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Some new estimates in the Dirichlet divisor problem

by

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1. Introduction and statement of results. For a fixed integer $k \geqslant 2$ the (general) Dirichlet divisor problem consists of the estimation of the function

$$\Delta_k(x) = \sum_{n\leqslant x} d_k(n) - \operatorname*{Res}_{s=1} x^s \zeta^k(s) s^{-1} = \sum_{n\leqslant x} d_k(n) - xP_{k-1}(\log x). \tag{1.1}$$

Here $d_k(n)$ is the divisor function which represents the number of ways n may be written as a product of k ($\geqslant 2$, fixed) factors, $P_{k-1}(t)$ is a suitable polynomial of degree $k-1$ in t, and $\zeta(s)$ is the Riemann zeta-function. The function $\Delta_k(x)$ in (1.1) is the error term in the asymptotic formula for $\sum_{n\leqslant x} d_k(n)$, that is, $\Delta_k(x) = o(x)$ as $x \to \infty$. Following standard notation, we define α_k and β_k as the infima of positive numbers a_k and b_k, respectively, for which

$$\Delta_k(x) \ll x^{a_k}, \qquad \int_1^x \Delta_k^2(y)\,dy \ll x^{1+2b_k}. \tag{1.2}$$

It is known that $(k-1)/(2k) \leqslant \beta_k \leqslant \alpha_k$ for all $k \geqslant 2$, and it was conjectured a long time ago that $\alpha_k = \beta_k = (k-1)/(2k)$ for all $k \geqslant 2$. For the time being the proof of this conjecture is hopeless, since $\beta_k = (k-1)/(2k)$ (for all $k \geqslant 2$) is equivalent to the Lindelöf hypothesis that $\zeta(\frac{1}{2}+it) \ll t^{\varepsilon}$ (see Ch. 13 of [8]). Many authors have given upper bound estimates for α_k and β_k, and for a comprehensive account of problems involving $\Delta_k(x)$, we refer the reader to Ch. 12 of [8] and Ch. 13 of [5]. The latter contains the sharpest known bounds, which for $k \geqslant 4$ are as follows:

$$\begin{gathered}\alpha_k \leqslant (3k-4)/(4k)\,(4 \leqslant k \leqslant 8),\ \alpha_9 \leqslant 35/54,\ \alpha_{10} \leqslant 41/60,\ \alpha_{11} \leqslant 7/10,\\ \alpha_k \leqslant (k-2)/(k+2)\,(12 \leqslant k \leqslant 25),\ \alpha_k \leqslant (k-1)/(k+4)\ (26 \leqslant k \leqslant 50),\\ \alpha_k \leqslant (31k-98)/(32k)\,(51 \leqslant k \leqslant 57),\ \alpha_k \leqslant (7k-34)/(7k)\ (k \geqslant 58).\end{gathered} \tag{1.3}$$

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Moreover, for k very large, the last bound is superseded by

$$\alpha_k \leqslant 1-\tfrac{1}{2}(Dk)^{-2/3}, \tag{1.4}$$

where $D > 0$ is such a constant for which

$$\zeta(\sigma+it) \ll t^{D(1-\sigma)^{3/2}} \log^{2/3} t \quad (t \geqslant t_0,\ 1/2 \leqslant \sigma \leqslant 1) \tag{1.5}$$

holds. From the work of H.-E. Richert [6] it is known that $D \leqslant 100$, and several authors (in unpublished works) have obtained smaller values of D. Explicit values of β_k are also contained in [5], and they are

$$\begin{aligned} &\beta_k = (k-1)/(2k) \quad \text{for } k = 2, 3, 4; \quad \beta_5 \leqslant 119/260 = 0.45769\ldots, \\ &\beta_6 \leqslant 1/2, \quad \beta_7 \leqslant 39/70 = 0.55714\ldots \end{aligned} \tag{1.6}$$

It is possible to obtain upper bounds for other β_k's also, but a general formula seems complicated. This is due to the fact that the bounds in question depend on the functions $M(A)$ and $m(\sigma)$, which are connected with power moments of $\zeta(s)$. These functions are defined as follows: For any fixed $A \geqslant 4$ the number $M(A)$ ($\geqslant 1$) is the infimum of all numbers M ($\geqslant 1$) such that

$$\int_1^T |\zeta(\tfrac{1}{2}+it)|^A\, dt \ll T^{M+\varepsilon}$$

for any $\varepsilon > 0$. Similarly, for $1/2 < \sigma < 1$ fixed we define $m(\sigma)$ ($\geqslant 4$) as the supremum of all numbers m ($\geqslant 4$) such that

$$\int_1^T |\zeta(\sigma+it)|^m\, dt \ll T^{1+\varepsilon}$$

for any $\varepsilon > 0$. Upper bounds for α_k and β_k in (1.3) and (1.6) were made in [5] to depend on upper bounds for $M(A)$ and lower bounds for $m(\sigma)$, especially on the latter. Thus in order to obtain new bounds for α_k and β_k we shall first refine the technique of Ch. 8 of [5] and obtain new lower bounds for $m(\sigma)$ (see § 3). Our results concerning α_k are contained in

