes the proof of the lemma. ■

Let f be a family containing an integer solution of (1.4), $h_1, \ldots, k_4$. Suppose that h_4 is the least of $h_1, \ldots, k_4$. (There are 8 similar cases.) Let $x_i = h_i - h_4$, $y_i = k_i - h_4$, for $i = 1, \ldots, 4$. We have, $0 \leqslant x_i \leqslant h_i < H$ and $0 \leqslant y_i \leqslant k_i < H$, for $i = 1, \ldots, 4$, and $x_4 = 0$. As $x_1, \ldots, y_4$ is a member of f it satisfies (1.1). Therefore, to count all such families as f at least once, it is sufficient to count the number of integer solutions of (1.1) and (1.4) of the form $x_1, x_2, x_3, 0, y_1, \ldots, y_4$.

To apply Lemma 1 we fix $y_1, \ldots, y_4$ and let $A = y_1^2 + \ldots + y_4^2$, $B = y_1 + \ldots + y_4$, so that

(2.1) $F = y_1^2 + \ldots + y_4^2 - y_1 y_2 - y_2 y_3 - y_3 y_4 - y_4 y_1 - y_1 y_3 - y_2 y_4$ and $F = F(y)$ is an indefinite quadratic form in $y_1, \ldots, y_4$ taking integer values at integer points.

If $\varepsilon > 0$, then $\omega(F) \leqslant \max\{1, 6d(F)\} \ll \max\{1, F^\varepsilon\} \ll H^\varepsilon$ and we see that there are $\ll H^\varepsilon$ solutions for x_1, x_2, x_3. Now we let the y_i range over the integers in $[0, H)$ to count a total of $\ll H^{4+\varepsilon}$ families. We obtain a bound $\ll H^4$, by showing that $\omega(F)$ is $\ll 1$ on average.

When F is a positive integer, we write $F = F_1 F_3$, where F_3 is the largest power of 3 that divides F. Suppose that $\omega(F) \neq 0$, so by Lemma 1

$$0 \neq \sum_{d|F}\left(\frac{d}{3}\right) = \sum_{d|F_1}\left(\frac{d}{3}\right) = \prod_{p^\beta \| F_1}\left\{\sum_{r=0}^{\beta}\left(\frac{p}{3}\right)^r\right\}$$

and

$$\sum_{r=0}^{\beta}\left(\frac{p}{3}\right)^r = \begin{cases} \beta+1, & p \equiv 1 \pmod 3, \\ (1+(-1)^\beta)/2, & p \equiv -1 \pmod 3, \end{cases}$$

where p is prime and the sign $\|$ indicates that β is the highest power of p dividing F_1. It follows that if $p^\beta \| F_1$, then either $p \equiv 1 \pmod 3$ or β is even and so $p^\beta \equiv 1 \pmod 3$ in both cases. This shows that $F_1 \equiv 1 \pmod 3$. Now,

$$\sum_{d|F_1} \left(\frac{d}{3}\right) = \sum_{\substack{d|F_1\\ d \leqslant \sqrt{F_1}}} \left(\frac{d}{3}\right) + \sum_{\substack{d|F_1\\ d > \sqrt{F_1}}} \left(\frac{d}{3}\right).$$

Let $e = F_1/d$, then $d|F_1,\ d > \sqrt{F_1} \Leftrightarrow e|F_1,\ e < \sqrt{F_1}$ and $\left(\frac{d}{3}\right)\left(\frac{e}{3}\right) = \left(\frac{F_1}{3}\right)$, so that $\left(\frac{d}{3}\right) = \left(\frac{e}{3}\right)$. By Lemma 1 it follows that

$$\omega(F) \leqslant 12 \sum_{\substack{d|F_1\\ d \leqslant \sqrt{F_1}}}{}' \left(\frac{d}{3}\right), \tag{2.2}$$

where the $\sum'$ indicates that weight 1/2 is used when $d = \sqrt{F_1}$.

We now define two densities associated with the quadratic form $F(y_1, y_2, y_3, y_4)$ given by (2.1).

$$D(b \bmod q; y_1, y_2) = \frac{1}{q^2}\Big\{ \sum_{\substack{y_3 \bmod q\ y_4 \bmod q\\ F(y) \equiv b \pmod q}} \sum 1 \Big\}, \tag{2.3}$$

$$D(b \bmod q) = \frac{1}{q^4}\Big\{ \sum_{\substack{y \bmod q\\ F(y) \equiv b \pmod q}} 1 \Big\} = \frac{1}{q^2}\Big\{ \sum_{\substack{y_1 \bmod q\ y_2 \bmod q\\ F(y) \equiv b \pmod q}} \sum D(b \bmod q; y_1, y_2) \Big\}. \tag{2.4}$$

These densities are multiplicative in q, by the Chinese Remainder Theorem, and will be evaluated using Gauss sums:

$$S(n, m) = \sum_{t \bmod m} e(nt^2/m), \quad \text{for } (n, m) = 1.$$

Here $e(x)$ denotes $\exp(2\pi i x)$, where $i = \sqrt{-1}$. The standard properties of Gauss sums are summarised as follows (see [5], Theorems 5.1 and 5.6 of Chapter 7).

LEMMA 2. *If* $(n, m) = c$, *then*

$$S(n, m) = cS(n/c, m/c).$$

If $(n, mm') = (m, m') = 1$, *then*

$$S(n, mm') = S(nm', m)\, S(nm, m').$$

If, further, m is odd and positive, then

$$S(n, m) = \left(\frac{n}{m}\right)\sqrt{\left(\frac{-1}{m}\right)}\sqrt{m}, \quad \text{where } \sqrt{-1} = i.$$

If, instead, m is a positive power of 2 and n is odd, then

$$S(n, m) = \begin{cases} 0, & \text{if } m = 2,\\ \left(\frac{m}{n}\right)\left(1+\left(\frac{-1}{n}\right)i\right)\sqrt{m}, & \text{otherwise,} \end{cases}$$

and consequently, $S(n, 4m) = 2S(n, m)$ if $4|m$. ■

LEMMA 3. *Let $c_q(a)$ denote Ramanujan's sum: $\sum^*_{n \bmod q} e(an/q)$, where the $\sum^*$ indicates that the sum is taken over those n coprime to q.*

If $(q, 3) = 1$, then

$$D(b \bmod q; v, w) = \frac{1}{q}\Big\{\sum_{r|q}\left(\frac{r}{3}\right)\frac{1}{r}c_r(3vw+b)\Big\}, \tag{2.5}$$

$$D(b \bmod q) = \frac{1}{q}\Big\{\sum_{r|q}\left(\frac{r}{3}\right)\frac{1}{r^2}c_r(b)\Big\}. \tag{2.6}$$

If q is a positive power of 3, then

$$D(b \bmod q; v, w) = \frac{1}{q}\Big\{1 + \sum_{1<r|q}\left(\frac{3(3vw+b)/r}{3}\right)\Big\}, \tag{2.7}$$

$$D(b \bmod q) = \frac{1}{q}\Big\{1 + \sum_{1<r|q}\left(\frac{3b/r}{3}\right)\frac{3}{r}\Big\}, \tag{2.8}$$

where $\left(\frac{a}{d}\right) = 0$ unless a is an integer coprime to d.

