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Metric Diophantine approximation with two restricted variables IV Miscellaneous results

by

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Dedicated to the memory of V. G. Sprindžuk

1. Introduction. An excellent introduction to the foundations of the metric theory of Diophantine approximation is provided by Sprindžuk's book [8]. In the first half the author deals with the question of whether the inequality

$$(1.1)\qquad |\alpha n-m| < \psi(n), \quad n,\, m\in \mathbf{Z},$$

has infinitely many solutions for almost all α (in the sense of Lebesgue measure on $\mathbf{R}$). Generalizations are also considered to several linear forms in many variables. Here $\psi(n)\in[0, \frac{1}{2})$ and the divergence of $\sum_{n=1}^{\infty}\psi(n)$ is always a necessary condition for there to be infinitely many solutions of (1.1) for almost all α. This condition is also sufficient when $\psi(n)$ is non-increasing, or if $\psi(n)=\psi'(n)\chi(n)$ where $\psi'(n)$ is non-increasing and $\chi(n)$ is the characteristic function of an infinite set of integers on which $\varphi(n)/n$ is bounded below on average by a positive constant. In this way one can produce 'non-linear' results, say by taking $\psi(n)=0$ unless n is a perfect square, or prime, etc. Sprindžuk remarks, however, at the end of the first half of his book, that the methods do not give completely non-linear results, because the variable m in (1.1) is always allowed to range over all integers. In [3], [4] and [5] we have sought to fill this gap in the theory by considering the inequality

$$(1.2)\qquad |\alpha n\pm m| < \psi(n), \quad n\in \mathcal{A},\, m\in \mathcal{B},$$

where $\mathcal{A}$ and $\mathcal{B}$ are given infinite sets of positive integers. For example, in [5] $\mathcal{A}$ and $\mathcal{B}$ are both the set of prime numbers. Here we shall continue these investigations to prove more general results.

In [3] we supposed that there exists a positive continuous function, $\varrho(m)$, which satisfies

$$\sum_{\substack{1\le m\le M\\ m\in\mathcal{B}}} 1 = \sum_{1\le m\le M}\varrho(m)+o\Big(\sum_{1\le m\le M}\varrho(m)\Big)$$

and for which there exist continuous functions f_1 and f_2 such that

$$0 < f_1(\theta) < \varrho(\theta m)/\varrho(m) < f_2(\theta)$$

for every $\theta > 0$ when $m \geqslant M(\theta)$. We then established that if

$$\sum_{n\in\mathscr{A}} \psi(n)\varrho(n) \tag{1.3}$$

converges, then there are only finitely many solutions of (1.2) for almost all α. As usual with such problems (see [8]) the difficulty lies in deducing that there are infinitely many solutions of (1.2) for almost all α when (1.3) diverges. It follows by the example constructed in [2] that at least one other condition, that is a moderately weak statement on the multiplicative nature of the sets $\mathscr{A}$, $\mathscr{B}$, is necessary.

In the main results of this paper, either $\varrho(m)$ is constant (so $\mathscr{B}$ has positive density), or $\mathscr{B}$ has positive lower asymptotic density (in which case ϱ may not even exist). We write

$$A(N) = \sum_{\substack{n\in\mathscr{A}\\ n\leqslant N}} 1, \qquad \Psi(N) = \sum_{\substack{n\in\mathscr{A}\\ n\leqslant N}} \psi(n),$$

$$\varphi(D, n) = \sum_{\substack{m=1\\ (m,n)\leqslant D}}^{n} 1.$$

We now list certain conditions which in suitable combinations will imply that (1.2) has infinitely many solutions for almost all α if $\Psi(\infty)$ diverges. In some cases (Theorem 1) the condition $(m, n) = 1$ can be added to (1.2).

