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A characterization of bi-invariant Schwartz space multipliers on nilpotent Lie groups

by

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Abstract. A simply connected nilpotent Lie group, N, has a naturally defined Schwartz space, $\mathscr{S}(N)$. A continuous endomorphism on $\mathscr{S}(N)$ that commutes with both the right and left action of N on $\mathscr{S}(N)$ is called a bi-invariant Schwartz multiplier. It is shown that a bi-invariant Schwartz multiplier is given as convolution by a tempered distribution whose Fourier transform is a smooth, Ad*-invariant function on the dual of the Lie algebra of N, all of whose derivatives have polynomial bounds. This characterization is used to discuss summability methods for the eigenfunction expansion of certain hypoelliptic differential operators on nilmanifolds, and to give a criterion for local solvability of invariant differential operators on N.

We recall some well-known facts from the Schwartz theory on Euclidean spaces. Let X denote a finite-dimensional vector space with a fixed positive-definite inner product, and let $\mathscr{S}(X)$ denote the Schwartz space on X. We let $\mathscr{M}\mathscr{S}(X)$ denote the space of continuous endomorphisms of $\mathscr{S}(X)$ that commute with the action of X on $\mathscr{S}(X)$, i.e. $E \in \mathscr{M}\mathscr{S}(X)$ if $f \to Ef$ is continuous from $\mathscr{S}(X)$ to $\mathscr{S}(X)$ and if for each $x \in X$, $f \in \mathscr{S}(X)$, $l_x(Ef) = E(l_x f)$, where $l_x f(y) = f(y-x)$. It follows from the continuity that for $E \in \mathscr{M}\mathscr{S}(X)$ the functional D_E defined on $\mathscr{S}(X)$ by $D_E(f) = Ef(0)$ is an element of $\mathscr{S}^*(X)$, the space of tempered distributions. The group invariance implies that $Ef(x) = l_{-x}Ef(0) = E(l_x\check{f})(0) = \langle D_E, l_x\check{f}\rangle := D_E * f(x)$, where $\check{f}(y) = f(-y)$. Conversely, if $D \in \mathscr{S}^*(X)$, then one can easily see that $E_D\colon f \to D*f$ is a mapping of $\mathscr{S}(X)$ into the smooth functions on X that commutes with translation. A natural question arises: For which D is $E_D \in \mathscr{M}\mathscr{S}(X)$? The answer is given in terms of the Fourier transform.

For $f \in \mathscr{S}(X)$, $\hat{f}$ is the function defined on X^*, the dual space of X, by

$$\hat{f}(\xi) = \int_X f(x)\, e^{-2\pi i\langle \xi, x\rangle}\, dx.$$

The mapping $f \to \hat{f}$ establishes an isomorphism between $\mathscr{S}(X)$ and $\mathscr{S}(X^*)$, and allows one to define, for $D \in \mathscr{S}^*(X)$, the element $\hat{D}$ in $\mathscr{S}^*(X^*)$ by $\langle \hat{D}, f\rangle = \langle D, \hat{f}\rangle$. In [Sc], Schwartz proves that for $D \in \mathscr{S}^*(X)$, $E_D \in \mathscr{M}\mathscr{S}(X)$ if, and only if, $\hat{D}$ is a smooth function on X^* which has polynomial bounds for all derivatives. Furthermore, in this case $(D*f)\hat{}\,(\xi) = \hat{D}(\xi)\hat{f}(\xi)$. In this note we announce analogues of these results for nilpotent Lie groups.

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Let N denote a connected, simply connected nilpotent Lie group, with Lie algebra $\mathfrak{n}$. The exponential mapping $\exp\colon \mathfrak{n} \to N$ is a diffeomorphism, and in terms of the corresponding coordinates the left and right translations on N are polynomial mappings. Thus, if $\mathcal{S}(N)$ denotes the image under composition with exp of $\mathcal{S}(\mathfrak{n})$, the right and left actions of N on $\mathcal{S}(N)$ are continuous endomorphisms. $\mathcal{S}(N)$ is topologized so that composition with exp is an isomorphism from $\mathcal{S}(\mathfrak{n})$ to $\mathcal{S}(N)$. We denote by $\mathcal{S}^*(N)$ the dual of $\mathcal{S}(N)$, the space of tempered distributions on N.