Proof. From (2.1) we obtain

$$4F(v, w, x, y) = (x+y-2v-2w)^2 + 3(x-y)^2 - 12vw.$$

Let

$$h = x+y-2v-2w \quad \text{and} \quad k = x-y, \tag{2.9}$$

then

$$4F(v, w, x, y) = h^2 + 3k^2 - 12vw. \tag{2.10}$$

From (2.3)

$$D(b \bmod q; v, w) = \frac{1}{q^2}\left\{\sum_{x \bmod q}\sum_{y \bmod q}\frac{1}{q}\sum_{a \bmod q} e(a(F-b)/q)\right\} \tag{2.11}$$

$$= \frac{1}{q}\left\{\sum_{r|q}\frac{1}{r^2}\sum_{A \bmod r}^{*}\sum_{x \bmod r}\sum_{y \bmod r} e(A(F-b)/r)\right\}$$

(here $\frac{A}{r}=\frac{a}{q}$). We now evaluate the sum over A in (2.11).

If q is odd, then so is r and we can rearrange the sum over A and use (2.9) and (2.10) to get

$$\sum_{A \bmod r}^{*}\sum_{x \bmod r}\sum_{y \bmod r} e\left(\frac{4A(F-b)}{r}\right) \tag{2.12}$$

$$= \sum_{A \bmod r}^{*} e\left(\frac{-4A(3vw+b)}{r}\right)\sum_{x \bmod r}\sum_{y \bmod r} e\left(\frac{A(h^2+3k^2)}{r}\right).$$

By (2.9) h and k are independent variables modulo r in the last double sum, which therefore equals

$$S(A, r)S(3A, r), \quad \text{for } (A, r) = 1. \tag{2.13}$$

If $(q, 3) = 1$, then, by Lemma 2, this is

$$\left(\frac{3}{r}\right)\sqrt{r}\left(\frac{-1}{r}\right)\sqrt{r} = \left(\frac{r}{3}\right)r. \tag{2.14}$$

As this is independent of A we may again rearrange the sum over A, (2.12), as

$$\left(\frac{r}{3}\right)r\sum_{A \bmod r}^{*} e(A(3vw+b)/r) = \left(\frac{r}{3}\right)rc_r(3vw+b). \tag{2.15}$$

If q is instead a positive power of 2, then we can rewrite the sum over A in (2.11) and use (2.9) and (2.10) to get

$$\sum_{A \bmod r}^{*}(1/4)\sum_{x \bmod 4r}(1/4)\sum_{y \bmod 4r} e(4A(F-b)/r) \tag{2.16}$$

$$= \sum_{A \bmod r}^{*} e(-A(3vw+b)/r)(1/16)\sum_{x \bmod 4r}\sum_{y \bmod 4r} e(A(h^2+3k^2)/4r).$$

By (2.9) we have $h \equiv k \pmod 2$ and also $(x, y) \leftrightarrow (h, k)$ is a two-to-one correspondence modulo $4r$. The sum over x and y in (2.16) can therefore be split up in the following form

$$2\sum_{\substack{h \bmod 4r \\ h \equiv k \pmod 2}}\sum_{k \bmod 4r}$$

$$= 2\left\{\sum_{h \bmod 4r}\sum_{k \bmod 4r} - \sum_{\substack{h \bmod 4r \\ 2|h}}\sum_{k \bmod 4r} - \sum_{h \bmod 4r}\sum_{\substack{k \bmod 4r \\ 2|k}} + 2\sum_{\substack{h \bmod 4r \\ 2|h}}\sum_{\substack{k \bmod 4r \\ 2|k}}\right\}.$$

The result is

$$2(S(A, 4r)S(3A, 4r) - 2S(A, r)S(3A, 4r) \tag{2.17}$$

$$- 2S(A, 4r)S(3A, r) + 8S(A, r)S(3A, r)).$$

If $4|r$, then, by Lemma 2, this is

$$2(4-4-4+8)S(A, r)S(3A, r) = 8S(A, r)S(3A, r) \tag{2.18}$$

$$= 8\left(\frac{r}{A}\right)\left(1+\left(\frac{-1}{A}\right)i\right)\sqrt{r}\left(\frac{r}{3A}\right)\left(1+\left(\frac{-1}{3A}\right)i\right)\sqrt{r}$$

$$= 8\left(\frac{r}{3}\right)\left|1+\left(\frac{-1}{A}\right)i\right|^2 r = 16r\left(\frac{r}{3}\right).$$

If $r = 2$, then, by Lemma 2, (2.17) is

$$2S(A, 8)S(3A, 8) = 2\left(\frac{8}{A}\right)\left(1+\left(\frac{-1}{A}\right)i\right)\sqrt{8}\left(\frac{8}{3A}\right)\left(1+\left(\frac{-1}{3A}\right)i\right)\sqrt{8}$$

$$= 16\left(\frac{8}{3}\right)\left|1+\left(\frac{-1}{A}\right)i\right|^2 = 16\left(\frac{2}{3}\right)2,$$

which is just (2.18). Hence, using (2.18) in (2.16) and replacing A by $-A$, we again see that the sum over A, (2.12), is given by (2.15).

If q is odd and coprime to 3, or if q is a positive power of 2, then from (2.15) and (2.11) we obtain (2.5). As the densities are multiplicative in q, we obtain (2.5) for any q coprime to 3. Then, by (2.4) and (2.5), we obtain

$$D(b \bmod q) = \frac{1}{q^3}\left\{\sum_{r|q}\left(\frac{r}{3}\right)\frac{1}{r}\sum_{A \bmod r}^{*}\sum_{v \bmod q}\sum_{w \bmod q} e((3vw+b)A/r)\right\}$$

$$= \frac{1}{q}\left\{\sum_{r|q}\left(\frac{r}{3}\right)\frac{1}{r^3}\sum_{A \bmod r}^{*} e(bA/r)\sum_{v \bmod r}\sum_{w \bmod r} e(3vwA/r)\right\},$$

if $(q, 3) = 1$.

In the above sum r will be coprime to $3A$, so that the sum over w equals zero, unless $r|v$, when it equals r. (2.6) follows, for any q coprime to 3.

Let q be a positive power of 3.

(2.19) When $r = 1$, the sum over A in (2.11) is just 1.

If $r > 1$, then the sum over A in (2.11) is given by (2.12) and, instead of (2.14), we get, by (2.13) and Lemma 2,

$$S(A, r)3S(A, r/3) = 3\left(\frac{A}{r}\right)\sqrt{\left(\frac{-1}{r}\right)}\sqrt{r}\left(\frac{A}{r/3}\right)\sqrt{\left(\frac{-1}{r/3}\right)}\sqrt{r/3} \qquad ((A, r) = 1)$$

$$= \sqrt{3}\left(\frac{A}{3}\right)\sqrt{\left(\frac{-1}{3}\right)}r = i\sqrt{3}\left(\frac{A}{3}\right)r.$$

The sum over A, (2.12), is then

$$i\sqrt{3r}\sum_{A \bmod r}^{*}\left(\frac{A}{3}\right)e(-4A(3vw+b)/r)$$

and rearranging this gives

$$-i\sqrt{3}r\sum_{A \bmod r}^{*}\left(\frac{A}{3}\right)e(A(3vw+b)/r). \tag{2.20}$$