THEOREM 1. *$\alpha_{10} < 27/40 = 0.675$, $\alpha_{11} < 0.6957$, $\alpha_{12} < 0.7130$, $\alpha_{13} < 0.7306$, $\alpha_{14} < 0.7461$, $\alpha_{15} < 0.75851$, $\alpha_{16} < 0.7691$, $\alpha_{17} < 0.7785$, $\alpha_{18} < 0.7868$, $\alpha_{19} < 0.7942$, $\alpha_{20} < 0.8009$, $\alpha_k \leqslant (63k-258)/(64k)$ for $79 \leqslant k \leqslant 119$, $\alpha_k \leqslant 1-165/(28k)$ for $k \geqslant 120$, and if (1.5) holds, then*

$$\alpha_k \leqslant 1-\tfrac{1}{3}\cdot 2^{2/3}(Dk)^{-2/3}. \tag{1.7}$$

The bounds of Theorem 1 improve, for $k \geqslant 10$, all the corresponding bounds in (1.3), which give e.g. $\alpha_{10} \leqslant 0.68333\ldots$, $\alpha_{11} \leqslant 0.7$, $\alpha_{12} \leqslant 0.71428\ldots$, $\alpha_{13} \leqslant 0.73333\ldots$, $\alpha_{14} \leqslant 0.75$ etc. Likewise (1.7) improves (1.4). As in [5], the bounds for α_k are not the optimal ones obtainable by our method, and small

improvements could be attained by further elaboration. It will also transpire from the proof of Theorem 1 that new bounds for α_k in the range $21 \leqslant k \leqslant 78$ may be obtained, but a general formula embodying the new estimates would be cumbersome, and it is for this reason that we omit it. We are also going to prove several new bounds for β_k. This is

THEOREM 2. *$\beta_5 \leqslant 0.45625$, $\beta_7 < 0.55469$, $\beta_8 < 0.60167$, $\beta_9 < 0.63809$, $\beta_{10} < 0.66717$, and if (1.5) holds, then*

$$\beta_k \leqslant 1-\tfrac{2}{3}(Dk)^{-2/3}. \tag{1.8}$$

New bounds for β_k when $k \geqslant 11$ may be also derived, but as in the case of upper bounds for α_k, a general formula appears to be complicated. Note that $\frac{2}{3} > \frac{1}{3}2^{2/3}$, so that the upper bound in (1.8) is smaller than the upper bound in (1.7).

Our last result concerns asymptotic formulas for the mean square of $\Delta_k(x)$ (see Ch. 13.6 of [5]). If we set

$$R_k(x) = \int_1^x \Delta_k^2(y)\, dy - ((4k-2)\pi^2)^{-1} \sum_{n=1}^{\infty} d_k^2(n)\, n^{-(k+1)/k} x^{(2k-1)/k}, \tag{1.9}$$

then it was established by K.-C. Tong [9] that under certain conditions, which involve power moments of $\zeta(s)$, $R_k(x)$ is of a lower order of magnitude than $x^{(2k-1)/k}$. In particular, it is known that

$$R_2(x) \ll x \log^5 x, \quad R_3(x) \ll x^{14/9+\varepsilon}.$$

It was stated in [5] that $R_k(x) \ll x^{(3k-3)/(2k)-\delta}$ cannot hold for any $\delta > 0$. We shall sharpen this result by proving

THEOREM 3. *If $R_k(x)$ is defined by (1.9), then for $k \geqslant 2$ fixed*

$$\begin{aligned} R_k(x) \ll\ & x^{(3k-3)/(2k)} (\log x)^{(k-1)(3-2k)/(2k)} \\ & \times (\log\log x)^{B_k} \exp(-D(\log\log\log x)^{1/2}) \end{aligned} \tag{1.10}$$

cannot hold if $B_k = (3k-3)(k\log k - k+1)/(2k) + 3k-3$ and $D > 0$ is a suitable constant.

It was conjectured in [5] that $R_k(x) \ll x^{(3k-3)/(2k)+\varepsilon}$ for $k \geqslant 2$, which in view of Theorem 3 would be essentially best possible. This conjecture, if true, is very strong, since by Lemma 2 of Section 5 it immediately implies the classical conjecture $\alpha_k = (k-1)/(2k)$ for $k \geqslant 2$.