For $f \in \mathcal{S}(N)$, the Fourier transform of f, $\hat{f}$, is defined on $\mathfrak{n}^*$, the dual of $\mathfrak{n}$, by

$$\hat{f}(\xi) = \int_{\mathfrak{n}} f(\exp(X))\, e^{-2\pi i \langle \xi, X\rangle}\, dX.$$

One has that $f \to \hat{f}$ is an isomorphism from $\mathcal{S}(N)$ onto $\mathcal{S}(\mathfrak{n}^*)$. For $D \in \mathcal{S}^*(N)$, $\hat{D}$ is defined on $\mathcal{S}(\mathfrak{n}^*)$ by $\langle \hat{D}, F\rangle = \langle D, \hat{F} \circ \log\rangle$, where log denotes the inverse of exp and, for $F \in \mathcal{S}(\mathfrak{n}^*)$ and $X \in \mathfrak{n}$,

$$\hat{F}(X) = \int_{\mathfrak{n}^*} F(\xi)\, e^{-2\pi i \langle \xi, X\rangle}\, d\xi.$$

Let Ad^* denote the coadjoint representation of N on $\mathfrak{n}^*$. A tempered distribution D on $\mathfrak{n}^*$ is said to be Ad^*-*invariant* if $\langle D, F \circ \mathrm{Ad}^*\rangle = \langle D, F\rangle$ for all $F \in \mathcal{S}(\mathfrak{n}^*)$. A tempered distribution D on N is said to be *bi-invariant* if $\langle D, r_{x^{-1}} f\rangle = \langle D, l_x f\rangle$ for all $f \in \mathcal{S}(N)$, where $r_x f(y) = f(yx)$ and $l_x f(y) = f(x^{-1} y)$ for all $x, y \in N$. A straightforward computation shows that an element $D \in \mathcal{S}^*(N)$ is bi-invariant if, and only if, $\hat{D}$ is Ad^*-invariant.

Let $\mathcal{MS}(N)$ denote the space of continuous endomorphisms on $\mathcal{S}(N)$ that commute with both right and left translations by elements of N. As in the Euclidean case, for each $E \in \mathcal{MS}(N)$ there is a $D_E \in \mathcal{S}^*(N)$ such that $Ef = D_E * f$, where, as before, $D_E * f(x) = \langle D_E, l_x \tilde{f}\rangle$, $\tilde{f}(y) = f(y^{-1})$. If $D \in \mathcal{S}^*(N)$ we denote by E_D the mapping defined on $\mathcal{S}(N)$ by $E_D f = D * f$.

THEOREM A. *For D in $\mathcal{S}^*(N)$, $E_D \in \mathcal{MS}(N)$ if, and only if, $\hat{D}$ is a smooth Ad^*-invariant function on $\mathfrak{n}^*$ with polynomial bounds on all derivatives.*

Let $\mathrm{PB}_N^\infty(\mathfrak{n}^*)$ denote the space of smooth Ad^*-invariant functions defined on $\mathfrak{n}^*$ with polynomial bounds on all derivatives. For integers $i, j \geq 0$, we define seminorms v_{ij} on $\mathrm{PB}_N^\infty(\mathfrak{n}^*)$ by

$$v_{ij}(\theta) = \sup_{|\alpha| \leq j} \sup_{\xi \in \mathfrak{n}^*} |\partial^\alpha \theta(\xi)|/(1 + \|\xi\|^2)^i,$$

where $d = \dim(\mathfrak{n})$, $\partial^\alpha = \partial_1^{\alpha_1} \ldots \partial_d^{\alpha_d}$, $\partial_1, \ldots, \partial_d$ are directional derivatives with respect to some basis of $\mathfrak{n}^*$, and $\|\ \|$ is a norm on $\mathfrak{n}^*$. The topology on $\mathrm{PB}_N^\infty(\mathfrak{n}^*)$ is determined by saying the sequence $\{\theta_k\}$ converges to zero if for each j there is an i such that $v_{ij}(\theta_k) \to 0$. The space $\mathcal{MS}(N)$ is topologized by saying a sequence $\{E_k\}$ converges to zero if for each $f \in \mathcal{S}(N)$, $E_k f \to 0$ in $\mathcal{S}(N)$.

