Similarly to the way we concluded from (5.8) that $\delta_1(\lambda_2-\lambda_1)\alpha \in \underline{K}$, we conclude from (5.9) that $\delta_1 \in \underline{K}$. This implies that $\alpha \in \underline{K}$.

As in [4] we can now prove that $\underline{K} = K$. Let $s \in S$. Repeating all our construction with $\underline{S}$ replaced by the semigroup generated by $\underline{S}$ and s, we observe that the latter semigroup is also contained in $\mathrm{PSC}(K)$. Thus $S \subseteq \mathrm{PSC}(K)$.

The proof is thereby completed.

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On Mellin–Ramanujan expansions

by

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1. Introduction. Ramanujan's trigonometrical sums are given by

$$c_q(n) = \sum_{\substack{1\le h\le q\\ (h,q)=1}} e^{-2\pi i nh/q} = \sum_{d\mid (n,q)} d\mu\left(\frac{q}{d}\right) \quad (q,\, n\in \boldsymbol{N}), \tag{1.1}$$

where $\mu(\cdot)$ is Möbius' μ-function. For fixed n the sequence $(c_q)_{q\ge 1}$ satisfies certain orthogonal relations. Thus in analogy to the Fourier theory of real functions the theory of Fourier–Ramanujan expansions

$$f(n) \sim \sum_{q\ge 1} a_q(f)\, c_q(n) \quad (n\in \boldsymbol{N}) \tag{1.2}$$

for arithmetical functions $f\colon \boldsymbol{N}\to \boldsymbol{C}$ (including the cases when f is multiplicative or additive) has been established by many authors [5], [6], [10], [12], [14]. The connection with the theory of Mellin integral transforms was studied by the author [7].

First special point-wise convergent expansions of the converse form

$$f(q) = \sum_{n\ge 1} a_n(f)\, c_q(n) \quad (q\in \boldsymbol{N}) \tag{1.3}$$

for arithmetical $f\colon \boldsymbol{N}\to \boldsymbol{C}$ are due to S. Ramanujan [9] and M. M. Crum ([4]; [11], pp. 10–12), e.g.

$$\sum_{d\mid q} \mu\left(\frac{q}{d}\right) d^{1-s} = \frac{1}{\zeta(s)} \sum_{n\ge 1} n^{-s} c_q(n) \tag{1.4}$$

($\mathrm{Re}\, s > 1$, $\zeta(s)$ being Riemann's zeta-function).

In contrast to (1.2) general criterions on the existence of the coefficients $a_n(f)$ in (1.3) even for special classes of f seem not to be known.

In the present paper we solve this open problem for the class of Dirichlet convolutions $g\colon \boldsymbol{N}\to \boldsymbol{C}$ defined for $q\in \boldsymbol{N}$ and $\mathrm{Re}\,\alpha \ge 0$ by

$$g_\alpha(q) = \sum_{d\mid q} \mu\left(\frac{q}{d}\right) d^{1-\alpha} w(d), \tag{1.5}$$

where $w = M^{-1}\{F(s)\}$ with $F(s) \in L(-\infty, +\infty)$ is an inverse Mellin transform. Using the Fourier analysis of [7] we prove that each g_α has an absolutely convergent expansion of the form (1.3) with coefficients $a_n(g_\alpha)$ defined by Mellin integrals.

In the applications we treat e.g. the sine and cosine integrals and the logarithms of Jacobi's elliptic theta-functions, including some new expansions for Euler's totient function φ, its Dirichlet inverse φ^{-1} and v. Mangoldt's Λ-function given in terms of the μ-function by

$$\varphi(n) = \sum_{d|n} \mu(d)\frac{n}{d}; \quad \varphi^{-1}(n) = \sum_{d|n} d\mu(d); \tag{1.6}$$
$$\Lambda(n) = \sum_{d|n} \mu(d)\log\frac{n}{d} \quad (n \in N).$$

2. Theorem. For $\mathrm{Re}\,\alpha \geq 0$ denote by C_α the class of all arithmetical functions $g_\alpha\colon N \to C$ defined by (1.5) where

(2.1) $w(x)$ is real-valued and piece-wise continuously differentiable on R^+,

(2.2) $F(s) = \int_0^\infty x^{s-1} w(x)\,dx$ absolutely convergent in the strip

$$\delta_1 < \sigma = \mathrm{Re}\,s < \delta_2 \quad (\delta_1, \delta_2 \in R),$$

(2.3) $F(s) \in L(-\infty, +\infty)$, i.e. $\int_{-\infty}^{+\infty} |F(\sigma+it)|\,dt < \infty \quad (\delta_1 < \sigma < \delta_2)$.