Density conditions:

$$A(KN)/A(N) > c+1 \quad \text{for all } N \geqslant 1, \tag{1.4}$$

(here K, c are constants depending only on $\mathscr{A}$),

$$A(2N)-A(N) < C \quad \text{for all } N \geqslant 1, \tag{1.5}$$

(C depends only on $\mathscr{A}$),

$$\lim_{\substack{m\to\infty\\ m\in\mathscr{A}}} \lim_{\substack{n\to\infty\\ n\in\mathscr{A}}} (m, n) = \infty. \tag{1.6}$$

Multiplicative conditions:

$$\lim_{D\to\infty} \liminf_{N\to\infty} \frac{1}{A(N)} \sum_{\substack{n\in\mathscr{A}\\ n\leqslant N}} \frac{\varphi(D, n)}{n} = 1, \tag{1.7}$$

$$\liminf_{N\to\infty} \frac{1}{A(N)} \sum_{\substack{n\in\mathscr{A}\\ n\leqslant N}} \frac{1}{n} \sum_{\substack{m\in\mathscr{B}\\ (m,n)=1\\ \beta n\leqslant m\leqslant \gamma n}} 1 > c(\gamma-\beta), \tag{1.8}$$

for all $\gamma > \beta > 0$, where c is a positive constant depending only on $\mathscr{A}$ and $\mathscr{B}$.

ψ conditions:

$$\psi(n)/n \text{ is non-increasing,} \tag{1.9}$$

$$0 < \sigma_1 < \psi(m)/\psi(n) < \sigma_2 \quad \text{for } n \leqslant m < 2n,\ n \geqslant 1, \tag{1.10}$$

where σ_1, σ_2 are constants depending only on ψ.

We remark that (1.8) is a generalization of the condition that $\varphi(n)/n$ should be bounded below on average by a positive constant for $n\in\mathscr{A}$. We also note that, by Lemma 7A of [7], Chapter 3, we have

$$\frac{1}{N}\sum_{n\leqslant N} \frac{\varphi(D, n)}{n} = 1 + O(1/D + (\log D)(\log N)/N), \tag{1.11}$$

so (1.7) holds for every set with positive lower asymptotic density.

THEOREM 1. *Let $\mathscr{A}$ and $\mathscr{B}$ be sets of positive integers, where $\mathscr{B}$ has positive lower asymptotic density. Suppose that at least one of (1.4)–(1.6) holds for $\mathscr{A}$ and at least one of (1.9), (1.10) holds for ψ, a function with $\psi(n)\in[0, \frac{1}{2})$ and such that $\Psi(\infty)$ diverges. Then, if (1.8) holds, there are infinitely many solutions to*

$$|\alpha n \pm m| < \psi(n), \quad n\in\mathscr{A},\ m\in\mathscr{B},\ (m, n) = 1, \tag{1.12}$$

for almost all real α.

One may deduce many corollaries from the above result by noting that (1.8) holds for any $\mathscr{B}$ with positive lower asymptotic density when $\mathscr{A}$ is the set of primes, sums of two squares, values taken by a polynomial with integer coefficients (having no fixed divisor), etc. To see this, swap the order of summation in (1.8). Clearly, (1.4) holds in all the quoted examples. Also, for example, one could take $\mathscr{B}$ as the set of square-free integers and

$$\mathscr{A} = \{r^b\colon b = 1, 2, 3, \ldots\}$$

where r is a fixed integer at least equal to two. In this case both (1.5) and (1.6) hold. We state just one corollary.

COROLLARY. *Given any set $\mathscr{B}$ with positive lower asymptotic density, there are infinitely many convergents to the continued fraction expansion of almost all α having the form m/n where $m\in\mathscr{B}$ and n is a prime.*

THEOREM 2. *Let the hypotheses of Theorem 1 be given, but with (1.7) substituted for (1.8) and $\mathscr{B}$ be required to have positive density. Then there are infinitely many solutions to (1.2) for almost all α.*

COROLLARY. *Let $\mathscr{A}$ be a set with positive lower asymptotic density, $\mathscr{B}$ a set with positive density, and suppose that ψ satisfies at least one of (1.9), (1.10). Then there are infinitely many solutions of (1.2) if and only if $\sum_{n=1}^{\infty} \psi(n)$ diverges.*

This corollary follows from Theorem 2 using (1.11). The 'only if' part follows from [3], where the reader may also find an example which shows that the condition that A has positive density cannot be relaxed without additional information.

Finally we mention an extension of the result proved in [5], which may be established by suitably modifying the argument presented there.