2. Estimates of α_k and β_k when k is large. First we prove (1.7) and (1.8), which are of interest when k is large. These estimates do not depend on power moment estimates for $\zeta(s)$ (i.e., $M(A)$ or $m(\sigma)$), but only on (1.5) (see

Ch. 6 of [5] for a derivation and discussion of (1.5)). We shall start from the standard Perron inversion formula (see the Appendix of [5]) applied to

$$A(s) = \zeta^k(s) = \sum_{n=1}^{\infty} d_k(n) n^{-s} \quad \text{for } \sigma = \operatorname{Re} s > 1.$$

We have, for $X^\varepsilon \ll T \ll X^{1-\varepsilon}$, $\frac{1}{2}X \leqslant x \leqslant X$, $b = 1+\varepsilon$,

$$\sum_{n \leqslant x} d_k(n) = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} \zeta^k(s) x^s s^{-1} ds + O(X^{1+\varepsilon} T^{-1}).$$

Now we replace the segment of integration in the above formula by the segment $[\sigma - iT, \sigma + iT]$, where $1/2 < \sigma < 1$ will be suitably chosen later. We pass over the pole $s = 1$ of the integrand, which gives rise to the main term in (1.1). Writing $G = XT^{-1}$ it follows that

$$(2.1) \quad \Delta_k(x) = (2\pi)^{-1} \int_{-X/G}^{X/G} \zeta^k(\sigma+it) \frac{x^{\sigma+it}}{\sigma+it} dt + O\Big(GX^\varepsilon + G \int_\sigma^{1+\varepsilon} |\zeta(\alpha + iXG^{-1})|^k x^{\alpha-1} d\alpha\Big).$$

Suppose now that G satisfies, besides $X^\varepsilon \ll G \ll X^{1-\varepsilon}$, the additional condition

$$(2.2) \quad \int_\sigma^{1+\varepsilon} |\zeta(\alpha + iXG^{-1})|^k X^{\alpha-1} d\alpha \ll X^\varepsilon.$$

We use then (1.5) to obtain from (2.1)

$$(2.3) \quad \Delta_k(X) \ll X^\varepsilon\Big(G + X^\sigma \int_1^{X/G} t^{kD(1-\sigma)^{3/2}-1} dt\Big) \ll X^\varepsilon\big(G + X^\sigma (X/G)^{kD(1-\sigma)^{3/2}}\big).$$

We choose G so that the last two terms in (2.3) are equal. Thus $G = X^{1-f(\sigma)}$, where

$$f(\sigma) = (1-\sigma)/(1 + kD(1-\sigma)^{3/2}),$$

hence $f'(\sigma) = 0$ for

$$\sigma = \sigma_0 = 1 - 2^{2/3}(Dk)^{-2/3}.$$

We have

$$1 - f(\sigma_0) = 1 - \tfrac{1}{3} 2^{2/3} (Dk)^{-2/3},$$

hence (1.7) follows with $\sigma = \sigma_0$ in (2.1), provided that (2.2) holds. To see this note that $\zeta(\sigma+it) \ll \log^{2/3}|t|$ uniformly for $\sigma \geqslant 1$, and it follows from (1.5) that

$$\max_{\sigma_0 \leqslant \alpha \leqslant 1} |\zeta(\alpha + iXG^{-1})|^k X^{\alpha-1} \ll \max_{\sigma_0 \leqslant \alpha \leqslant 1} \{(X/G)^{kD(1-\alpha)^{3/2}} X^{\alpha-1}\} \log^k x \ll X^\varepsilon.$$

This is because

$$\max_{\sigma_0 \leqslant \alpha \leqslant 1} \exp\{(\tfrac{1}{3}\cdot 2^{2/3}(Dk)^{1/3}(1-\alpha)^{3/2} + \alpha - 1)\log X\} \leqslant 1,$$

since

$$\tfrac{1}{3}\cdot 2^{2/3}(Dk)^{1/3}(1-\alpha)^{3/2} + \alpha - 1 \leqslant 0$$

reduces to

$$1 \geqslant \alpha \geqslant 1 - 9\cdot 2^{-4/3}(Dk)^{-2/3},$$

and we have

$$\alpha \geqslant \sigma_0 = 1 - 2^{2/3}(Dk)^{-2/3} > 1 - 9\cdot 2^{-4/3}(Dk)^{-2/3}.$$

This proves (1.7).

The bound for β_k given by (1.8) will follow from

$$(2.4) \quad I = \int_{X/2}^{X} \Delta_k^2(x) dx \ll X^{1+2\eta+\varepsilon}, \quad \eta = 1 - \tfrac{2}{3}(Dk)^{-2/3},$$

on replacing X by $X2^{-j}$ and summing over $j = 0, 1, 2, \ldots$ We use (2.1), supposing again that (2.2) holds. This gives

$$I \ll X^{1+\varepsilon}G^2 + \int_{X/2}^{X} \Big| \int_{-X/G}^{X/G} \zeta^k(\sigma+it) \frac{x^{\sigma+it}}{\sigma+it} dt \Big|^2 dx$$

$$= X^{1+\varepsilon}G^2 + \int_{-X/G}^{X/G} \int_{-X/G}^{X/G} \frac{\zeta^k(\sigma+it)\zeta^k(\sigma-iu)}{(\sigma+it)(\sigma-iu)} \Big(\int_{X/2}^{X} x^{2\sigma+it-iu} dx\Big) dt\, du.$$