Note that the Dirichlet inverse of the divisor function $\sigma_s(n)$ is given by ([1], p. 39)

$$\sigma_s^{-1}(n) = \sum_{d|n} d^s \mu(d)\mu\left(\frac{n}{d}\right) \quad (n \in N,\ s \in C). \tag{2.4}$$

For the class C_α we prove the following

THEOREM. *Let $g_\alpha \in C_\alpha$. Then*

$$g_\alpha(q) = \sum_{n\geq 1} a_n(g_\alpha) c_q(n) \tag{2.5}$$

with

$$a_n(g_\alpha) = \frac{1}{2\pi i}\int_{(c)} F(s)\{\zeta(s+\alpha)\}^{-1} n^{-s-\alpha}\,ds, \tag{2.6}$$

(c) denoting the vertical line $(c-i\infty, c+i\infty)$, $c > 1$ and the Mellin–Ramanujan series in (2.5) being absolutely convergent for $q \in N$.

Conversely define by (2.6)

$$u_\alpha(q) = \sum_{d|q} \mu\left(\frac{q}{d}\right) d a_d(g_\alpha). \tag{2.7}$$

Then

$$u_\alpha(q) = \sum_{n\geq 1} n^{-\alpha} w(n) e_q(n) \tag{2.8}$$

with

$$e_q(n) = \sum_{d|n} \sigma_0^{-1}(d)\, c_q\left(\frac{n}{d}\right), \tag{2.9}$$

the series in (2.8) converging absolutely for $q \in N$.

COROLLARY. *Define for $q, n \in N$*

$$b_q(n) = \sum_{d|q} c_{q/d}(n) \tag{2.10}$$

and

$$h_q(n) = \sum_{d|q} e_{q/d}(n). \tag{2.11}$$

Then

$$q^{1-\alpha} w(q) = \sum_{n\geq 1} a_n(g_\alpha) b_q(n) \tag{2.12}$$

and

$$q a_q(g_\alpha) = \sum_{n\geq 1} n^{-\alpha} w(n) h_q(n) \tag{2.13}$$

with absolute convergence of the trigonometric series in (2.12), (2.13).

3. Proofs. In order to prove the Theorem observe that by (2.1)–(2.3) Mellin's inversion theorem ([3], p. 88) furnishes that $w(x)$ is the inverse Mellin transform of $F(s)$. Hence

$$w(x) = M^{-1}\{F(s)\} = \frac{1}{2\pi i}\int_{(c)} F(s) x^{-s}\,ds \quad (x \in R^+,\ c > 1). \tag{3.1}$$

By definition (1.5) we get for $\mathrm{Re}\,\alpha \geq 0$, $q \in N$

$$g_\alpha(q) = \frac{1}{2\pi i}\int_{(c)} F(s) \sum_{d|q} \mu\left(\frac{q}{d}\right) d^{1-s-\alpha}\,ds. \tag{3.2}$$

By (1.4) we have for $\mathrm{Re}(s+\alpha) > 1$

$$\zeta(s+\alpha)\sum_{d|q} \mu\left(\frac{q}{d}\right) d^{1-s-\alpha} = \sum_{n\geq 1} n^{-s-\alpha} c_q(n). \tag{3.3}$$

Hence

$$g_\alpha(q) = \frac{1}{2\pi i}\int_{(c)} F(s)\{\zeta(s+\alpha)\}^{-1}\sum_{n\geq 1} n^{-s-\alpha}c_q(n)\,ds. \tag{3.4}$$

Now let $\alpha = \alpha_1 + i\alpha_2$, $\alpha_1 \geq 0$ and $s = \sigma + it$, $\sigma > 1$. By (1.1) we have $|c_q(n)| \leq q$, and since $\{\zeta(s)\}^{-1}$ is bounded on any vertical line (c), $c > 1$, we get by (2.3)

$$\int_{-\infty}^{+\infty} |F(c+it)|\,|\{\zeta(c+\alpha_1+i(t+\alpha_2))\}^{-1}|\sum_{n\geq 1} n^{-c-\alpha_1}|c_q(n)|\,dt < \infty. \tag{3.5}$$