THEOREM 3. *Let $\mathcal{P}$ be the set of primes, and $\mathcal{S}$ the set of numbers properly represented as the sum of two squares. Let $\psi(n) \in (0, \frac{1}{2})$ and suppose that at least one of (1.9), (1.10) holds. Then there are infinitely many solutions of (1.2) for almost all α, where $\mathcal{A}$, $\mathcal{B} \in \{\mathcal{P}, \mathcal{S}\}$, if and only if*

$$\sum_{n \geqslant 2} \frac{\psi(n)}{(\log n)^{1+\theta}}$$

diverges, where

$$\theta = \begin{cases} 0 & \text{if } \mathcal{A} = \mathcal{B} = \mathcal{S}, \\ \frac{1}{2} & \text{if } \mathcal{A} = \mathcal{P},\ \mathcal{B} = \mathcal{S} \text{ or vice versa}, \\ 1 & \text{if } \mathcal{A} = \mathcal{B} = \mathcal{P}. \end{cases}$$

COROLLARY. *Almost all real numbers have infinitely many convergents to their continued fraction expansion with numerator and denominator simultaneously the sum of two squares.*

We remark that the words 'almost all' in the corollary cannot be replaced with 'all irrational'. For example, all the convergents to $3+\sqrt{1/5}$ $(=[3, 2, \dot{4}])$ have numerator congruent to 3 (mod 4).

2. Proof of Theorem 1. The basic plan of the proof is the same as in [5]. It suffices to consider $\alpha > 0$, and so we write $\boldsymbol{R}^+ = \{x \in \boldsymbol{R}\colon x > 0\}$. Combining Lemmas 1 and 2 of [5] then gives the following result, where we use λ to denote Lebesgue measure on $\boldsymbol{R}$.

LEMMA 1. *Let $\mathcal{D}_m$ be a sequence of subsets of $\boldsymbol{R}^+$ and write*

$$\mathcal{D} = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} \mathcal{D}_m,$$

(thus $\mathcal{D}$ is the subset of $\boldsymbol{R}^+$ belonging to infinitely many $\mathcal{D}_m$). Suppose that for every finite open interval $\mathcal{J} \subset \boldsymbol{R}^+$, there is a sequence of subsets $\mathcal{B}_m$ of $\boldsymbol{R}^+$ with

$$\mathcal{B}_m \subseteq \mathcal{D}_m \wedge \mathcal{J},$$

$$\sum_{m=1}^{\infty} \lambda(\mathcal{B}_m) = \infty, \tag{2.1}$$

and

$$\limsup_{N\to\infty} \Big(\sum_{m=1}^{N} \lambda(\mathcal{B}_m)\Big)^2 \Big(\sum_{1 \leqslant m,n \leqslant N} \lambda(\mathcal{B}_m \wedge \mathcal{B}_n)\Big)^{-1} \geqslant \delta\lambda(\mathcal{J}) \tag{2.2}$$

for some positive constant δ (independent of $\mathcal{J}$). Then $\lambda(\boldsymbol{R}^+ \setminus \mathcal{D}) = 0$, in other words, almost all α belong to infinitely many $\mathcal{D}_m$.

We note that in problems where m can range over all integers it suffices to prove that (2.2) holds for one finite interval $\mathcal{J}$. This is not the case here however, as, although it is known that a 'zero-one' law operates when m is not restricted, we have given an example in [3] where no such law operates.

To prove Theorem 1 we shall apply Lemma 1 with

$$\mathcal{D}_m = \begin{cases} \varnothing & \text{if } m \notin \mathcal{A}, \\ \displaystyle\bigcup_{\substack{r \in \mathcal{B} \\ (m,r)=1}} \left(\frac{r-\psi(m)}{m}, \frac{r+\psi(m)}{m}\right) & \text{if } m \in \mathcal{A}, \end{cases} \tag{2.3}$$

and put $\mathcal{B}_m = \mathcal{D}_m \wedge \mathcal{J}$.