Summing over $B = A+3$ shows that, if r does not divide $3(3vw+b)$, then (2.20) is zero. If $r \mid 3vw+b$, then summing over $B = -A$ shows that (2.20) is zero. The remaining case is when $r/3$ is the highest power of 3 dividing $3vw+b$. (2.20) then becomes

$$-i\sqrt{3}r\frac{r}{3}\sum_{A \bmod 3}^{*}\left(\frac{A}{3}\right)e((A(3vw+b)/(r/3))/3)$$

$$= -i(r^2/\sqrt{3})\left(\frac{(3vw+b)/(r/3)}{3}\right)\sum_{B \bmod 3}^{*}\left(\frac{B}{3}\right)e(B/3).$$

The sum over B is $2i\sin 2\pi/3 = i\sqrt{3}$, so that (2.20) is

$$\left(\frac{(3vw+b)/(r/3)}{3}\right)r^2 = \left(\frac{3(3vw+b)/r}{3}\right)r^2. \tag{2.21}$$

With the convention that $\left(\frac{a}{d}\right) = 0$, unless a is an integer coprime to d, we see that (2.21) is also true in the cases where $r/3$ was not the highest power of 3 to divide $3vw+b$. From (2.19), (2.21) and (2.11) we obtain (2.7).

By (2.4) we may obtain $D(b \bmod q)$ by summing $D(b \bmod q; v, w)$ over v and w modulo q, then dividing by q^2. We may carry this out on (2.20), to obtain, for $r > 1$

$$\frac{1}{q^2}(-i\sqrt{3}r)\sum_{A \bmod r}^{*}\left(\frac{A}{3}\right)e(Ab/r)(q/(r/3))^2\sum_{v \bmod (r/3)}\sum_{w \bmod (r/3)}e(Avw/(r/3)).$$

Here $(A, 3) = 1$, so that the sum over w equals zero, unless $v \equiv 0 (\bmod r/3)$, when it equals $r/3$. We obtain

$$-i3\sqrt{3}\sum_{A \bmod r}^{*}\left(\frac{A}{3}\right)e(Ab/r). \tag{2.22}$$

The sum over A here is of the same form as that occurring in (2.20), so that we may treat (2.22) in the same way to get

$$\frac{3}{r}\left(\frac{3b/r}{3}\right)r^2 = 3\left(\frac{3b/r}{3}\right)r. \tag{2.23}$$

By (2.19) the contribution from $r = 1$ is

$$\frac{1}{q^2}\left\{\sum_{v \bmod w}\sum_{w \bmod q}1\right\} = 1. \tag{2.24}$$

Summing the results from (2.23) and (2.24) over $r \mid q$, with weight $q/(r^2)$, we obtain (2.8) and the proof of the lemma is complete. ∎

By short-circuiting the proof in [3], we obtain the following less refined result.

LEMMA 4. *Let $\mathscr{R}$ be a measurable subset of $\overset{n}{\underset{1}{\times}}[A_i, A_i+I_i]$, with the property, that every line in $\mathbf{R}^n$ which is parallel to one of the coordinate axes intersects $\mathscr{R}$ on a set consisting of at most c disjoint connected components in the line. Then*

$$\left|\sum_{\mathbf{Z}^n}\chi_{\mathscr{R}}(x) - \int_{\mathbf{R}^n}\chi_{\mathscr{R}}(x)\,dx\right| \leqslant c\left\{\prod_{1}^{n}(I_i+1) - \prod_{1}^{n}I_i\right\},$$

where $\chi_{\mathscr{R}}(x)$ is the characteristic function of $\mathscr{R}$. ∎

LEMMA 5. *Let q be a positive integer, $F(X_1, \ldots, X_n)$ a non-constant polynomial with the degree in each variable being $\leqslant d$. Let*

$$\mathscr{R} = \{x \in \mathbf{R}^n;\ F(x) \geqslant b;\ 0 \leqslant x_i \leqslant I,\ i = 1, \ldots, n\}$$

and let $V(\mathscr{R})$ denote the volume of $\mathscr{R}$. If $w(x)$ is any weight function on $\mathscr{R}$ with $0 \leqslant w(x) \leqslant 1$ and $w(x) = 1$ if $F(x) > b$, then, for any $\mathbf{a} \in \mathbf{R}^n$,

$$\left|\left\{\sum_{\substack{x \in \mathscr{R} \\ x \equiv \mathbf{a} (\bmod q)}} w(x)\right\} - \frac{1}{q^n}V(\mathscr{R})\right| \leqslant [(d+2)/2]\left\{\sum_{r=0}^{n-1}\binom{n}{r}(I/q)^r\right\},$$

where $[u]$ denotes the integer part of u.

Proof. Let $\mathscr{D} = \{x \in \mathscr{R};\ F(x) > b\}$. $\mathscr{D}$ is a finite intersection of measurable sets and is therefore a measurable set. Now

$$\sum_{\substack{x \in \mathscr{R} \\ x \equiv \mathbf{a} (\bmod q)}} 1 = \sum_{y \in \mathbf{Z}^n}\chi_{\mathscr{A}}(y),$$

where $\mathscr{A} = \dfrac{1}{q}(\mathscr{D}-\boldsymbol{a})$. $\mathscr{A}$ is a measurable subset of $\overset{n}{\underset{1}{\times}}[-a_i/q,\ -a_i/q+I/q]$. To show that $\mathscr{A}$ satisfies the conditions of Lemma 4, let $1 \leqslant j \leqslant n$, let

$$y_i = b_i, \qquad \text{for } i = 1, \ldots, n,\ i \neq j$$

be the equation of a line parallel to the j-axis and let z be a point on that line. z is a point in $\mathscr{A}$ if and only if $-a_j/q \leqslant z_j \leqslant -a_j/q+I/q$ and $F(q\boldsymbol{z}+\boldsymbol{a}) > b$.

$F(q\boldsymbol{z}+\boldsymbol{a})-b = g(z_j)$, a polynomial in z_j of degree at most d.

$\mathscr{S} = \{z_j;\ z \in \mathscr{A}\}$ is the intersection of an open subset of $\boldsymbol{R}$ with $[-a_j/q,\ -a_j/q+I/q]$. The boundary points of $\mathscr{S}$ in $\boldsymbol{R}$ are either roots of g, or $-a_j/q$, or $-a_j/q+I/q$. Every disjoint connected component of $\mathscr{S}$ has two boundary points in $\boldsymbol{R}$. Each boundary point bounds at most two disjoint connected components of $\mathscr{S}$. Two disjoint connected components of $\mathscr{S}$ share the same boundary point only when that point is a multiple root of g. As g has at most d roots (counting multiplicity), it follows that $\mathscr{S}$ has at most $[(d+2)/2]$ disjoint connected components.