Hence by Lebesgue's dominated convergence theorem it is permissible to invert the order of summation and integration in (3.4) and we have

$$g_\alpha(q) = \sum_{n\geq 1} a_n(g_\alpha)\,c_q(n),$$

where $a_n(g_\alpha)$ is given by (2.6) with $a_n(g_\alpha) = O(n^{-c})$ $(n \to \infty,\ c > 1)$.

We now prove (2.8). By (2.6) we have

$$na_n(g_\alpha) = \frac{1}{2\pi i}\int_{(c)} F(s)\{\zeta(s+\alpha)\}^{-1} n^{1-s-\alpha}\,ds. \tag{3.6}$$

Hence by (2.7) and (3.3)

$$u_\alpha(q) = \frac{1}{2\pi i}\int_{(c)} F(s)\{\zeta(s+\alpha)\}^{-2}\sum_{n\geq 1} n^{-s-\alpha}c_q(n)\,ds. \tag{3.7}$$

But

$$\{\zeta(s)\}^{-k} = \sum_{n\geq 1} b_n(k)\,n^{-s} \quad (k \in \boldsymbol{N},\ \operatorname{Re} s > 1)$$

where the coefficients $b_n(k)$ are determined by

$$\{\zeta(s)\}^{-k} = \prod_p (1-p^{-s})^k = \prod_p \left(\sum_{\mu=0}^{k} (-1)^\mu \binom{k}{\mu} p^{-s\mu}\right) \quad (p \text{ prime}).$$

In the case $k = 2$ we have by (2.4)

$$b_n(2) = \sigma_0^{-1}(n) = \sum_{d|n} \mu(d)\,\mu\left(\frac{n}{d}\right).$$

Thus (3.7) becomes

$$u_\alpha(q) = \frac{1}{2\pi i}\int_{(c)} F(s)\sum_{n\geq 1} n^{-s-\alpha}\sigma_0^{-1}(n)\sum_{n\geq 1} n^{-s-\alpha}c_q(n)\,ds.$$

Now for $c > 1$

$$\sum_{n\geq 1} |\sigma_0^{-1}(n)|\,n^{-c} < \infty, \qquad \sum_{n\geq 1} n^{-c}|c_q(n)| < \infty.$$

Thus Dirichlet's multiplication rule and (2.9) yield

$$u_\alpha(q) = \frac{1}{2\pi i}\int_{(c)} F(s)\sum_{n\geq 1} n^{-s-\alpha}e_q(n)\,ds,$$

where the last series again converges absolutely for $\operatorname{Re}(s+\alpha) > 1$. Hence by (2.3), Lebesgue's dominated convergence theorem and (3.1) we get (2.8).

The Corollary follows by Möbius' inversion formula ([1], Th. 2.9). By (1.5) we get the inversion

$$q^{1-\alpha}w(q) = \sum_{d|q} g_\alpha\left(\frac{q}{d}\right),$$

and (2.12) with (2.10) result from (2.5) of the Theorem. Similarly we get by (2.7) the inversion

$$qa_q(g_\alpha) = \sum_{d|q} u_\alpha\left(\frac{q}{d}\right),$$

and (2.13) with (2.11) follow from (2.8).

4. Examples. We here consider some characteristic examples from the class C_1. By (1.5) and the Theorem

$$g_1(q) = \sum_{d|q} \mu\left(\frac{q}{d}\right)w(d) = \sum_{n\geq 1} a_n(g_1)\,c_q(n) \tag{4.1}$$

and

$$\sum_{d|q} \mu\left(\frac{q}{d}\right)da_d(g_1) = \sum_{n\geq 1} n^{-1}w(n)\,e_q(n) \tag{4.2}$$

with

$$na_n(g_1) = \frac{1}{2\pi i}\int_{(c)} F(s)\{\zeta(s+1)\}^{-1} n^{-s}\,ds \quad (c > 1). \tag{4.3}$$