THEOREM B. *The mapping $\mathcal{MS}(N) \to \mathrm{PB}_N^\infty(\mathfrak{n}^*)$: $E \to \hat{D}_E$ is a homeomorphism and an algebra isomorphism, the products being composition in $\mathcal{MS}(N)$ and pointwise multiplication in $\mathrm{PB}_N^\infty(\mathfrak{n}^*)$.*

For $\xi \in \mathfrak{n}^*$, let π_ξ denote the irreducible unitary representation of N corresponding to the Ad^*-orbit of ξ by the Kirillov theory. For $\theta \in \mathrm{PB}_N^\infty(\mathfrak{n}^*)$, let D_θ be the tempered distribution on N with Fourier transform θ.

THEOREM C. *For $\theta \in \mathrm{PB}_N^\infty(\mathfrak{n}^*)$, $f \in \mathcal{S}(N)$, and $\xi \in \mathfrak{n}^*$,*

$$\pi_\xi(D_\theta * f) = \theta(\xi)\, \pi_\xi(f).$$

As an application of these results, we consider the question of local solvability. Recall that a left-invariant differential operator L on N is said to be *locally solvable* if there is an open set $U \subset N$ such that $C_c^\infty(U) \subset L(C^\infty(U))$.

Let $o(\xi)$ denote the Ad^*-orbit in $\mathfrak{n}^*$ that contains ξ, and, having fixed a norm on $\mathfrak{n}^*$, set $|o(\xi)| = \inf\{\|\xi'\|\colon \xi' \in o(\xi)\}$. There is a linear subspace $V \subset \mathfrak{n}^*$ and a Zariski open subset $V_0 \subset V$ such that the elements in V_0 parametrize an open dense set of orbits in $\mathfrak{n}^*$. Representations corresponding to elements of V_0 are said to be *in general position.*

Suppose that N contains a discrete cocompact subgroup Γ. Then $L^2(\Gamma\backslash N)$ is a direct sum of subspaces $\mathcal{H}_\xi$ such that the restriction to $\mathcal{H}_\xi$ of right translation is a finite multiple of π_ξ. We denote by $(\Gamma\backslash N)\hat{}_0$ the elements of $\hat{N}$ appearing in this decomposition that are in general position.

THEOREM D. *Let L be a left-invariant differential operator on N. Suppose that for each $\pi_\xi \in (\Gamma\backslash N)\hat{}_0$, $\pi_\xi(L)$ has a bounded right inverse A_ξ on $\mathcal{H}_\xi$, and that the norm of A_ξ is bounded by a polynomial in $|o(\xi)|$. Then L is locally solvable.*

Although Theorems A and B are stated in terms of convolution between elements of $\mathcal{S}(N)$ and $\mathcal{S}^*(N)$, their proofs require the introduction of somewhat more general spaces. Let $\mathfrak{h}$ be a subspace of the center of $\mathfrak{n}$, and let $\lambda \in \mathfrak{h}^*$. We define the unitary character χ_λ on $H = \exp(\mathfrak{h})$ by $\chi_\lambda(\exp(X)) = e^{2\pi i \langle \lambda, X\rangle}$, and denote by $\mathcal{S}(N/H, \chi_\lambda)$ the space of all smooth functions f defined on N such that $f(xy) = \chi_\lambda(y) f(x)$ for all $x \in N$, $y \in H$, and such that $f \circ \exp|_{\mathfrak{k}} \in \mathcal{S}(\mathfrak{k})$, where $\mathfrak{k}$ is a complement to $\mathfrak{h}$ in $\mathfrak{n}$. The topology of $\mathcal{S}(N/H, \chi_\lambda)$ is defined by requiring that the mapping $f \to f \circ \exp|_{\mathfrak{k}}$ be a homeomorphism. Define $P_\lambda\colon \mathcal{S}(N) \to \mathcal{S}(N/H, \chi_\lambda)$ by

$$P_\lambda f(\exp(X)) = \int_{\mathfrak{h}} f(\exp(X+Y))\, \chi_\lambda(\exp(-Y))\, dY.$$

P_λ is an open surjection and thus its adjoint P_λ^* is an isomorphism of $\mathcal{S}^*(N/H, \chi_\lambda)$ into $\mathcal{S}^*(N)$.