Using $|ab| \leqslant \frac{1}{2}(|a|^2+|b|^2)$, it further follows that

$$(2.5) \quad I \ll X^{1+\varepsilon}G^2 + X^{1+2\sigma} \int_{-X/G}^{X/G} |\zeta(\sigma+it)|^{2k} (\sigma^2+t^2)^{-1} \Big(\int_{-X/G}^{X/G} \frac{du}{1+|t-u|}\Big) dt$$

$$\ll X^{1+\varepsilon}G^2 + X^{1+2\sigma}\log X \Big(1 + \int_2^{X/G} |\zeta(\sigma+it)|^{2k} t^{-2} dt\Big)$$

$$\ll X^{1+\varepsilon}G^2 + X^{1+2\sigma+\varepsilon}\Big(1 + \int_2^{X/G} t^{2Dk(1-\sigma)^{3/2}-2} dt\Big)$$

$$\ll X^\varepsilon\big(XG^2 + X^{1+2\sigma} + X^{2\sigma+2Dk(1-\sigma)^{3/2}} G^{1-2Dk(1-\sigma)^{3/2}}\big),$$

provided that

$$(2.6) \quad 2Dk(1-\sigma)^{3/2} > 1.$$

This time we choose G to make the first and the third term in the above estimate equal. We obtain

$$G = X^{1-g(\sigma)}, \quad g(\sigma) = 2(1-\sigma)/(1 + 2Dk(1-\sigma)^{3/2}),$$

so that $g'(\sigma) = 0$ for $\sigma = \sigma_1 = 1-(Dk)^{-2/3}$, and (2.6) holds. Hence we choose $G = X^{1-g(\sigma_1)}$, where $1-g(\sigma_1) = \eta$, as given by (2.4). Since $\sigma_1 < 1-g(\sigma_1)$, (1.8) follows from (2.5), provided that (2.2) holds. This will in turn follow from

$$\max_{\sigma_1 \leqslant \alpha \leqslant 1} \{(XG^{-1})^{Dk(1-\alpha)^{3/2}} X^{\alpha-1}\}$$
$$= \max_{\sigma_1 \leqslant \alpha \leqslant 1} \exp \{(\tfrac{2}{3}(Dk)^{1/3}(1-\alpha)^{3/2}+\alpha-1)\log X\} \leqslant 1.$$

The inequality

$$\tfrac{2}{3}(Dk)^{1/3}(1-\alpha)^{3/2}+\alpha-1 \leqslant 0$$

reduces to $1 \geqslant \alpha \geqslant 1-\frac{9}{4}(Dk)^{-2/3}$, and we have

$$1 \geqslant \alpha \geqslant \sigma_1 = 1-(Dk)^{-2/3} > 1-\tfrac{9}{4}(Dk)^{-2/3},$$

so that (1.8) is proved.

3. New bounds for $m(\sigma)$. In this section we shall derive some new bounds for the function $m(\sigma)$ (defined in Section 1), which will lead then to bounds for α_k and β_k in Theorem 1 and Theorem 2. We shall refine the method which is exploited in Ch. 8 of [5]. Therein one of the key ingredients in estimating $m(\sigma)$ was the following

LEMMA 1. *Let $t_1 < \ldots < t_R$ be real numbers such that $T \leqslant t_r \leqslant 2T$ for $r = 1, \ldots, R$ and $|t_r - t_s| \geqslant \log^4 T$ for $1 \leqslant r \neq s \leqslant R$. If*

$$T^{\varepsilon} < V \leqslant \Big|\sum_{M<n\leqslant 2M} a(n) n^{-\sigma - it_r}\Big|$$

where $a(n) \ll M^{\varepsilon}$ for $M < n \leqslant 2M$, $1 \ll M \ll T^C$ ($C > 0$ a fixed number), then

$$R \ll T^{\varepsilon}(M^{2-2\sigma}V^{-2}+TV^{-f(\sigma)}),$$

where

$$f(\sigma) = \begin{cases} 2/(3-4\sigma) & \textit{for } 1/2 < \sigma \leqslant 2/3, \\ 10/(7-8\sigma) & \textit{for } 2/3 \leqslant \sigma \leqslant 11/14, \\ 34/(15-16\sigma) & \textit{for } 11/14 \leqslant \sigma \leqslant 13/15, \\ 98/(31-32\sigma) & \textit{for } 13/15 \leqslant \sigma \leqslant 57/62, \\ 5/(1-\sigma) & \textit{for } 57/62 \leqslant \sigma \leqslant 1-\varepsilon. \end{cases} \tag{3.1}$$