Note further that by (1.4) and (2.9)

$$\sum_{n\geq 1} n^{-s}e_q(n) = \{\zeta(s)\}^{-1}\sum_{d|q} \mu\left(\frac{q}{d}\right)d^{1-s} \quad (\operatorname{Re} s > 1). \tag{4.4}$$

(a) The sine and cosine integrals are defined for $x \in \boldsymbol{R}^+$ by ([8], p. 267)

$$\operatorname{Si}(x) = \sum_{n\geq 0} (-1)^n\{(2n+1)(2n+1)!\}^{-1}x^{2n+1} = \int_0^x t^{-1}\sin t\,dt,$$

$$\mathrm{Ci}(x) = \gamma + \log x + \sum_{n\geq 1} (-1)^n \{(2n)(2n)!\}^{-1} x^{2n}$$
$$= -\int_x^\infty t^{-1} \cos t\, dt,$$
$$\mathrm{si}(x) = -\int_x^\infty t^{-1} \sin t\, dt = \mathrm{Si}(x) - \pi/2,$$

γ being Euler's constant.

By [8], pp. 193, 68, and Cauchy's theorem we have for $1 < \mathrm{Re}\, s < 2$ the Mellin transforms

(4.5) $$\frac{2}{\pi}\{\mathrm{Ci}(x)\sin x - \mathrm{si}(x)\cos x + x^{-1}\} = M^{-1}\{\sec(\pi s/2)\,\Gamma(s)\}$$

and

(4.6) $$w(x) = \log\left\{x^{1/2}\frac{\Gamma(x)}{\Gamma(x+\frac{1}{2})}\right\} + \tfrac{1}{8}x^{-1}$$
$$= M^{-1}\{\tfrac{1}{2}\pi^{-s}2^{-2s}\sec(\pi s/2)\,\Gamma(s)(2^{s+1}-1)\zeta(s+1)\}.$$

Hence by (4.1)–(4.3) we get in view of (4.4)–(4.6), (1.4) and (1.6) after some obvious computations the expansions

(4.7) $$\tfrac{1}{2}\Lambda(q) + \sum_{d|q}\mu\left(\frac{q}{d}\right)\log\frac{\Gamma(d)}{\Gamma(d+\frac{1}{2})}$$
$$= \frac{1}{\pi}\sum_{n\geq 1} n^{-1}\{\mathrm{si}(4\pi n) - 2\mathrm{si}(2\pi n)\}\, c_q(n) - \frac{1}{8q}\varphi^{-1}(q)$$

and conversely

(4.8) $$\frac{1}{\pi}\sum_{d|q}\mu\left(\frac{q}{d}\right)\{\mathrm{si}(4\pi d) - 2\,\mathrm{si}(2\pi d)\}$$
$$= \sum_{n\geq 1} n^{-1}\log\left\{n^{1/2}\frac{\Gamma(n)}{\Gamma(n+\frac{1}{2})}\right\} e_q(n) + \frac{3}{4\pi^2 q}\varphi^{-1}(q).$$

Note the special case $q = 1$ in (4.7). Since $\varphi^{-1}(1) = c_1(n) = 1$, $\Lambda(1) = 0$, and $\Gamma(\frac{3}{2}) = \frac{1}{2}\sqrt{\pi}$ we simply get

(4.9) $$\tfrac{1}{8} + \pi\log 2 - \tfrac{1}{2}\pi\log\pi = \sum_{n\geq 1} n^{-1}\{\mathrm{si}(4\pi n) - 2\,\mathrm{si}(2\pi n)\}.$$

(b) For $\tau \in H = \{z \in \mathbf{C} |\ \mathrm{Im}\, z > 0\}$ Dedekind's eta-function is defined by

(4.10) $$\eta(\tau) = q^{1/12}\prod_{n\geq 1}(1-q^{2n}), \quad q = e^{\pi i\tau}$$

and the logarithms of the elliptic theta-functions $\vartheta_i(\tau|0) = \vartheta_i(\tau)$ $(i = 2, 3, 4)$ of zero argument are given by ([7], p. 522)

(4.11) $$\log\vartheta_2(\tau) = \log 2 + \frac{\pi i}{12} + 5\log\eta(\tau) - 2\log\eta\left(\frac{\tau+1}{2}\right) - 2\log\eta\left(\frac{\tau}{2}\right),$$