$\mathscr{A}$ satisfies the conditions of Lemma 4 and we obtain

$$\Big|\sum_{\boldsymbol{Z}^n} \chi_{\mathscr{A}}(y) - \int_{\boldsymbol{R}^n} \chi_{\mathscr{A}}(y)\,dy\Big| \leqslant [(d+2)/2]\big(((I/q)+1)^n-(I/q)^n\big).$$

Putting this in terms of x, we obtain

$$\left|\Big\{\sum_{\substack{x \in \mathscr{D} \\ x \equiv \boldsymbol{a} \pmod q}} 1\Big\} - \frac{1}{q^n}V(\mathscr{D})\right| \leqslant [(d+2)/2]\Big\{\sum_{r=1}^{n-1}\binom{n}{r}(I/q)^r\Big\}.$$

By a similar argument, with minor adjustments, we obtain the above result, with $\mathscr{D}$ replaced by $\mathscr{R}$. Now $\mathscr{R}-\mathscr{D}$ has measure zero, so $V(\mathscr{R}) = V(\mathscr{D})$ and

$$\Big\{\sum_{\substack{x \in \mathscr{D} \\ x \equiv \boldsymbol{a} \pmod q}} 1\Big\} \leqslant \Big\{\sum_{\substack{x \in \mathscr{R} \\ x \equiv \boldsymbol{a} \pmod q}} w(x)\Big\} \leqslant \Big\{\sum_{\substack{x \in \mathscr{R} \\ x \equiv \boldsymbol{a} \pmod q}} 1\Big\}$$

so that we may combine the results for $\mathscr{R}$ and $\mathscr{D}$ to complete the proof of the lemma. ∎

LEMMA 6. *The integer solutions of* (1.1) *and* (1.4) *fall into* $O(H^4)$ *families.*

Proof. After the discussion following Lemma 1, we may use Lemma 1 and (2.2) to see that the number of such families is

$$\ll \Big\{\sum_{\substack{y \\ F(y)=0}} 1\Big\} + \sum_{\substack{y \\ F(y)>0 \\ F_1 \equiv 1 \pmod 3}} \Big\{\frac{1}{2} + \sum_{\substack{d \mid F_1 \\ d < \sqrt{F_1}}} \Big(\frac{d}{3}\Big)\Big\},$$

where $\sum_y$ denotes the sum over the integers $y_1, \ldots, y_4$ in $[0, H)$. In this region $F(y) < H^2$. The above expression is at most

(2.25) $$\left\{\begin{array}{l} \Big\{\sum_y 1\Big\} + \sum_m \sum_{1 \leqslant d < H/(\sqrt{3})^m} \Big(\frac{d}{3}\Big) \sum_{\substack{y \in \mathscr{R} \\ F(y) \equiv b \pmod q}} 1, \\ \text{where } F_3 = 3^m,\ q = 3^{m+1}d,\ b \text{ is the least positive integer satisfying} \\ \text{the congruences } b \equiv 3^m \pmod{3^{m+1}} \text{ and } b \equiv 0 \pmod d \text{ and} \\ \mathscr{R} = \{x \in \boldsymbol{R}^4;\ F(x) > 3^m d^2;\ 0 \leqslant x_i \leqslant H-1,\ i = 1, \ldots, 4\}. \end{array}\right.$$

The first sum over y is H^4. The last sum over y is

$$\sum_{\substack{\boldsymbol{a} \bmod q \\ F(\boldsymbol{a}) \equiv b \pmod q}} \sum_{\substack{y \in \mathscr{R} \\ y \equiv \boldsymbol{a} \pmod q}} 1.$$

$\mathscr{R}$ satisfies the conditions of Lemma 5, with $n = 4$, $d = 2$, so that the inner sum here is

$$\frac{1}{q^4}V(\mathscr{R}) + O\big((H/q)^3+1\big).$$

If $q \leqslant 3H$, then we sum over $\boldsymbol{a}$ to see, that

$$\sum_{\substack{y \in \mathscr{R} \\ F(y) \equiv b \pmod q}} 1 = \big(V(\mathscr{R}) + O(qH^3)\big)D(b \bmod q).$$

If $q > 3H$, then, as $d < H/(\sqrt{3})^m \leqslant H$, there is a unique integer t with

(2.26) $$1 \leqslant t \leqslant m \quad \text{and} \quad H < q_1 \leqslant 3H, \quad \text{where } q_1 = 3^t d.$$

Arguing as above and using (2.25) and (2.26), we obtain

$$\begin{aligned} \sum_{\substack{y \in \mathscr{R} \\ F(y) \equiv b \pmod q}} 1 \leqslant \sum_{\substack{y \in \mathscr{R} \\ F(y) \equiv 0 \pmod{q_1}}} 1 &= \big(V(\mathscr{R}) + O(q_1 H^3)\big)D(0 \bmod q_1) \\ &\ll H^4 D(0 \bmod q_1). \end{aligned}$$

It now remains to estimate three sums.

First Sum:

$$S_1 = \sum_m \sum_{1 \leqslant d < H/3^m} \Big(\frac{d}{3}\Big) V(\mathscr{R}) D(b \bmod q).$$

By the multiplicativity of the densities and Lemma 3,

$$S_1 = \sum_m \frac{1}{3^{m+1}}\Big(1+\frac{1}{3^m}\Big) \sum_{1 \leqslant d < H/3^m} \Big(\frac{d}{3}\Big) V(\mathscr{R}) \frac{1}{d} \sum_{r \mid d} \Big(\frac{r}{3}\Big) \frac{1}{r^2} \varphi(r).$$

Let $d = re$ and use $\varphi(r) \leqslant r$, to see that

$$S_1 \ll \sum_m \frac{1}{3^m} \sum_r \frac{1}{r^2} \left| \sum_{1 \leqslant e < H/3^m r} \left(\frac{e}{3}\right) \frac{V(\mathscr{M})}{e} \right|.$$

For fixed m and r, $V(\mathscr{M})$ is non-increasing, as e increases, so that the innermost sum is bounded below, by zero, and above, by its first term ($e = 1$), which is at most H^4. The sums over m and r converge and $S_1 \ll H^4$.

Second Sum:

$$S_2 = \sum_m \sum_{\substack{1 \leqslant d < H/3^m \\ (d,3)=1}} qH^3 D(b \bmod q).$$

We proceed as above, with $d = re$, and also use $\left|\left(\frac{r}{3}\right)\right| \leqslant 1$ to show that

$$S_2 \ll H^3 \sum_m \sum_r \frac{1}{r} \sum_{1 \leqslant e < H/3^m r} 1 \ll H^4,$$

since the sum over e is $\ll H/3^m r$.

Third Sum:

$$S_3 = \sum_m \sum_{\substack{H/3^m < d < H/(\sqrt{3})^m \\ (d,3)=1}} H^4 D(0 \bmod q_1).$$

As the densities are non-negative, we may use (2.26) to replace one H with $3^t d$. Then, as above, we let $d = re$ to obtain the bound

$$S_3 \ll H^3 \sum_m \sum_r \frac{1}{r} \sum_{e < H/(\sqrt{3})^m r} 1 \ll H^4,$$

since the sum over e is $\ll H/(\sqrt{3})^m r$.

This completes the proof of the lemma. ■

3. Using the inequality. Given a solution of (1.1), $h_1, \ldots, k_4$, in a family, f, we define the polynomials

$$P(x) = (x-h_1)\ldots(x-h_4), \qquad Q(x) = (x-k_1)\ldots(x-k_4)$$

and the products $P_i(f) = P(k_i)$, $Q_i(f) = Q(h_i)$, for $i = 1, \ldots, 4$. These products are determined by f.