We shall indicate how for σ relatively close to 1 the last expression for $f(\sigma)$ may be replaced by a better one. Namely, one can take

$$f(\sigma) = \frac{2^l(l-2)+2}{2^l-1-2^l\sigma} \quad \text{for } 1-\frac{l-1}{2^l-2} \leqslant \sigma \leqslant 1-\frac{l}{2^{l+1}-2} \tag{3.2}$$

for any $l = 3, 4, \ldots,$ and also for $k \geqslant 3$

$$f(\sigma) = \frac{k}{1-\sigma} \quad \text{for } 1-\frac{k}{2^{k+1}-2} \leqslant \sigma \leqslant 1-\varepsilon \tag{3.3}$$

for any fixed $\varepsilon > 0$. Therefore the last value of $f(\sigma)$ in (3.1) may be replaced by an arbitrary number of values furnished by (3.2) for $l \geqslant 6$, plus a value of $f(\sigma)$ furnished by (3.3) with a suitable k. The proof is analogous to the proof of (3.1) given in [5], and therefore the details will be omitted. If as usual one defines

$$\mu(\sigma) = \inf\{c \geqslant 0\colon \zeta(\sigma+it) \ll |t|^c\}$$

for a given real σ, and $c(\sigma)$ is an upper bound for $\mu(\sigma)$, then it was shown in [5] that $f(\sigma)$ of Lemma 1 may be determined by the equations

$$2c(\theta)+1+\theta-2(1+c(\theta))\sigma = 0, \tag{3.4}$$

$$f(\sigma) = \frac{2(1+c(\theta))}{c(\theta)}. \tag{3.5}$$

Using the classical estimates (see [8]) $\mu(\sigma) \leqslant 1/(2L-2)$ for $\sigma = 1-l/(2L-2)$, $L = 2^{l-1}$, $l \geqslant 3$, and convexity of $\mu(\sigma)$ it follows that one may take

$$c(\theta) = \frac{2^{l-1}-1-2^{l-1}\theta}{l2^{l-1}-2^l+2} \quad \text{for } 1-\frac{l-1}{2^{l-1}-2} \leqslant \theta \leqslant 1-\frac{l}{2^l-2}, \tag{3.6}$$

and similarly one can take

$$c(\theta) = \frac{1-\theta}{k} \quad \text{for } 1-\frac{k}{2^k-2} \leqslant \theta \leqslant 1. \tag{3.7}$$

Substituting (3.6) and (3.7) in (3.4) and (3.5), we obtain (3.2) and (3.3), respectively.

We are now going to bound the function $m(\sigma)$ for the values $\sigma = \frac{27}{40}, \frac{5}{7}, \frac{3}{4}, \frac{5}{6}, \frac{7}{8}$ and $\frac{14}{15}$. It was shown in [5], Ch. 8 that to obtain bounds for $m(\sigma)$ it suffices to obtain bounds of the form $R \ll T^{1+\varepsilon}V^{-m(\sigma)}$, where R is the number of points t_r $(r = 1, \ldots, R)$ such that $|t_r| \leqslant T$, $|t_r - t_s| \geqslant \log^4 T$ for $1 \leqslant r \neq s \leqslant R$ and $|\zeta(\sigma+it_r)| \geqslant V > 0$ for any given V. Moreover, by (8.97) of [5] we have (with T^{ε} omitted for brevity)

$$R \ll TV^{-2f(\sigma)} + T^{\frac{(4-4\sigma)}{(1+2\sigma)}} V^{\frac{-12}{(1+2\sigma)}} + T^{\frac{4(1-\sigma)(\varkappa+\lambda)}{((2+4\lambda)\sigma-1+2\varkappa-2\lambda)}} V^{\frac{-4(1+2\varkappa+2\lambda)}{((2+4\lambda)\sigma-1+2\varkappa-2\lambda)}}$$
$$= R_1+R_2+R_3,$$

say. Here $f(\sigma)$ has the same meaning as in Lemma 1, and $(\varkappa, \lambda)$ is an exponent pair (see e.g. Ch. 2 of [5] for the definition and properties of exponent pairs). To avoid unwieldy expressions, we shall work primarily with the exponent pair $(\varkappa, \lambda) = (\frac{11}{53}, \frac{33}{53}) = BABA^2BA^2B(0, 1)$ in the usual notation of the A- and B-process in the theory of exponent pairs, since we found this exponent pair very convenient for our purposes.

For $\sigma = \frac{27}{40}$ we obtain $f(\frac{27}{40}) = \frac{25}{4}$, $c(\frac{27}{40}) = \frac{4}{45}$, hence $R_1 = TV^{-25/2}$, $R_2 = T^{26/47}V^{-240/47} \ll TV^{-x}$ for $V \ll T^{21/(47x-240)}$, which is certainly satisfied for

$$c\left(\frac{27}{40}\right) = \frac{4}{45} \leqslant \frac{21}{47x-240}, \quad x \leqslant \frac{1905}{188} = 10.1329\ldots,$$

whence $R_2 \ll TV^{-10.1329\ldots}$. With $(\varkappa, \lambda) = (\frac{11}{53}, \frac{33}{53})$ we obtain