(4.12) $$\log\vartheta_3(\tau) = 5\log\eta(\tau) - 2\log\eta(\tau/2) - 2\log\eta(2\tau),$$

(4.13) $$\log\vartheta_4(\tau) = \frac{\pi i}{12} + 5\log\eta(\tau) - 2\log\eta\left(\frac{\tau+1}{2}\right) - 2\log\eta(2\tau),$$

where $\log\eta(\tau) = \pi i\tau/12 + o(1)$ $(\tau \to i\infty)$. Take $\tau = ix$, $x \in \mathbf{R}^+$. Set $\eta(ix) = \tilde{\eta}(x)$, $\vartheta_i(ix) = \tilde{\vartheta}_i(x)$ and define

(4.14) $$\Psi(s) = \Gamma(s)\zeta(s)\zeta(s+1)(2\pi)^{-s} \quad (\mathrm{Re}\, s > 1).$$

Then we have the Mellin transform ([7], p. 522)

(4.15) $$w_1(x) = \log\{e^{\pi x/12}\tilde{\eta}(x)\} = M^{-1}\{-\Psi(s)\}.$$

Hence by (4.11)–(4.13) we get ([7], p. 523)

(4.16) $$w_2(x) = \log\{\tfrac{1}{2}e^{\pi x/4}\tilde{\vartheta}_2(x)\} = M^{-1}\{(1-2^{1-s})\Psi(s)\},$$

(4.17) $$w_3(x) = \log\tilde{\vartheta}_3(x) = M^{-1}\{-(1-2^{1-s})(1-2^{s+1})\Psi(s)\},$$

(4.18) $$w_4(x) = \log\tilde{\vartheta}_4(x) = M^{-1}\{(1-2^{s+1})\Psi(s)\}.$$

By (4.15)–(4.18) define the arithmetical functions

(4.19) $$g_1^{(k)}(q) = \sum_{d|q}\mu\left(\frac{q}{d}\right)w_k(d) \quad (k = 1, 2, 3, 4;\ q \in \mathbf{N}).$$

Thus by (4.1) and (1.6) we get the expansions

(4.20) $$\frac{\pi}{12}\varphi(q) + \sum_{d|q}\mu\left(\frac{q}{d}\right)\log\tilde{\eta}(d) = \sum_{n\geq 1} a_n(g_1^{(1)})\, c_q(n),$$

(4.21) $$-\log 2\sum_{d|q}\mu(d) + \frac{\pi}{4}\varphi(q) + \sum_{d|q}\mu\left(\frac{q}{d}\right)\log\tilde{\vartheta}_2(d) = \sum_{n\geq 1} a_n(g_1^{(2)})\, c_q(n),$$

(4.22) $$\sum_{d|q}\mu\left(\frac{q}{d}\right)\log\tilde{\vartheta}_3(d) = \sum_{n\geq 1} a_n(g_1^{(3)})\, c_q(n),$$

(4.23) $$\sum_{d|q}\mu\left(\frac{q}{d}\right)\log\tilde{\vartheta}_4(d) = \sum_{n\geq 1} a_n(g_1^{(4)})\, c_q(n),$$

where $\sum_{d|q}\mu(d) = 0$ $(q > 1)$, $= 1$ $(q = 1)$ and the coefficients (4.3) are given by ([7], p. 522)

(4.24) $$n a_n(g_1^{(1)}) = -\tfrac{1}{2}\{\coth(\pi n) - 1\},$$

(4.25) $$na_n(g_1^{(2)}) = \tfrac{1}{2}\{\coth(\pi n)-1\} - \{\coth(2\pi n)-1\},$$

(4.26) $$na_n(g_1^{(3)}) = \tfrac{1}{2}\{1-5\coth(\pi n)\} + \coth(\pi n/2) + \coth(2\pi n),$$

(4.27) $$na_n(g_1^{(4)}) = \tfrac{1}{2}\{\coth(\pi n)-1\} - \{\coth(\pi n/2)-1\}.$$

Hence by (4.2) we get conversely

(4.28) $$\sum_{d|q}\mu\left(\frac{q}{d}\right)da_d(g_1^{(1)}) = \sum_{n\geq 1} n^{-1}\log\{e^{\pi n/12}\tilde{\eta}(n)\}\,e_q(n),$$

