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The second Peano derivative as a composite derivative

by

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Abstract. Differentiable functions $f\colon R\to R$ which simultaneously have a second derivative in the Peano sense, f_2, and a second derivative in the composite sense, $(f')'_c$, are investigated. It is shown that $\{x\colon (f')'_c(x)\neq f_2(x)\}$ is a scattered set, i.e. a countable set not dense in any perfect set. As a corollary it follows that f_2 is the derivative of f' in the composite sense.

1. One of the long outstanding problems concerning Peano derivatives is the lack of a precise description of in what sense an $(n+1)$th Peano derivative can be considered as a derivative of the associated (n)th Peano derivative. In this paper we provide an answer to that problem in the case when $n=1$ and the derivative is taken in the composite sense. To make the presentation as readily intelligible as possible requires a little background information.

There is a wealth of information about certain aspects of the class of Peano derivatives. The interested reader should see for example the excellent survey [2]. It is also safe to say that all known properties of these functions are also properties of approximate derivatives, see [4], [7]. However, for approximately differentiable functions $f\colon R\to R$ and its approximate derivative, g, the following property is known to hold, [6],:

For any fixed perfect set P, there is an open interval, (a, b) having nonempty intersection with P, such that for any x in $(a, b)\cap P$,

$$\operatorname*{limit}_{\substack{h\to 0\\ x+h\in P}} \frac{f(x+h)-f(x)}{h} = g(x).$$

It is naturally reasonable to hope an analogous situation holds for the class of Peano derivatives. In [6], the above enclosed relationship for a pair of functions f and g was formalized by saying f was compositely differentiable to g and that g was a composite derivative of f.

Using that terminology, we can rephrase the previously mentioned problem as: Does the nth Peano derivative compositely differentiate to the $(n+1)$th Peano derivative?

Historically, Denjoy has provided partial answers to that problem, [1]. He established that if besides the $(n+1)$th P-derivative, the $(n+2)$th Peano is also assumed to exist, then the nth Peano derivative does indeed compositely differentiate to the $(n+1)$th Peano derivative. If this condition on $n+2$ is removed, the enclosed relationship above can only be verified pointwise for a residual subset of $(a, b)\cap P$.

Here we take an alternate approach for the first and second Peano derivatives. Namely we assume that we have a function $f\colon R\to R$ which has a derivative, f', a second Peano derivative, f_2 and that in addition, f' has a composite derivative, $(f')'_c$. We will also call this $(f')'_c$ the second composite derivative of f. It would be natural to expect that $f_2 = (f')'_c$. Unfortunately, this is not the case in general. Indeed in [4] the authors have constructed examples where f' is even approximately differentiable and still the function $(f')'_{ap} \neq f_2$. Based on the existence of such examples it is also natural to investigate whether the example of [4] can be modified to provide a counterexample to f_2 being a composite derivative of f'. Clearly such an example requires the existence a perfect set E such that $\{x\colon (f')'_c(x) \neq f_2(x)\}$ is dense in E. We show this is an impossibility and obtain as a corollary that f' compositely differentiates to f_2 if f' has a composite derivative.

2. We will prove the following theorem:

THEOREM. *Let $f\colon R\to R$ have a finite derivative f', a second Peano derivative f_2 and let f' have a composite derivative, $(f')'_c$ for all x in R. Then the set*

$$\{x\colon f_2(x) \neq (f')'_c(x)\}$$

is a scattered set, that is a countable G_δ. (Equivalently, it is not dense in any perfect set.)

Proof. Assume the contrary. That is, assume that there is a perfect set $Q\neq\emptyset$ such that $W = \{x\colon f_2(x)\neq (f')'_c(x)\}$ is dense in Q. Based on previously proven results in [3] and the definition of composite derivative we can make a number of preliminary statements and reductions.

(1) We may assume that Q has been suitably chosen so that for every $x\in Q$,

$$\lim_{\substack{h\to 0\\ x+h\in Q}} \frac{f'(x+h)-f'(x)}{h} = (f')'_c(x).$$

That is, by reducing Q, if necessary, we may assume f' compositely differentiates to $(f')'_c$ over Q.

(2) By results in [3], the function f' restricted to Q is extendable to a differentiable function g.

(3) If we define $G_{(x)} = \int_0^x g(t)\,dt$, we obtain a twice differentiable function on R.

(4) Next we consider the new function $F(x) = f(x)-G(x)$.

(a) $F(x)$ has a derivative $F' = f'-g$,

(b) $F(x)$ has a second derivative in the Peano sense

$$F_2 = f_2 - g',$$

(c) $F'(x)$ has a composite derivative $(F')'_c = (f')'_c - g'$.

(5) For all x in Q, $F'(x) = f'(x)-g(x) = 0$ and $(F')'_c(x) = 0$ and in particular,

$$\lim_{h\to 0}\frac{F(x+h)-F(x)}{h^2/2} = F_2(x).$$

(6) Suppose x_0 is a proof of Q at which $F_2(x_0)\neq 0$. Then, dependent on the sign of $F_2(x_0)$, F has a strict local maxima or minima at x_0. Since for any function the set of local maxima or minima points is countable we have that:

$$\{x\colon F_2(x)\neq 0\}\cap Q \text{ is countable}.$$

However $\{x\colon F_2(x)\neq 0\}\cap Q = \{x\colon f_2(x)\neq g'(x)\}\cap Q = \{x\colon f_2(x)\neq (f')'_2(x)\}\cap Q$.

(7) By known properties of Peano derivatives we may assume Q is nowhere dense; though our arguments following do not require that observation.

This ends the preliminary statements.