$$R_3 \ll T^{1144/1273}V^{-11280/1273} \ll TV^{-y}$$

for $4/45 \leqslant 129/(1273y-11280)$, which gives $y \leqslant 50925/5092 = 10.000981\ldots$ and proves that $m(\frac{27}{40}) \geqslant 10.000981\ldots$ By a similar procedure we obtain for $\sigma = 5/7$ (using $c(\frac{5}{7}) = \frac{1}{14}$) that $m(\frac{5}{7}) \geqslant x$ for

$$x = \min\left(\frac{210}{17}, \frac{14(5+10\varkappa+2\lambda)}{3+14\varkappa+6\lambda}\right)$$

and for $\sigma = \frac{3}{4}$ (using $c(\frac{3}{4}) = \frac{1}{16}$) that $m(\frac{3}{4}) \geqslant x$ for

$$x = \min\left(\frac{72}{5}, \frac{8(3+6\varkappa+2\lambda)}{1+4\varkappa+2\lambda}\right).$$

Taking $(\varkappa, \lambda) = (\frac{4}{18}, \frac{11}{18})$ and $(\frac{11}{53}, \frac{33}{53})$ respectively, we obtain

$$m\left(\frac{5}{7}\right) \geqslant \frac{133}{11} = 12.090909\ldots, \quad m\left(\frac{3}{4}\right) \geqslant \frac{2328}{163} = 14.28220859\ldots$$

Similar calculations with $(\varkappa, \lambda) = (\frac{11}{53}, \frac{33}{53})$ yield $m(\frac{5}{6}) \geqslant 26.881578\ldots$, $m(\frac{7}{8}) \geqslant 39.8181\ldots$, $m(\frac{14}{15}) \geqslant 93.5880\ldots$ All these values improve the corresponding ones in Ch. 8 of [5], and for intermediate values of σ one may use the properties of $m(\sigma)$. Namely by Th. 8.1 of [5] one has, for $1/2 \leqslant \sigma_1 < \sigma < \sigma_2 < 1$,

$$m(\sigma) \geqslant \frac{m(\sigma_1)m(\sigma_2)(\sigma_2-\sigma_1)}{m(\sigma_2)(\sigma_2-\sigma)+m(\sigma_1)(\sigma-\sigma_1)}. \tag{3.8}$$

Further slight improvements on the above estimates could be obtained by using the recent algorithm of S. W. Graham [2] for minimizing certain expressions involving exponent pairs. For values of σ between 14/15 and 1, we can use the bound

$$c(\sigma) = \frac{1}{6}(1-\sigma) \quad \left(\frac{28}{31} \leqslant \sigma \leqslant 1\right)$$

and

$$R \ll TV^{-f(\sigma)} + T^{(2-2\sigma)/(4\sigma-1)}V^{-6/(4\sigma-1)} + T^{(12-12\sigma)/(34\sigma-15)}V^{-38/(34\sigma-15)}.$$

This is (8.99) and (8.100) of [5], and it gives $R \ll TV^{-x}$ for

$$x = \min\left(f(\sigma), \frac{30\sigma-12}{(4\sigma-1)(1-\sigma)}, \frac{238\sigma-124}{(34\sigma-15)(1-\sigma)}\right).$$

Hence using (3.2) with $l = 6$ and (3.3) with $k = 6$ we obtain

$$m(\sigma) \geqslant \begin{cases} \dfrac{258}{63-64\sigma} & \text{for } 14/15 \leqslant \sigma \leqslant c_0, \\ \dfrac{30\sigma-12}{(4\sigma-1)(1-\sigma)} & \text{for } c_0 \leqslant \sigma \leqslant 1-\varepsilon, \end{cases} \tag{3.9}$$

where $c_0 = \frac{1}{222}(171+\sqrt{1602}) = 0.95056\ldots$

4. Proof of other bounds for α_k and β_k. To obtain the remaining bounds for α_k in Theorem 1 we use

$$\Delta_k(x) \ll x^{\sigma+\varepsilon}, \tag{4.1}$$

which is the estimate proved in Ch. 13.3 of [5]. Here $1/2 < \sigma < 1$ is a constant for which $m(\sigma) = k$, where for $m(\sigma)$ one may take lower bounds for this function, such as those furnished by Section 3 and convexity. All the latter are easily seen to satisfy $m(\sigma) \leqslant 1/c(\sigma)$, where $c(\sigma)$ is given by (3.6) and (3.7), and this condition is necessary for (4.1) to hold. Using only $m(\frac{27}{40}) > 10$, $m(\frac{5}{7}) \geqslant 133/11$ and the bound in (3.8) we obtain

$$m(\sigma) > \frac{1463}{581-644\sigma} \quad \text{for } \frac{27}{40} \leqslant \sigma \leqslant \frac{5}{7}.$$