(3.1) We say that the family is "trivial" if and only if

$$P_i(f) = Q_i(f) = 0, \qquad \text{for } i = 1, \ldots, 4.$$

LEMMA 7. *A solution of* (1.1) *is trivial (in the sense of* (1.6)) *if and only if it is a member of a trivial family (in the sense of* (3.1)). *The number of trivial*

solutions of (1.1) *and* (1.4) *in integers is*

$$A(H) = 24H^4 - 72H^3 + 82H^2 - 33H. \tag{3.2}$$

Proof. If a solution of (1.1) is a trivial member of a family, f, then $P(x) = Q(x)$. Therefore $P_i(f) = P(k_i) = Q(k_i) = 0$, for $i = 1, \ldots, 4$ and similarly for the $Q_i(f)$. By (3.1), f is trivial.

If a solution of (1.1) is a member of a trivial family, f, then

$$0 = P_i(f) = P(k_i) \Rightarrow k_i \in \{h_1, \ldots, h_4\}, \qquad \text{for } i = 1, \ldots, 4.$$

By symmetry, $\{h_1, \ldots, h_4\} = \{k_1, \ldots, k_4\}$, as unordered sets. There are now four cases to consider.

Case 1: $h_1, \ldots, h_4$ take 4 distinct values. $k_1, \ldots, k_4$ are a permutation of these values, so the solution is trivial. There are $4!\,H!/(H-4)!$ such integer solutions of (1.1) and (1.4).

Case 2: $h_1, \ldots, h_4$ take 3 distinct values, A, B, C. The linear equation in (1.1) is of one of the forms $2A+B+C = A+2B+C$ or $2A+B+C = 2A+B+C$. From the former equation we find $A = B$, a contradiction. From the latter equation it is seen that the solution is trivial and there are $\binom{4}{2}\frac{4!\,H!}{2!\,(H-3)!}$ such integral solutions of (1.1) and (1.4).

Case 3: $h_1, \ldots, h_4$ take 2 distinct values, A, B. The linear equation in (1.1) takes the form $nA+(4-n)B = mA+(4-m)B$, $n \in \{1, 2\}$. Whence $(m-n)A = (m-n)B \Rightarrow m = n$ and the solution is trivial. For $n = 1$, 2 there are, respectively,

$$\binom{4}{3}\frac{4!\,H!}{3!\,(H-2)!}, \qquad \frac{1}{2}\binom{4}{2}\frac{4!\,H!}{2!\,2!\,(H-2)!}$$

such integral solutions of (1.1) and (1.4).

Case 4: If $h_1, \ldots, h_4$ all take the same value, A, then so do $k_1, \ldots, k_4$. The solution is trivial and there are H such integral solutions of (1.1) and (1.4). This completes the proof of the lemma. ■

LEMMA 8. *If δ, $K > 0$ and $|x_j|$, $|y_j| \leqslant K$ for $j = 1, \ldots, N$ and*

$$\sum_1^N (x_j^\beta - y_j^\beta) \ll \delta K^\beta, \qquad \text{for } \beta = 1, \ldots, N,$$

then

$$\prod_1^N (x_j^r - y_i^r) \ll \delta K^{Nr}, \qquad \text{for } i = 1, \ldots, N,\ 1 \leqslant r \in \mathbf{Z}.$$

The implied constant depends on r, N and the implied constant in the hypothesis.

Proof. For $0 \leqslant \beta \leqslant N$ we define the polynomials

$$S_\beta = S_\beta(X) = S_\beta(X_1, \ldots, X_N) = \sum_1^N X_j^\beta$$

and the polynomials $E_\beta = E_\beta(X) = E_\beta(X_1, \ldots, X_N)$

$$\prod_1^N (\lambda - X_j) = \sum_0^N E_{N-j} \lambda^j.$$

We then have

$$(3.3) \quad \left|\prod_{j=1}^N (x_j - y_i)\right| = \left|\prod_{j=1}^N (y_i - x_j) - \prod_{j=1}^N (y_i - y_j)\right| = \left|\sum_{j=0}^N (E_{N-j}(x) - E_{N-j}(y)) y_i^j\right|$$
$$\leqslant \sum_{j=0}^{N-1} |E_{N-j}(x) - E_{N-j}(y)| K^j.$$

Now $E_1(x) - E_1(y) = S_1(x) - S_1(y) \ll \delta K$, which proves the lemma in the case $r = N = 1$. For $N > 1$, we may take, as an induction hypothesis,

$$(3.4) \qquad |E_r(x) - E_r(y)| \ll 3N^{r-1} \delta K^r \quad \text{for } 1 \leqslant r < m \leqslant N.$$

From [7], p. 166, we have the equation, true for $1 \leqslant m \leqslant N$,

$$S_m + E_1 S_{m-1} + \ldots + E_{m-1} S_1 + E_m m = 0.$$

We find

$$|E_m(x) - E_m(y)| \leqslant \frac{1}{m} \sum_{r=0}^{m-1} |E_r(x) S_{m-r}(x) - E_r(y) S_{m-r}(y)|$$
$$\leqslant \frac{1}{m} \sum_{r=0}^{m-1} (|E_r(x) - E_r(y)| |S_{m-r}(x)| + |E_r(y)| |S_{m-r}(x) - S_{m-r}(y)|)$$

and, by the hypotheses of the lemma and the induction hypothesis, (3.4), this is

$$\leqslant \frac{1}{m} \sum_{r=0}^{m-1} \left(3N^{r-1} \delta K^r N K^{m-r} + \binom{N}{r} K^r \delta K^{m-r}\right)$$
$$\leqslant \delta K^m \sum_{r=0}^{m-1} \left(3N^r + \binom{N}{r}\right) / m$$
$$\leqslant \delta K^m \sum_{r=0}^{m-1} 4N^r/m \leqslant \delta K^m N^{m-1} (4(1 - N^{-m}))/(m(1 - N^{-1}))$$
$$\leqslant \delta K^m N^{m-1} (4(1 - N^{-2}))/(2(1 - N^{-1})),$$

since $1 < m \leqslant N$. Therefore, for $N \geqslant 2$, we get

$$|E_m(x) - E_m(y)| \leqslant \delta K^m N^{m-1} (2(1 + N^{-1})) \leqslant 3N^{m-1} \delta K^m$$

and the induction hypothesis, (3.4), is true with $m = N$.

By (3.3) and (3.4), with $m = N$,

$$\left|\prod_{j=1}^{N-1} (x_j - y_i)\right| \ll \sum_{j=0}^{N-1} 3N^{N-j-1} \delta K^{N-j} K^j \ll 3\delta K^N N^{N-1} (1 - N^{-1}) \ll 6N^{N-1} \delta K^N,$$

for $N \geqslant 2$. This proves the lemma in the case $r = 1$. From the above result we deduce

$$\left|\prod_{j=1}^N (x_j^r - y_i^r)\right| = \left|\prod_{j=1}^N (x_j - y_i)(x_j^{r-1} + x_j^{r-2} y_i + \ldots + y_i^{r-1})\right|$$
$$\leqslant (rK^{r-1})^N \left|\prod_{j=1}^N (x_j - y_i)\right| \ll 6r^N N^{N-1} \delta K^{rN},$$

which completes the proof of the lemma. ■

Our next lemma carries the burden of the proof. First we introduce some notation:

Given a family, f, and a member of f, $(h_1, \ldots, k_4)$, we can parametrise the family as $f = \{(h_1 + t, \ldots, k_4 + t);\ t \in \mathbf{R}\}$. Then we can define the real function of t,

$$(3.5) \qquad M_\beta(f, t) = \sum_1^4 ((h_i + t)^\beta - (k_i + t)^\beta),$$

where

$$\beta > 0 \quad \text{and} \quad t \geqslant -\min\{h_1, \ldots, k_4\}.$$

The graph of $M_\beta(f, t)$ is determined by β and f, up to a translation parallel to the t-axis.