Now let $\varepsilon > 0$ be given. At any point x_0 in Q where $F_2(x_0) = 0$ there is a $\delta(x_0) > 0$ such that: $|F(x)-F(x_0)| \leqslant \varepsilon(x-x_0)^2$ whenever we have $0<|x-x_0| < \delta(x_0)$. For each $n = 1, 2, \ldots$, we define

$$A_n = \{x_0\colon |F(x)-F(x_0)| \leqslant \varepsilon(x-x_0)^2 \quad \text{whenever } 0<|x-x_0|<x_n\}\cap Q.$$

Since F is continuous, A_n is closed for each n. Further we have that $Q\setminus\bigcup_{n=1} A_n$ is a subset of $\{x\colon F_2(x)\neq 0\}\cap Q$, and is at most countable. Therefore by the Baire category theorem, there is an N and (a, b) such that $\emptyset\neq (a, b)\cap Q\subset A_N$. We may assume that $b-a<\frac{1}{n}$. Since we are assuming $\{x\colon F_2(x)\neq 0\}$ is dense in Q, we may select a point x_∞ from $(a, b)\cap Q\subset A_N$ such that $F_2(x_\infty)\neq 0$ and without loss of generality also assume

(i) $x_\infty = 0$,
(ii) $F(x_\infty) = 0$,
(iii) $F_2(x_\infty) > 0$.

Further we will assume x_∞ is a limit point of Q from the right at least.

We have $\lim\limits_{x\to 0}\left(\dfrac{F(x)}{x^2}\right) = \gamma > 0$, $0\in A_N$ which implies $\gamma\leqslant\varepsilon$. We claim we may assume that $\gamma<\varepsilon$. This follows from the argument that since $(a, b)\cap Q$ can be assumed to be perfect and $\{x\colon F_2(x) = 0\}$ is residual in $(a, b)\cap Q$, the Baire class one function F_2 must have $F_2(x) = 0$ at any point of relative continuity in Q. Hence, by making a further reduction, if necessary, we may assume that $F_2(x) < \varepsilon$ over Q. So $\gamma < \varepsilon$ can be assumed.

Hindsight into the upcoming arguments causes us to wish to choose three positive numbers α, β, Γ in such a way that the following hold simultaneously:

$$\beta = \gamma + \Gamma, \quad 0 < \Gamma < \sqrt{\gamma^2+\varepsilon^2} - \varepsilon,$$
$$\alpha = \gamma - \Gamma, \quad \beta < \varepsilon.$$

We next determine a $\delta > 0$ such that if $0 < x < \delta < b$ then

$$* \qquad ax^2 < F(x) < \beta x^2 .$$

Inside the open interval $(0, \delta)$ we have for any $x_0 \in A_N$ that

$$-\varepsilon(x-x_0)^2 \leqslant F(x) - F(x_0) \leqslant \varepsilon(x-x_0)^2 ;$$

so that

$$** \qquad F(x) \leqslant F(x_0) + \varepsilon(x-x_0)^2 < \beta x_0^2 + \varepsilon(x-x_0)^2 .$$

Since 0 is a right limit point of A_N we may select an x_0 in $A_N \cap (0, f)$ so that $0 < x_0 \left(\frac{\varepsilon}{\varepsilon-\alpha}\right) < \delta$. Then at $\left(\frac{\varepsilon}{\varepsilon-\alpha}\right) x_0$, ** above gives the inequality

$$F\left(\frac{\varepsilon}{\varepsilon-\alpha} \cdot x_0\right) < \beta x_0^2 + \varepsilon \left(\frac{\varepsilon}{\varepsilon-\alpha} \cdot x_0 - x_0\right)^2$$
$$= \beta x_0^2 + \varepsilon x_0^2 \left[\frac{\alpha}{\varepsilon-\alpha}\right]^2 .$$

If we now consider the point value $\beta x_0^2 + \dfrac{\varepsilon \cdot x_0^2 \cdot \alpha^2}{(\varepsilon-\alpha)^2}$, we claim this value is less than $\alpha\left(\dfrac{\varepsilon}{\varepsilon-\alpha} x_0\right)^2$.

Proof of claim.

$$\beta + \frac{\varepsilon\alpha^2}{(\varepsilon-\alpha)^2} < \frac{\alpha\varepsilon^2}{(\varepsilon-\alpha)^2}$$

if and only if

$$\beta < \frac{\alpha\varepsilon(\varepsilon-\alpha)}{(\varepsilon-\alpha)^2} = \frac{\alpha\varepsilon}{\varepsilon-\alpha} .$$

By our choices of α, β, Γ above we have:

$$(\gamma+\Gamma)[\varepsilon-(\gamma-\Gamma)] = \varepsilon\gamma + \varepsilon\Gamma - (\gamma+\Gamma)(\gamma-\Gamma) < \varepsilon(\gamma-\Gamma) ;$$

so substituting in α and β and simplifying we have

$$\beta < \frac{\varepsilon(\alpha)}{\varepsilon-\alpha} .$$

This implies that

$$F\left(\frac{\varepsilon}{\varepsilon-\alpha} x_0\right) < \alpha \left(\frac{\varepsilon}{\varepsilon-\alpha} x_0\right)^2$$

however we have this point in $(0, \delta)$ and by $*$ above this implies in turn that

$$F\left(\frac{\varepsilon}{\varepsilon-\alpha} x_0\right) > \alpha \left(\frac{\varepsilon}{\varepsilon-\alpha} x_0\right)^2$$

which is the contradiction we have been seeking.

COROLLARY. *Under the hypothesis of the theorem above, the function f' has the second Peano derivative of f as a composite derivative.*

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