Let $\mathfrak{h}^\perp$ be the annihilator of $\mathfrak{h}$ in $\mathfrak{n}^*$. For $\lambda \in \mathfrak{h}^*$ (identified with a subspace of $\mathfrak{n}^*$), there is a natural Schwartz space on $\mathfrak{h}^\perp + \lambda$, $\mathcal{S}(h^\perp + \lambda)$, given by composing elements of $\mathcal{S}(\mathfrak{h}^{\perp})$ with translation by $-\lambda$. Considering $\mathcal{S}(N/H, \chi_\lambda)$ and $\mathcal{S}(\mathfrak{h}^{\perp}+\lambda)$ as subspaces of $\mathcal{S}^*(N)$ and $\mathcal{S}^*(\mathfrak{n}^*)$ respectively, the Fourier transform is defined on these spaces and $f \to \hat{f}$ is an isomorphism of $\mathcal{S}(N/H, \chi_\lambda)$ onto $\mathcal{S}(\mathfrak{h}^{\perp}+\lambda)$ and of $\mathcal{S}(\mathfrak{h}^{\perp}+\lambda)$ onto $\mathcal{S}(N/H, \chi_{-\lambda})$. Also, for $D \in \mathcal{S}^*(N/H, \chi_\lambda)$, $(P_\lambda^* D)^\wedge = R_\lambda^* \hat{D}$, where R_λ: $\mathcal{S}(\mathfrak{n}^*) \to \mathcal{S}(\mathfrak{h}^{\perp}+\lambda)$ is restriction. Thus $(P_\lambda^* D)^\wedge$ is supported on $\mathfrak{h}^{\perp}-\lambda$ and has no normal derivatives.

For $f \in \mathcal{S}(N/H, \chi_\lambda)$ and $D \in \mathcal{S}^*(N/H, \chi_{-\lambda})$, the convolution $D * f$ is defined by setting $D*f(x) = \langle D, l_x \check{f} \rangle$ for each $x \in N$. Suppose now that $D \in \mathcal{S}^*(N)$ and $f \in \mathcal{S}(N)$. One can use Abelian Fourier analysis to study the mapping defined on $\mathfrak{z}$, the center of $\mathfrak{n}$, by $Y \to D*f(\exp(X+Y))$. If this mapping is in $\mathcal{S}(\mathfrak{z})$, then

$$D * f(\exp(X)) = \int_{\mathfrak{z}^*} P_\lambda(D*f)(\exp(X))\, d\lambda,$$

for appropriately normalized Lebesgue measure $d\lambda$. Furthermore, $P_\lambda(D*f) = D_\lambda * P_\lambda f$, where D_λ is the element of $\mathcal{S}^*(N/H, \chi_{-\lambda})$ whose Fourier transform is the restriction to $\mathfrak{h}^{\perp}+\lambda$ of $\hat{D}$. Thus, convolution between elements of $\mathcal{S}^*(N)$ and $\mathcal{S}(N)$ decomposes into convolution between elements of $\mathcal{S}^*(N/H, \chi_{-\lambda})$ and $\mathcal{S}(N/H, \chi)$ in such a way that smoothness and growth conditions on $\hat{D}$, $D \in \mathcal{S}^*(N)$, are inherited by $\hat{D}_\lambda$, $D_\lambda \in \mathcal{S}^*(N/H, \chi_{-\lambda})$. One then proceeds by induction on the dimension of N/H. Of course, this requires maintaining considerable control of the various seminorm estimates that appear in the decompositions.

Remarks. The sufficiency of the condition in Theorem A was first proved by R. Howe in [Ho], and indeed, the ideas presented there are the foundation of this work. Theorem C was proved for the case where θ is a polynomial by A. Kirillov in [K]. In [CG], L. Corwin and F. Greenleaf proved Theorem D with the additional assumption that all the representations in general position were induced from a common normal subgroup. One-sided Schwartz multipliers have been studied by L. Corwin in [C].