Setting the right-hand side equal to 11 and 12 and solving for σ we obtain $\alpha_{11} \leqslant 0.695652\ldots$ and $\alpha_{12} \leqslant 0.712862\ldots$ In general, from (3.8) and (4.1) we obtain

$$\alpha_k \leqslant \frac{k(m(\sigma_2)\sigma_2 - m(\sigma_1)\sigma_1) - m(\sigma_1)m(\sigma_2)(\sigma_2-\sigma_1)}{k(m(\sigma_2)-m(\sigma_1))} \tag{4.2}$$

for $13 \leqslant k \leqslant 26$, where $\sigma_1 = 5/7$, $\sigma_2 = 3/4$ or $\sigma_1 = 3/4$, $\sigma_2 = 5/6$. Hence from (4.2) we easily obtain the remaining upper bounds stated in Theorem 1 for $13 \leqslant k \leqslant 20$. It is obvious that, using the remaining values of $m(\sigma)$ calculated in Section 3 and (4.2), one can improve all the bounds given in (1.3). In particular, from the first bound in (3.9) one has

$$m(\sigma) \geqslant \frac{258}{63-64\sigma} \quad (14/15 \leqslant \sigma \leqslant c_0),$$

implying by (4.1)

$$\alpha_k \leqslant \frac{63k-258}{64k} \quad (79 \leqslant k \leqslant 119). \tag{4.3}$$

Likewise for $\sigma \geqslant 19/20 = 0.95$ we have $(30\sigma-12)/(4\sigma-1) \geqslant 165/28$, hence

$$m(\sigma) \geqslant \frac{165}{28(1-\sigma)} \quad (c_0 \leqslant \sigma \leqslant 1-\varepsilon),$$

implying by (4.1)

$$\alpha_k \leqslant \frac{28k-165}{28k} \quad (k \geqslant 120). \tag{4.4}$$

The bounds in (4.3) and (4.4) complete the proof of Theorem 1.

To obtain upper bounds for β_k one may note that $\beta_k \leqslant \sigma_1 = \sigma_1(k)$, if σ_1 satisfies

$$\int_T^{2T} |\zeta(\sigma_1+it)|^{2k}\,dt \ll T^{2-\delta} \tag{4.5}$$

for some $\delta = \delta(k) > 0$. This follows e.g. from Lemma 13.1 of [5], and was used in the proof of Th. 13.4 of [5]. To prove $\beta_5 \leqslant 73/160$ we observe first that, from $m(27/40) > 10$ and the functional equation for $\zeta(s)$, we have

$$\int_T^{2T} |\zeta(\tfrac{13}{40}+it)|^{10}\,dt \ll T^{11/4+\varepsilon},$$

while

$$\int_T^{2T} |\zeta(\tfrac{1}{2}+it)|^{10}\,dt \ll T^{7/4+\varepsilon}$$

by Th. 8.3 of [5]. Combining the preceding estimates by convexity we obtain

$$\int_T^{2T} |\zeta(\sigma+it)|^{10}\,dt \ll T^{(129-160\sigma)/28+\varepsilon} \quad (13/40 \leqslant \sigma \leqslant 1/2).$$

Since $(129-160\sigma)/28 < 2$ for $\sigma > 73/160$, one obtains $\beta_5 \leqslant 73/160 = 0.45625$ from (4.5). For the time being it does not seem possible to improve the bound $\beta_6 \leqslant 1/2$ of [5], but for $k > 6$ one can improve all the existing upper bounds for β_k by using the improved estimates for $m(\sigma)$, which were derived in Section 3. For k fixed let $c = c(k)$ be such a constant for which $M(2k) \leqslant 1+c$, and let $\sigma_0 = \sigma_0(k) > \frac{1}{2}$ satisfy $m(\sigma_0) \geqslant 2k$. Then we can show that

$$\beta_k \leqslant \frac{(c-1)\sigma_0+1/2}{c}. \tag{4.6}$$

Indeed, if

$$F(\sigma) = \frac{2c(\sigma_0-\sigma)+2\sigma_0-1}{2\sigma_0-1},$$

then $F(\frac{1}{2}) = 1+c$ and $F(\sigma_0) = 1$. Hence by convexity

$$\int_T^{2T} |\zeta(\sigma+it)|^{2k}\,dt \ll T^{F(\sigma)+\varepsilon} \quad (1/2 \leqslant \sigma \leqslant \sigma_0),$$

and $F(\sigma) < 2$ for $\sigma > (c\sigma_0-\sigma_0+\frac{1}{2})/c$, so that (4.6) follows from (4.5).