Hence (2.1) follows from (1.8), the conditions on ψ, and the divergence of $\Psi(\infty)$. If $\mathcal{J} = (\beta, \gamma)$ we obtain

$$\sum_{n=1}^{N} \lambda(\mathcal{B}_n) \geqslant 2c(\gamma-\beta)\Psi(N), \tag{2.4}$$

for all sufficiently large N. It is at this point that we require 'lim inf' in (1.8) rather than 'lim sup'. It then suffices to show that, for some absolute constant C,

$$\sum_{1 \leqslant m,n \leqslant N} \lambda(\mathcal{B}_m \wedge \mathcal{B}_n) \leqslant (C(\gamma-\beta)+o(1))\,\Psi(N)^2, \tag{2.5}$$

since $\mathcal{D}$ is the set of real numbers α for which there are infinitely many solutions of (1.12).

To obtain (2.5) we note that

$$\lambda(\mathcal{B}_m \wedge \mathcal{B}_n) \leqslant 2 \min\left(\frac{\psi(m)}{m}, \frac{\psi(n)}{n}\right) \sum_{\substack{r \sim m \\ r \in \mathcal{B} \\ (r,m)=(s,n)=1 \\ |rn-sm|<A}} \sum_{\substack{s \sim n \\ s \in \mathcal{B}}} 1, \tag{2.6}$$

where $A = \max(m\psi(n), n\psi(m))$, and $a \sim b$ means that $a/b \in (\beta, \gamma)$. We shall henceforth suppose that (1.9) holds, the proof when (1.10) holds is similar but the constant C in (2.5) will then depend on σ_1 and σ_2. We then suppose that $n \leqslant m$, $M \leqslant m < 2M$ which enables us to replace A by $2M\psi(n)$. Clearly that part of (2.6) with $n > m$ gives a similar order contribution. We then require the following result.

LEMMA 2. *Let $A > 0$, $M \geqslant 1$, and n a positive integer be given with $n < 2M$. Let $\mathcal{A}$ be a set of positive integers and (β, γ) an open interval in $\boldsymbol{R}^+$. Then the number of solutions of*

$$|ms - rn| < A \tag{2.7}$$

with $r \sim m$, $s \sim n$, $(n, s) = (r, m) = 1$, $m \neq n$, $m \in \mathscr{A}$, $M \leqslant m < 2M$, *is*

$$\leqslant AK(\gamma-\beta) \sum_{\substack{m \in \mathscr{A} \\ M \leqslant m < 2M}} 1 + O(E) \tag{2.8}$$

where K *is absolute, and*

$$E = \begin{cases} A(2M)-A(M) & \textit{if (1.6) holds,} \\ \min\bigl(M, n(A(2M)-A(M))\bigr) & \textit{otherwise.} \end{cases} \tag{2.8$'$}$$

Here the constant implied by the O *notation depends only on* (β, γ), *except when* (1.6) *holds when it also depends on the speed of growth of* (m, n).

We now complete the proof of Theorem 1 and defer the proof of Lemma 2 until the next section. Suppose, firstly, that (1.4) holds. Then, by (2.6) and (2.8), taking $E = M$, for some constant K' we have, for fixed n,

$$\sum_{n \leqslant m \leqslant N} \lambda(\mathscr{B}_m \wedge \mathscr{B}_n) \leqslant K'(\gamma-\beta) \sum_{\substack{n \leqslant m < 2N \\ m \in \mathscr{A}}} \psi(m)\psi(n) + O\Bigl(\sum_{n \leqslant m < 2N} \frac{\psi(m)}{m}\Bigr). \tag{2.9}$$

The first term on the right of (2.9) when summed over n gives the $C(\gamma-\beta)\Psi(N)^2$ of (2.5) (note that $\Psi(2N) \leqslant 2\Psi(N)$), while

$$\sum_{\substack{n \leqslant N \\ n \in \mathscr{A}}} \sum_{n \leqslant m < 2N} \frac{\psi(m)}{m} \leqslant \sum_{m \leqslant 2N} \frac{\psi(m)}{m} A(m) = O(\Psi(2N)) = O(\Psi(N)),$$

using (1.4). Since $\Psi(N) \to \infty$, this establishes (2.5).