LEMMA 9. *The number of non-trivial families, f, which contain a solution of (1.4) in integers, $(h_1, \ldots, k_4)$, and for which there is a value of t satisfying*

$$(3.6) \qquad 0 \leqslant h_i + t,\ k_i + t \leqslant H, \quad \textit{for } i = 1, \ldots, 4$$

and

$$(3.7) \qquad |M_{3/2}(f, t)| \leqslant \delta H^{3/2}, \quad |M_{1/2}(f, t)| \leqslant \delta H^{1/2}$$

is $\ll \delta H^4 \log^2 H$.

Proof. We first dispose of those f for which some $P_i(f)$ or $Q_i(f)$ is zero. In this case some h_i equals some k_j. There are 16 possible similar cases. We assume that $h_4 = k_4$. h_4 and k_4 eliminate each other from the equations (1.1) and (3.7). An appeal to Lemma 8, with $N = 3$, $x_i = \sqrt{h_i + t}$, $y_i = \sqrt{k_j + t}$, $K = \sqrt{H}$ (by (3.6)), δ the same as in (3.7) and $r = 2$, shows that

$$(3.8) \qquad (h_1 - k_3)(h_2 - k_3)(h_3 - k_3) \ll \delta H^3.$$

If this product were zero, then some h_i equals k_3, for $i \leqslant 3$. There are 3 possible similar cases. We assume that $h_3 = k_3$. Equations (1:1) reduce to

$h_1^2+h_2^2 = k_1^2+k_2^2$ and $h_1+h_2 = k_1+k_2$. Whence (k_1, k_2) is a permutation of (h_1, h_2) and $(h_1, \ldots, k_4)$ is a trivial solution of (1.1). As f is non-trivial, Lemma 7 shows that it contains no trivial solutions of (1.1). Therefore, the above product, (3.8), cannot be zero. f contains a member $(u_1, \ldots, v_4)$, where $u_i = h_i-k_3$, $v_i = k_i-k_3$, $-H \leqslant u_i, v_i \leqslant H$ (by (1.4)), for $i = 1, \ldots, 4$, $v_3 = 0$, $v_4 = u_4$ (as $k_4 = h_4$), $0 \neq u_1 u_2 u_3 \ll \delta H^3$ (by (3.8)) and v_1, v_2 are the roots of a quadratic polynomial with coefficients which are functions of u_1, u_2, u_3 only.

There are

$\ll H$ choices for $v_4 = u_4$,

$\ll \delta H^3 \log^2 H$ choices for u_1, u_2, u_3

and then

$\ll 1$ choices for v_1, v_2.

Taking account of all 16 similar cases, we see that

(3.9) there are $\ll \delta H^4 \log^2 H$ families, f, which satisfy the conditions of the lemma and have some $P_i(f)$ or $Q_i(f)$ equal to zero.

We now count those f with no $P_i(f)$ or $Q_i(f)$ equal to zero. We apply Lemma 8, with $N = 4$, $x_j = \sqrt{h_j+t}$, $y_j = \sqrt{k_j+t}$, $K = \sqrt{H}$ (by (3.6)), δ the same as in (3.7) and $r = 2$, to see that

$$0 \neq P_i(f), Q_i(f) \ll \delta H^4, \quad \text{for } i = 1, \ldots, 4. \tag{3.10}$$

We let $a = (h_1+ \ldots +h_4)/4 = (k_1+ \ldots +k_4)/4$, by (1.1). We suppose, that

$$|k_4-a| = \max\{\max_i |h_i-a|, \max_i |k_i-a|\}. \tag{3.11}$$

This is one of 8 similar cases. We may suppose, that

$$k_4 \leqslant a. \tag{3.12}$$

Otherwise, we count $-f$, replacing $(h_1, \ldots, k_4)$ by $(H-1-h_1, \ldots, H-1-k_4)$, so that (1.1), (1.4), (3,10), (3.11) and (3.12) are still all satisfied.

By (3.11) and (3.12) $k_4 \leqslant h_i, k_i$, for $i = 1, \ldots, 4$ and, by (3.10), $h_i \neq k_j$, for $i, j = 1, \ldots, 4$, so that

$$k_4 < h_i, \quad \text{for } i = 1, \ldots, 4. \tag{3.13}$$

We now dispose of those f for which $F(h_1-k_4, \ldots, h_4-k_4) = 0$. By Lemma 1

$$k_1 = k_2 = k_3. \tag{3.14}$$

We suppose that $h_4 \leqslant h_i$, for $i = 1, \ldots, 4$. There are 4 such similar cases. Let $y_i = h_i-(h_4-1)$, $x_i = k_i-(h_4-1)$, for $i = 1, \ldots, 4$. By (3.13), $1 \leqslant y_i \leqslant h_i-k_4$, for $i = 1, \ldots, 4$. Hence, by (3.10), $1 \leqslant y_1 y_2 y_3 y_4 \leqslant P_4(f) \ll \delta H^4$ and, as

$y_4 = 1$, we get $1 \leqslant y_1 y_2 y_3 \ll \delta H^4$. By (3.14) $x_1 = x_2 = x_3$. (1.1) becomes

$$3x_1^2+x_4^2 = y_1^2+y_2^2+y_3^2+1^2,$$

$$3x_1+x_4 = y_1+y_2+y_3+1.$$

There are $\ll \delta H^4 \log^2 H$ choices for the positive integers y_1, y_2, y_3. Then there are at most 2 choices for x_1, as a root of a known quadratic equation. x_4 is then determined. f contains the member $(y_1, y_2, y_3, 1, x_1, \ldots, x_4)$ and so (taking account of the 4 similar cases)

(3.15) there are $\ll \delta H^4 \log^2 H$ families, f, for which the conditions of the lemma, (3.10), (3.11), (3.12) and $F(h_1-k_4, \ldots, h_4-k_4) = 0$ all hold.

We now treat those f for which $F(h_1-k_4, \ldots, h_4-k_4) \neq 0$. By (3.13), we may suppose that $1 \leqslant h_4-k_4 \leqslant h_3-k_4 \leqslant h_2-k_4 \leqslant h_1-k_4$. There are 4! such similar cases. Suppose it were true that $h_2-k_4 < (h_1-k_4)/3$. Then

$$\begin{aligned} a-k_4 &= ((h_1-k_4)+ \ldots +(h_4-k_4))/4 \\ &< (h_1-k_4)(1+\tfrac{1}{3}+\tfrac{1}{3}+\tfrac{1}{3})/4 = (h_1-a)/2+(a-k_4)/2, \end{aligned}$$

so $a-k_4 < h_1-a$, which contradicts (3.11) with (3.12). Therefore,

$$h_2-k_4 \geqslant (h_1-k_4)/3.$$

By (1.4) and (3.13), $1 \leqslant h_i-k_4 < H$, for $i = 1, \ldots, 4$. Writing $x = h_1-k_4$, $y = h_2-k_4$, $v = h_3-k_4$, $w = h_4-k_4$, we have $F = F(x, y, v, w) \neq 0$, $1 \leqslant w \leqslant v \leqslant y \leqslant x < H$ and $y \geqslant x/3$. From (3.10), $P_4(f) \ll \delta H^4$, so $xyvw \ll \delta H^4$ and $x(x/3)vw \ll \delta H^4$. Putting

$$T = T(vw) = \min\{H, \sqrt{3c\delta H^4/vw}\}, \tag{3.16}$$

where c is the implicit constant in the order of magnitude bound, we have $x \leqslant T$. Therefore, by (2.2) and the discussion following Lemma 1, the number of these f is

$$\ll \sum_{1 \leqslant v,w \leqslant T} \sum_{\substack{0 \leqslant x,y \leqslant T \\ F>0 \\ F_1 \equiv 1 \pmod 3}} \sum_{\substack{d|F_1 \\ d \leqslant \sqrt{F_1}}}{}' \left(\frac{d}{3}\right). \tag{3.17}$$