Following the proof of Th. 8.3 of [5] and using the new bound $\mu(\frac{1}{2}) \leqslant 9/56$ of E. Bombieri and H. Iwaniec [1], we obtain

$$M(2k) \leqslant 1+\tfrac{9}{28}(k-3) \quad (k \geqslant 7), \tag{4.7}$$

whence $c = c(k) = \frac{9}{28}(k-3)$. From the proof of the upper bounds for α_k we readily find that

$$\sigma_0(7) = 0.7461, \quad \sigma_0(8) = 0.7691, \quad \sigma_0(9) = 0.7868, \quad \sigma_0(10) = 0.8009.$$

It follows then immediately from (4.6) and (4.7) that

$$\beta_7 \leqslant 0.554688\ldots, \quad \beta_8 \leqslant 0.60166\ldots, \quad \beta_9 \leqslant 0.638088\ldots, \quad \beta_{10} \leqslant 0.667166\ldots,$$

and upper bounds for β_k when $k \geqslant 11$ may be calculated analogously.

5. Proof of Theorem 3. For the proof of Theorem 3 we need the following

LEMMA 2. *For $x^\varepsilon \leqslant H \leqslant x$ and $k \geqslant 2$ fixed we have uniformly*

$$\Delta_k(x) = H^{-1}\int_x^{x+H} \Delta_k(y)\,dy + O(H\log^{k-1}x). \tag{5.1}$$

Proof. We have

$$\begin{aligned} H^{-1}\int_x^{x+H} \Delta_k(y)\,dy - \Delta_k(x) &= H^{-1}\int_x^{x+H} (\Delta_k(y)-\Delta_k(x))\,dy \\ &\ll H\log^{k-1}x + H^{-1}\int_x^{x+H} \sum_{x<n\leqslant y} d_k(n)\,dy \\ &\ll H\log^{k-1}x + H^{-1}\int_x^{x+H} \sum_{x<n\leqslant x+H} d_k(n)\,dy \\ &\ll H\log^{k-1}x. \end{aligned}$$

Here we used (1.1) and the estimate

$$\sum_{x<n\leqslant x+H} d_k(n) \ll H\log^{k-1}x \qquad (x^\varepsilon \leqslant H \leqslant x),$$

which follows from the work of P. Shiu [7].

We proceed now to the proof of Theorem 3. Suppose that we have

$$(5.2)\qquad \int_1^x \Delta_k^2(y)\,dy = A_k x^{(2k-1)/k} + O\left(x^{(3k-3)/(2k)} G_k(x)\right),$$

where

$$A_k = (4k-2)^{-1}\pi^{-2}\sum_{n=1}^{\infty} d_k^2(n)\, n^{-(k+1)/k},$$

and $G_k(x)$ is a decreasing function for $x \geqslant x_0(k)$ such that $\log^{1-k}x \ll G_k(x) \ll 1$. We use (5.1) and the Cauchy–Schwarz inequality. Then (5.2) gives, for $x^\varepsilon \leqslant H \leqslant x$,

$$(5.3)\qquad \begin{aligned} \Delta_k^2(x) &\ll H^{-1}\int_x^{x+H} \Delta_k^2(y)\,dy + H^2\log^{2k-2}x \\ &= H^{-1}A_k\left((x+H)^{(2k-1)/k} - x^{(2k-1)/k}\right) \\ &\quad + O\left(x^{(3k-3)/(2k)}G_k(x)H^{-1} + H^2\log^{2k-2}x\right) \\ &\ll x^{(k-1)/k} + x^{(3k-3)/(2k)}G_k(x)H^{-1} + H^2\log^{2k-2}x. \end{aligned}$$

Choosing

$$H = x^{(k-1)/(2k)}\left(G_k(x)\log^{2-2k}x\right)^{1/3}$$

we obtain from (5.3)

$$(5.4)\qquad \begin{aligned} \Delta_k(x) &\ll x^{(k-1)/(2k)}\left(1 + (G_k(x)\log^{k-1}x)^{1/3}\right) \\ &\ll x^{(k-1)/(2k)}\left(G_k(x)\log^{k-1}x\right)^{1/3}. \end{aligned}$$

On the other hand, it is known (see J. L. Hafner [3], [4]) that, for $k \geqslant 2$,

$$(5.5)\qquad \Delta_k(x) = \Omega_+\left\{(x\log x)^{(k-1)/(2k)}(\log\log x)^{\gamma_k}\exp\left(-C(\log\log\log x)^{1/2}\right)\right\},$$

where $\gamma_k = (k-1)(k\log k - k + 1)/(2k) + k - 1$, $C > 0$. Comparing (5.4) and (5.5) we obtain

$$(5.6)\qquad (\log x)^{(k-1)/(2k)}(\log\log x)^{\gamma_k}\exp\left(-C(\log\log\log x)^{1/2}\right) \ll \left(G_k(x)\log^{k-1}x\right)^{1/3}.$$

Thus if we choose

$$G_k(x) = (\log x)^{(k-1)(3-2k)/(2k)}(\log\log x)^{3\gamma_k}\exp\left(-D(\log\log\log x)^{1/2}\right)$$

then $G_k(x)$ is decreasing for $x \geqslant x_0(k, D)$ and satisfies $\log^{1-k}x \ll G_k(x) \ll 1$, but (5.6) is false with a suitable $D > 0$. Hence we obtain the assertion of the theorem.

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