Now suppose that (1.5) holds. In this case we take $E = n\bigl(A(2M)-A(M)\bigr)$. We then obtain (2.9), but now with an error term

$$n \sum_{\substack{n \leqslant m < 2N \\ m \in \mathscr{A}}} \frac{\psi(m)}{m}. \tag{2.10}$$

By (1.5)

$$\sum_{\substack{1 \leqslant n \leqslant m \\ n \in \mathscr{A}}} n \ll m.$$

Thus summing (2.10) over n gives an error which is $O(\Psi(N))$ as required. The proof when (1.6) holds follows similarly.

3. Proof of Lemma 2. We first need a lemma on the Fourier series of an approximation to the characteristic function of an interval (mod 1). As usual, $\|x\|$ denotes the distance from x to a nearest integer, and $e(x)$ represents $\exp(2\pi i x)$.

LEMMA 3. *Let* η, μ *real*, $0 < \eta < \frac{1}{2}$, *and* T *a positive integer be given. Then there is a function* $f(x)$ *such that:*

$$f(x) \geqslant 1 \quad \textit{if } \|x+\mu\| < \eta;$$

$$f(x) \geqslant 0 \quad \textit{if } \|x+\mu\| \geqslant \eta;$$

$$f(x) = 2\eta + T^{-1} + \sum_{\substack{n=-T \\ n \neq 0}}^{T} a_n e(nx);$$

$$a_n \ll \min(\eta, |n|^{-1}).$$

Proof. See Lemma 2.7 of [1].

Proof of Lemma 2. Write $\theta = \gamma-\beta$. We may suppose that $n^{-1} < \theta < \frac{1}{2}$, and $A < n$, for otherwise the proof can be greatly simplified. Let h be a divisor of n and consider the contribution of those m belonging to

$$\mathscr{A}_h = \{m \in \mathscr{A}: (m, n) = h,\ M \leqslant m < 2M\}.$$

We note that the conditions $m \neq n$, $(n, s) = (m, r) = 1$ ensure that there is no solution with $ms = rn$, so we must have $A \geqslant h$ for solutions to exist. The number of solutions of (2.7) is then no more than the number of solutions for v, s, a of

$$vs \equiv a \pmod g, \quad |a| \leqslant A/h, \quad a \neq 0, \quad vh \in \mathscr{A}_h, \quad s \sim n, \tag{3.1}$$

where $g = n/h$. If $h\theta \geqslant 1$ we can simply count the solutions in s for each pair v, a to obtain a bound

$$\frac{2A}{h}(h\theta+1) \sum_{m \in \mathscr{A}_h} 1 \leqslant 4A\theta \sum_{m \in \mathscr{A}_h} 1. \tag{3.2}$$

If $h\theta < 1$, write $B = [A/h]$ (where [ ] indicates integer part) and we then count solutions in v, s, a of

$$vs \equiv a \pmod g, \quad |a| \leqslant 2B, \quad vh \in \mathscr{A}_h, \quad s \sim n, \tag{3.3}$$

weighted with a factor $\nu(a) = (2B-|a|)/B$. Clearly this provides an upper bound for the solutions of (3.1). Now write $\mu = -(\beta+\gamma)h/2$ and let $\bar{v}$ denote a solution x of $vx \equiv 1 \pmod g$ (note that $vh \in \mathscr{A}_h$ implies that $(v, g) = 1$, while x only appears in functions having period g so any solution suffices). Solving (3.3) is then equivalent to solving

$$\left\|\frac{\bar{v}a}{g} + \mu\right\| < \frac{\theta h}{2}, \quad |a| \leqslant 2B, \quad vh \in \mathscr{A}_h.$$

We now apply Lemma 3 with $T = [(\theta h)^{-1}]$ to show that the number of solutions of (3.3) weighted with the factor $\nu(a)$ is no more than

$$2\theta h \sum_{|a| \leqslant 2B} \nu(a) \sum_{m \in \mathscr{A}_h} 1 + O\Bigl(\sum_{vh \in \mathscr{A}_h} h\theta \sum_{r=1}^{T} \delta(B, r\bar{v}/g)\Bigr), \tag{3.4}$$

where $\delta(B, x) = \min(B, B^{-1}\|x\|^{-2})$. To obtain (3.4) we have noted that, for $\|x\| \neq 0$,

$$\sum_{|a| \leqslant 2B} \nu(a) e(ax) = \frac{1}{B}\left(\frac{\sin 2\pi Bx}{\sin \pi x}\right)^2 \leqslant 4\delta(B, x).$$