Let $\mathscr{R} = \{(x, y) \in \boldsymbol{R}^2 \colon 0 \leqslant x, y \leqslant T, F(x, y, v, w) \geqslant 3^m d^2\}$. As in Lemma 6, we rewrite the sum over x, y, d as

$$\sum_m \sum_{1 \leqslant d < T/(\sqrt{3})^m} \left(\frac{d}{3}\right) \sum_{\substack{(x,y) \in \mathscr{R} \\ F \equiv b \pmod q}}{}' 1.$$

The sum over (x, y) is

$$\sum_{\substack{(\theta,\varphi) \bmod q \\ F(\theta,\varphi,v,w) \equiv b \pmod q}} \sum_{\substack{(x,y) \in \mathscr{R} \\ (x,y) \equiv (\theta,\varphi) \pmod q}} w(x, y),$$

where

$$w(x, y) = \begin{cases} 1, & F \geqslant 3^m d^2, \\ 1/2, & F = 3^m d^2. \end{cases}$$

We may apply Lemma 5, with $n = 2$, $d = 2$ to see that the inner sum here is

$$\frac{1}{q^2} V(\mathscr{M}) + O\left(\frac{T}{q} + 1\right).$$

If $q \leqslant 3T$, then we sum over (θ, φ) to see that

$$\sum_{\substack{(x,y) \in \mathscr{M} \\ F \equiv b \pmod q}}' 1 = \big(V(\mathscr{M}) + O(qT)\big) D(b \bmod q;\, v, w).$$

If $q > 3T$, then, as $d < t$, there is a unique integer t, with

(3.18) $\qquad 1 \leqslant t \leqslant m \quad$ and $\quad T < q_1 \leqslant 3T, \quad$ where $q_1 = 3^t d$.

$$\sum_{\substack{(x,y) \in \mathscr{M} \\ F \equiv b \pmod q}}' 1 \leqslant \sum_{\substack{(x,y) \in \mathscr{M} \\ F \equiv 0 \pmod{q_1}}} 1 = \big(V(\mathscr{M}) + O(q_1 T)\big) D(0 \bmod q_1;\, v, w) \quad \text{(as above)}$$

$$\ll T^2 D(0 \bmod q_1;\, v, w).$$

As in Lemma 6, we now have three sums to estimate.

First Sum:

$$S_1 = \sum_m \sum_{1 \leqslant d \leqslant T/3^m} \left(\frac{d}{3}\right) V(\mathscr{M}) D(b \bmod q;\, v, w).$$

By the multiplicativity of the (non-negative) densities and Lemma 3,

$$S_1 \ll \sum_m \frac{1}{3^m} \sum_{1 \leqslant d \leqslant T/3^m} \left(\frac{d}{3}\right) V(\mathscr{M}) \frac{1}{d} \sum_{r|d} \left(\frac{r}{3}\right) \frac{1}{r} c_r(3vw).$$

Note that, $c_r(3vw) = \sum_{s|(r,|3vw|)} \mu(r/s)s$ and $v, w > 0$, $(r, 3) = 1$ (when $\left(\frac{r}{3}\right) \neq 0$), so $(r, |3vw|) = (r, vw)$. We substitute $r = st$ and $d = re = ste$, to rewrite the sum over d as

$$\sum_{\substack{s|vw \\ (s,3)=1}} \frac{1}{s} \sum_{(t,3)=1} \frac{\mu(t)}{t^2} \sum_{1 \leqslant e \leqslant T/3^m st} \left(\frac{e}{3}\right) \frac{V(\mathscr{M})}{e}.$$

As in Lemma 6, we see that the sum over e is $\ll T^2$. As $|\mu(t)| \leqslant 1$, we may complete the sums over m and t, to see that

$$S_1 \ll T^2 \sum_{s|vw} (1/s) = T^2 \sigma(vw)/vw.$$

Second Sum:

$$S_2 = \sum_m \sum_{\substack{1 \leqslant d \leqslant T/3^m \\ (d,3)=1}} qTD(b \bmod q;\, v, w).$$

We repeat the substitutions made in S_1, to obtain the bound

$$S_2 \ll T \sum_m \sum_{s|vw} \sum_t \frac{1}{t} \sum_{1 \leqslant e \leqslant T/3^m st} 1.$$

The sum over e is $\leqslant T/3^m st$. As above, we may complete the sums over m and t to see that $S_2 \ll T^2 \sigma(vw)/vw$.

Third Sum:

$$S_3 = \sum_m \sum_{\substack{T/3^m < d \leqslant T/(\sqrt{3})^m \\ (d,3)=1}} T^2 D(0 \bmod q_1;\, v, w).$$

As the densities are non-negative, we may use (3.18) to replace one T by $3^t d$. We then make the substitutions made in S_1, to obtain the bound

$$S_3 \ll T \sum_m \sum_{s|vw} \sum_t (1/t) \sum_{T/3^m st < e \leqslant T/(\sqrt{3})^m st} 1.$$

The sum over e is $\leqslant T/(\sqrt{3})^m st$. We complete the sums over m and t, to see that $S_3 \ll T^2 \sigma(vw)/vw$.

From the bounds for the three sums and (3.16), we see that the right-hand side of (3.17) is

$$\ll \sum_{1 \leqslant v,w \leqslant T} T^2 \frac{\sigma(vw)}{vw} \ll \sum_{1 \leqslant v,w \leqslant H} \frac{\delta H^4}{vw} \frac{\sigma(v)\sigma(w)}{vw} \tag{3.19}$$

$$\ll \delta H^4 \Big\{ \sum_{1 \leqslant v \leqslant H} \frac{\sigma(v)}{v^2} \Big\}^2 \ll \delta H^4 \log^2 H.$$

The bounds (3.9), (3.15) and (3.19) complete the proof of the lemma. ■

Theorem 1. *The number of integer solutions of* (1.1), (1.2) *and* (1.4) *is* $\ll H^4 + \Delta H^5 \log^3 H$, *where H is a positive integer and* $\Delta \geqslant 0$.

Proof. By Lemma 7,

(3.20) the number of trivial solutions of (1.1) and (1.4) is $A(H) \ll H^4$ and every non-trivial solution of (1.1) is a member of a non-trivial family.