(4.29) $$\sum_{d|q}\mu\left(\frac{q}{d}\right)da_d(g_1^{(2)}) = \sum_{n\geq 1} n^{-1}\log\{\tfrac{1}{2}e^{\pi n/4}\tilde{\vartheta}_2(n)\}\,e_q(n),$$

(4.30) $$\sum_{d|q}\mu\left(\frac{q}{d}\right)da_d(g_1^{(3)}) = \sum_{n\geq 1} n^{-1}\log\tilde{\vartheta}_3(n)\,e_q(n),$$

(4.31) $$\sum_{d|q}\mu\left(\frac{q}{d}\right)da_d(g_1^{(4)}) = \sum_{n\geq 1} n^{-1}\log\tilde{\vartheta}_4(n)\,e_q(n).$$

(c) Consider Jacobi's relation ([13], pp. 470–472)

(4.32) $$\vartheta_1'(\tau) = \vartheta_2(\tau)\,\vartheta_3(\tau)\,\vartheta_4(\tau) \qquad (\tau\in H),$$

where

$$\vartheta_1'(\tau) := \left.\frac{\partial\vartheta_1(\tau|z)}{\partial z}\right|_{z=0}.$$

For $\tau = ix$, $x\in \boldsymbol{R}^+$ set

$$\vartheta_1'(ix) = \tilde{\vartheta}_1'(x)$$

and

$$g_1^{(5)}(q) = \sum_{d|q}\mu\left(\frac{q}{d}\right)w_5(d) \qquad \text{with } w_5(x) = \log\{\tfrac{1}{2}e^{\pi x/4}\tilde{\vartheta}_1'(x)\}.$$

Then (4.20)–(4.31) yield

(4.33) $$-\log 2\sum_{d|q}\mu(d) + \frac{\pi}{4}\varphi(q) + \sum_{d|q}\mu\left(\frac{q}{d}\right)\log\tilde{\vartheta}_1'(d) = \sum_{n\geq 1} n^{-1}a_n(g_1^{(5)})\,c_q(n)$$

and

(4.34) $$\sum_{d|q}\mu\left(\frac{q}{d}\right)da_d(g_1^{(5)}) = \sum_{n\geq 1} n^{-1}\log\{\tfrac{1}{2}e^{\pi n/4}\tilde{\vartheta}_1'(n)\}\,e_q(n)$$

with

(4.35) $$na_n(g_1^{(5)}) = 3na_n(g_1^{(1)}) = -\tfrac{3}{2}\{\coth(\pi n)-1\}.$$

(d) Consider the discriminant $\Delta(\tau) = g_2^3(\tau) - 27g_3^2(\tau)$ $(\tau\in H)$, where g_2, g_3 are the invariants of the Weierstrass $\wp$-function.

Note the well-known fact ([2], p. 14) $\Delta(\tau) = (2\pi)^{12}\eta^{24}(\tau)$. For $\tau = ix$, $x\in\boldsymbol{R}^+$ set $\Delta(ix) = \tilde{\Delta}(x)$ and

$$g_1^{(6)}(q) = \sum_{d|q}\mu\left(\frac{q}{d}\right)w_6(d) \qquad \text{with } w_6(x) = \log\{(2\pi)^{-12}e^{2\pi x}\tilde{\Delta}(x)\}.$$

Then by (4.20) and (4.28) we get

(4.36) $$-12\log 2\pi\sum_{d|q}\mu(d) + 2\pi\varphi(q) + \sum_{d|q}\log\tilde{\Delta}(d) = \sum_{n\geq 1} n^{-1}a_n(g_1^{(6)})\,c_q(n)$$

and

(4.37) $$\sum_{d|q}\mu\left(\frac{q}{d}\right)da_d(g_1^{(6)}) = \sum_{n\geq 1} n^{-1}\log\{(2\pi)^{-12}e^{2\pi n}\tilde{\Delta}(n)\}\,e_q(n)$$

with

(4.38) $$na_n(g_1^{(6)}) = 24na_n(g_1^{(1)}) = -12\{\coth(\pi n)-1\}.$$

(e) The above formulae become more sophisticated if we consider the behaviour of η, ϑ_i and Δ under the generator $S\tau = \tau^{-1}$ $(\tau\in H)$ of the modular group ([13], pp. 475–476; [2], pp. 48–50)

$$\vartheta_2(\tau) = (-i\tau)^{-1/2}\vartheta_4(-\tau^{-1}); \qquad \vartheta_3(\tau) = (-i\tau)^{-1/2}\vartheta_3(-\tau^{-1})$$

$$\eta(\tau) = (-i\tau)^{-1/2}\eta(-\tau^{-1}); \qquad \Delta(\tau) = (-i\tau)^{-12}\Delta(-\tau^{-1}).$$