Now the main term in (3.4) is

$$8A\theta \sum_{m \in \mathscr{A}_h} 1.$$

To bound the second term in (3.4) we note that its value is not decreased by changing the summation to include all v with $(v, g) = 1$ and $M \leqslant vh < 2M$ (this summation condition will be tacitly assumed below). Swapping the summation order then gives

$$\theta h \sum_{r=1}^{T} \sum_{v} \delta(B, \bar{v}r/g). \tag{3.5}$$

From $n\theta > 1$ we deduce $T < g$, so that $(r, g) < g$. Thus, if $1 \leqslant r \leqslant T$, for any V, we have

$$\sum_{\substack{v=V \\ (v,g)=1}}^{V+g-1} \delta(B, \bar{v}r/g) \leqslant 2(r, g) \sum_{v=1}^{g/(r,g)} \min\left(B,(g/(r, g))^2 B^{-1} v^{-2}\right) \leqslant 8g.$$

Hence (3.5) is bounded above by

$$8g(M/(hg)+1)T\theta h \leqslant 24M/h,$$

since $gh = n < 2M$.

A different bound for the second term of (3.4) may be obtained by noting that $\delta(B, \bar{v}r/g) \leqslant g$, and so the sum is bounded above by

$$\frac{n}{h} \sum_{m \in \mathscr{A}_h} 1. \tag{3.6}$$

Assembling our results so far then gives the following upper estimate for the number of solutions to (2.7):

$$\sum_{h|n} 8A\theta \sum_{m \in \mathscr{A}_h} 1 + O\left(\sum_{\substack{h|n \\ h\theta < 1}} \min\left(\frac{n}{h} \sum_{m \in \mathscr{A}_h} 1, \frac{M}{h}\right)\right).$$

Since

$$\sum_{h|n} \sum_{m \in \mathscr{A}_h} 1 = A(2M) - A(M)$$

this establishes (2.8) with (2.8)′ except in the case when (1.6) holds. To deal with this case we note that if (1.6) is valid then, given θ, there is a number R such that $(m, n) \geqslant \theta^{-1}$ for $m, n \geqslant R$. If we take the term (3.6) when $n < R$ and note

that $A_h = \varnothing$ for $h < \theta^{-1}$ when $n \geqslant R$, we obtain a bound from (3.2) and (3.4) of

$$8A\theta \sum_{m \in \mathscr{A}_h} 1 + O\big(R(A(2M) - A(M))\big),$$

as required. (R depends only on θ and the speed of growth of (m, n) as claimed.)

4. Proof of Theorem 2. The outline of the proof is the same as for Theorem 1. We write

$$\mathscr{D}_m = \begin{cases} \varnothing & \text{if } m \notin \mathscr{A}, \\ \bigcup_{r \in \mathscr{B}} \left(\dfrac{r-\psi(m)}{m}, \dfrac{r+\psi(m)}{m}\right) & \text{if } m \in \mathscr{A}. \end{cases}$$

Now we must be careful in our choice of the sets $\mathscr{B}_m$ and avail ourselves of Schmidt's technique used in [7], Chapter 3. Let $\mathscr{J} = (\beta, \gamma) \subset \boldsymbol{R}^+$, and suppose $\lambda(J) < 1$ for otherwise the proof could be simplified. Suppose that $\mathscr{B}$ has density c. Then, since (1.6) holds, we can pick D so that

$$\lim_{N \to \infty} \frac{1}{A(N)} \sum_{\substack{n \in \mathscr{A} \\ n \leqslant N}} \frac{\varphi(D, n)}{n} > 1 - \frac{c}{2}(\gamma - \beta).$$

Hence

$$\lim_{N \to \infty} \frac{1}{\Psi(N)} \sum_{\substack{m \in \mathscr{A} \\ m \leqslant N}} \frac{\psi(m)}{m} \sum_{\substack{n \sim m \\ n \in \mathscr{B} \\ (m,n) \leqslant D}} 1 > \frac{c}{2}(\gamma - \beta). \tag{4.1}$$