Let f be a non-trivial family containing an integer solution of (1.1) and (1.4), $(h_1, \ldots, k_4)$. The members of f are of the form $\boldsymbol{m}(t) = (h_1 + t, \ldots, k_4 + t)$, where $t \in \boldsymbol{R}$.

(3.21) The set of values of t, for which $\boldsymbol{m}(t)$ is a solution of (1.4), is an interval $\lceil L, U)$, of length $< H$.

Given $a \in \mathbf{R}$, we let

$$C(a: X_1, \ldots, X_8) = \prod_{\varepsilon_i = \pm 1} (\varepsilon_1 X_1 + \ldots + \varepsilon_8 X_8 - a).$$

This is a polynomial in $X_1, \ldots, X_8$ which is even in each X_i, has degree 2^8 and has the two factors $(X_1 + \ldots + X_4 - X_5 - \ldots - X_8 \pm a)$. Hence, $C(\Delta H^{3/2}; (h_1+t)^{3/2}, \ldots, (k_4+t)^{3/2})$ is a polynomial of degree at most 384 in t, with zeros when $M_{3/2}(f, t) = \pm \Delta H^{3/2}$.

(3.22) Therefore, the set of values of $t \geqslant L$ for which $m(t)$ is a solution of (1.2) is a closed set $\mathcal{S}$, with at most 385 boundary points.

$m(t)$ is a solution of (1.2) and (1.4) in integers if and only if $t \in \mathbf{Z} \cap [L, U) \cap \mathcal{S}$. By (3.22), $[L, U) \cap \mathcal{S}$ is a union of at most 385 disjoint intervals. Therefore, if $T(f)$ is the maximum number of integers contained in any one of these intervals, then

(3.23) The number of members of f which are integer solutions of (1.2) and (1.4) is $\leqslant 385T(f)$.

If $0 \leqslant T(f) \leqslant 1$, then we use (3.23) and Lemma 6 to see that

(3.24) the total number of solution from such "separated" families is $\ll H^4$.

If $T(f) \geqslant 2$, then there is an interval $I \subset [L, U) \cap \mathcal{S}$ containing $T(f)$ integers. We may suppose that $(h_1, \ldots, k_4)$ was chosen so that $t = 0$ is the least integer in I. Therefore $[0, T(f)-1] \subset [L, U) \cap \mathcal{S}$. By (3.21) and (3.23),

(3.25) $m(t)$ is solution of (1.2) and (1.4), for $t \in [0, T(f)-1]$.

Now, $2 \leqslant T(f) \leqslant H$, so we may divide up the range for $T(f)$ into the ranges

(3.26) $L \leqslant T(f) < 2L$, where L is a power of 2 and $2 \leqslant L \leqslant H$.

(3.27) There are $\ll \log H$ of these ranges.

Now,

$$\frac{d}{dt}(M_{3/2}(f, t)) = \tfrac{3}{2} M_{1/2}(f, t),$$

so that, by the First Mean-Value Theorem, there exists $\tau \in [0, T(f)-1]$ with

$$\big(M_{3/2}(f, T(f)-1) - M_{3/2}(f, 0)\big)/(T(f)-1) = \tfrac{3}{2} M_{1/2}(f, \tau).$$

Therefore, by (3.25) and (3.26),

$$|M_{1/2}(f, \tau)| \leqslant \tfrac{4}{3} \Delta H^{3/2}/(T(f)-1) \leqslant \left(\frac{8H}{3L}\Delta\right) H^{1/2},$$

$$|M_{3/2}(f, \tau)| \leqslant \Delta H^{3/2} \leqslant \left(\frac{8H}{3L}\Delta\right) H^{1/2}$$

and

$$0 \leqslant h_i + \tau,\ k_i + \tau < H, \quad \text{for } i = 1, \ldots, 4.$$

These non-trivial f satisfy the conditions of Lemma 9 with $\delta = (8H/3L)\Delta$ Therefore,

(3.28) there are $\ll (H/L)\Delta H^4 \log^2 H$ such f.

By (3.23),

(3.29) each such f has at most $770L$ members which are integer solutions of (1.2) and (1.4).

By (3.20), (3.27), (3.28) and (3.29), we see that the total number of non-trivial, integer solutions of (1.1), (1.2) and (1.4) from non-"separated" families is $\ll \Delta H^5 \log^3 H$. Combining this with (3.20) and (3.24) completes the proof of the theorem. ∎

THEOREM 2. *The number of integer solutions of* (1.1), (1.2), (1.4) *and* (1.5) *is* $A(H) + O(\Delta H^5 \log^3 H)$, *where* H *is a positive integer,* $\Delta \geqslant 0$ *and* $0 \leqslant \Delta_2 \leqslant \Delta H \log H$.

Proof. By Lemma 7, the number of trivial solutions is $A(H)$. The proof continues, as in Theorem 1, except that, when $0 \leqslant T(f) \leqslant 1$, we can apply Lemma 9 with $\delta = \Delta_2$. ∎

Added in proof. In that part of the proof of Theorem 1 which leads up to statement (3.22) I have failed to consider the possibility that the polynomial in t,

$$C(\Delta H^{3/2}; (h_1+t)^{3/2}, \ldots, (k_4+t)^{3/2}),$$

has all its coefficients equal to zero. In fact this cannot occur when $\Delta > 0$ (the only case we need consider). This can be seen by using the binomial theorem to investigate the asymptotic behaviour of the factors

$$\varepsilon_1 (h_1+t)^{3/2} + \ldots + \varepsilon_8 (k_4+t)^{3/2} - \Delta H^{3/2},$$

for large positive values of t.

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The distribution of powerful integers of type 4

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Let $k \geqslant 2$ be a fixed integer. A natural number n_k is said to be *powerful of type k* if $n_k = 1$ or if each prime factor of n_k divides it at least to the kth power. This paper is concerned only with the distribution of powerful integers n_4 of type 4. Such a number can be uniquely represented by

$$n_4 = a_0^4 a_1^5 a_2^6 a_3^7,$$

where a_1, a_2, a_3 are square-free numbers and $(a_i, a_j) = 1$ for $1 \leqslant i < j \leqslant 3$ We put

$$f_4(n) = \begin{cases} 1 & \text{for } n = n_4, \\ 0 & \text{for } n \neq n_4. \end{cases}$$

Let $N_4(x)$ denote the number of powerful integers of type 4 not exceeding x. Then

$$N_4(x) = \sum_{n_4 \leqslant x} 1 = \sum_{n \leqslant x} f_4(n).$$

For the Dirichlet series

$$F_4(s) = \sum_{n_4=1}^{\infty} \frac{1}{n_4^s} = \sum_{n=1}^{\infty} \frac{f_4(n)}{n^s}$$

we obtain

$$F_4(s) = \prod_p \left(1 + \frac{p^{-4s}}{1-p^{-s}}\right) = \frac{\zeta(4s)\zeta(5s)\zeta(6s)\zeta(7s)}{\zeta(10s)} \sum_{n=1}^{\infty} \frac{c_{11}(n)}{n^s}, \tag{1}$$

where the Dirichlet series $\sum c_{11}(n) n^{-s}$ is absolutely convergent for $\mathrm{Re}(s) > 1/11$. This shows that an asymptotic representation for $N_4(x)$ may be written in the form

$$N_4(x) = \sum_{\nu=4}^{7} \gamma_{\nu,4} x^{1/\nu} + \Delta_4(x), \tag{2}$$