Take $\tau = ix$, $x\in\boldsymbol{R}^+$. Then we get e.g. from (4.20)–(4.23), (4.36) and (1.6) the expansions

(4.39) $$\frac{\pi}{12}\varphi(q) - \tfrac{1}{2}\Lambda(q) + \sum_{d|q}\mu\left(\frac{q}{d}\right)\log\tilde{\eta}(d^{-1}) = \sum_{n\geq 1} a_n(g_1^{(1)})\,c_q(n),$$

(4.40) $$-\log 2\sum_{d|q}\mu(d) + \frac{\pi}{4}\varphi(q) - \tfrac{1}{2}\Lambda(q) + \sum_{d|q}\mu\left(\frac{q}{d}\right)\log\tilde{\vartheta}_4(d^{-1}) = \sum_{n\geq 1} a_n(g_1^{(2)})\,c_q(n),$$

(4.41) $$-\tfrac{1}{2}\Lambda(q) + \sum_{d|q}\mu\left(\frac{q}{d}\right)\log\tilde{\vartheta}_3(d^{-1}) = \sum_{n\geq 1} a_n(g_1^{(3)})\,c_q(n),$$

(4.42) $$-\tfrac{1}{2}\Lambda(q) + \sum_{d|q}\mu\left(\frac{q}{d}\right)\log\tilde{\vartheta}_2(d^{-1}) = \sum_{n\geq 1} a_n(g_1^{(4)})\,c_q(n)$$

and finally

(4.43) $$-12\log 2\pi \sum_{d|q} \mu(d) + 2\pi\varphi(q) - 12\Lambda(q) + \sum_{d|q} \log \tilde{\Delta}(d^{-1}) = \sum_{n\geq 1} a_n(g_1^{(6)})\, c_q(n).$$

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Bilinear form of the remainder term in the Rosser–Iwaniec sieve of dimension $\varkappa \in (1/2, 1)$

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1. Introduction. It is well known that the remainder term in the linear sieve can be expressed in terms of bilinear forms $\sum_{m\leq M}\sum_{n\leq N} a_m b_n r(\mathcal{A}, mn)$. This result due to H. Iwaniec was established in 1977 (see [4]). This shape of the remainder term is more flexible than the conventional one and usually improves the estimates for the sifting function since the level of uniform distribution may be increased. On the other hand, it seems that an application of Rosser's weights would lead to the best sieving limit when the dimension of the sieve lies in the interval $(\frac{1}{2}, 1)$ (see [3]). In such circumstances it is natural to ask for the analogous result to that of paper [4] in the case when $1/2 < \varkappa < 1$. The aim of this paper is to prove that the remainder term in the latter case can be expressed in terms of bilinear forms defined on the product $[-1, 1]^{[M]} \times [-1, 1]^{[N]}$, where $M, N > 1$ satisfy

$$MN^{\beta-1} = \Delta.$$

Here $\beta = \beta(\varkappa)$ is the sieving limit and Δ reflects the level of uniform distribution.

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Notation. Let $\mathcal{A} = \{a_1, a_2, \ldots\}$ be a finite sequence of positive integers; $a_i \in \mathcal{A}$ means that a_i is an element of the sequence $\mathcal{A}$. For a given set $\mathcal{P}$ of primes and $z \geq 2$ we write

$$P(z) = \prod_{\substack{p\in\mathcal{P}\\ p<z}} p.$$

The main object in sieve theory is the sifting function $S(\mathcal{A}, \mathcal{P}, z)$ which represents the number of elements $a_i \in \mathcal{A}$ such that $(a_i, P(z)) = 1$.

For any $d \mid P(z)$ we consider the subsequence $\mathcal{A}_d$ which consists of those elements $a_i \in \mathcal{A}$ for which $a_i \equiv 0 \pmod d$.