We put

$$\mathscr{B}_m = \begin{cases} \varnothing & \text{if } m \notin \mathscr{A}, \\ \bigcup_{\substack{r \in \mathscr{B} \\ (m,r) \leqslant D \\ r \sim m}} \left(\dfrac{r-\psi(m)}{m}, \dfrac{r+\psi(m)}{m}\right) & \text{if } m \in \mathscr{A}. \end{cases}$$

Thus, by (4.1),

$$\lim_{N \to \infty} \frac{1}{\Psi(N)} \sum_{m=1}^{N} \lambda(\mathscr{B}_m) > c(\gamma - \beta).$$

We must now obtain a satisfactory estimate for

$$\sum_{1 \leqslant m,n \leqslant N} \lambda(\mathscr{B}_m \wedge \mathscr{B}_n). \tag{4.2}$$

We may work as in the previous two sections, replacing $(r, m) = (n, s) = 1$ with $(r, m) \leqslant D$, $(s, n) \leqslant D$. This only brings in certain solutions of (2.7) with $ms = rn$ since we only used the co-primeness condition to show that no such solutions existed. We must therefore deal with an additional term

$$\sum_{\substack{1 \leqslant n \leqslant N \\ n \in \mathscr{A}}} \sum_{n < m \leqslant N} \frac{\psi(m)}{m} \sum_{\substack{r \sim m \\ r \in \mathscr{B} \\ nr = sm \\ (r,m),(s,n) \leqslant D}} \sum_{\substack{s \sim n \\ s \in \mathscr{B}}} 1. \tag{4.3}$$

Now, if we fix a pair m, n and consider $rn = sm$ subject to $(r, m), (s, n) \leqslant D$, then it is clear that unless $(m, n) \geqslant m/D$ there will be no solutions. When solutions do exist, then there are no more than $2n$ of them. Hence (4.3) is bounded above by

$$2 \sum_{\substack{1 \leqslant m \leqslant N \\ m \in \mathscr{A}}} \frac{\psi(m)}{m} \sum_{\substack{m \geqslant n \geqslant m/D \\ (m,n) \geqslant m/D}} n \leqslant 2D^2 \Psi(N),$$

since

$$\sum_{\substack{m \geqslant n \geqslant m/D \\ (m,n) \geqslant m/D}} n \leqslant \sum_{\substack{d \mid m \\ d \leqslant D}} md \leqslant D^2 m.$$

It follows that the sum (4.2) is bounded above by

$$K(\gamma - \beta)\, \Psi(N)^2 + O(\Psi(N))$$

for some absolute constant K and Theorem 2 then follows from Lemma 1. It should be noted that, as in [7], the parameter D is vital to exclude from $\mathscr{D}_m$ certain intervals which would have made $\lambda(\mathscr{B}_m \wedge \mathscr{B}_n)$ too large on average.

References

[1] R. C. Baker, *Diophantine Inequalities*, Clarendon Press, Oxford 1986.

[2] R. J. Duffin and A. C. Schaeffer, *Khintchine's problem in metric Diophantine approximation*, Duke Math. J. 8 (1941), 243–255.

[3] G. Harman, *Metric Diophantine approximation with two restricted variables I: Two square-free integers, or integers in arithmetic progressions*, Math. Proc. Camb. Phil. Soc. 103 (1988), 197–206.

[4] — *Metric Diophantine approximation with two restricted variables II: A prime and a square-free*, Mathematika 35 (1988), 59–68.

[5] — *Metric Diophantine approximation with two restricted variables III: Two prime numbers*, J. Number Theory 29 (1988), 364–375.

[6] A. Khintchine, *Eine Sätze über Kettenbrüche, mit Anwendungen auf die Theorie der Diophantischen Approximationen*, Math. Ann. 92 (1924), 115–125.

[7] W. M. Schmidt, *Diophantine Approximation*, Lecture Notes in Mathematics 785, Springer, Berlin 1980.

[8] V. G. Sprindžuk, *Metric Theory of Diophantine Approximation*, translated by R. A. Silverman, Winston/Wiley, New York 1979.

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