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1. Preliminaries. Let N denote a connected, simply connected nilpotent Lie group with Lie algebra $\mathfrak{n}$. Let $[X, Y]$ denote the Lie bracket of elements $X, Y \in \mathfrak{n}$. Denote by ad the adjoint representation of $\mathfrak{n}$ on $\mathfrak{n}$, i.e. $\operatorname{ad}(X)(Y) = [X, Y]$. The *rank* of $\mathfrak{n}$, r, is the smallest integer s such that $(\operatorname{ad}(X))^s = 0$ for all $X \in \mathfrak{n}$.

The exponential mapping, denoted by exp, is a diffeomorphism of $\mathfrak{n}$ onto N. For $X, Y \in \mathfrak{n}$, define $C(X, Y)$ by $\exp(C(X, Y)) = \exp(X)\exp(Y)$. The

Campbell–Hausdorff formula (cf. [S]) gives

$$C(X, Y) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sum \frac{\tau(X^{p_1}, Y^{q_1}, \ldots, X^{p_n}, Y^{q_n})}{A(p_1, q_1, \ldots, p_n, q_n)}$$

where the second sum runs over the integers $p_i, q_i \geq 0$ with $p_i + q_i \geq 1$ for $i = 1, \ldots, n$ and

$$A(p_1, q_1, \ldots, p_n, q_n) = \sum_{i=1}^{n} (p_i+q_i)\, p_1!\, q_1! \ldots p_n!\, q_n!,$$

$$\tau(X^{p_1}, Y^{q_1}, \ldots, X^{p_n}, Y^{q_n}) = [[\ldots[[[X^{p_1}, Y^{q_1}]\, X^{p_2}]\, Y^{q_2}] \ldots X^{p_n}]\, Y^{q_n}],$$

where, by definition,

$$[X^p, Y^q] = [[\ldots[[[X, X], X] \ldots X], Y], \ldots, Y],$$

where X occurs p times and Y occurs q times, and $\tau(X) = X$, $\tau(Y) = Y$. It is clear that the coefficient of t in $C(Y, tX)$ is given by

$$\sum_{j=1}^{r-1} c_j (\operatorname{ad}(Y))^j (X) = E(Y)(X), \tag{1.1}$$

where the c_j are universal constants. Since the matrix representation of E with respect to a Jordan–Hölder basis is upper triangular with ones on the diagonal, E is an invertible endomorphism on $\mathfrak{n}$.

Given $X \in \mathfrak{n}$, define the differential operator ∂_X on $\mathfrak{n}$ by

$$\partial_X f(Y) = \frac{d}{dt} f(Y+tX)\Big|_{t=0}.$$

The mapping $X \to \partial_X$ extends to an isomorphism from the symmetric algebra of $\mathfrak{n}$, $S(\mathfrak{n})$, to the algebra of constant coefficient differential operators on $\mathfrak{n}$.

Recall that $S(\mathfrak{n})$ is a graded algebra with grading

$$S(\mathfrak{n}) = \bigoplus_{j=0}^{\infty} S^j(\mathfrak{n}),$$

where $S^j(\mathfrak{n})$ is the span of products of j elements from $\mathfrak{n}$. There is also the associated filtration given by

$$S^{(k)} = \bigoplus_{j=0}^{k} S^j.$$

Given $X \in \mathfrak{n}$, define the differential operator ϱ_X on N by

$$\varrho_X f(\exp(Y)) = \frac{d}{dt} f(\exp(Y)\exp(tX))\Big|_{t=0}$$

The mapping $X \to \varrho_X$ lifts to an isomorphism between the universal enveloping algebra of $\mathfrak{n}$, $\mathcal{U}(\mathfrak{n})$, and the algebra of differential operators on N that are invariant under left translation by elements from N.

The algebra $\mathcal{U}(\mathfrak{n})$ is a filtered algebra

$$\mathcal{U}(\mathfrak{n}) = \bigcup_{k\geqslant 0} \mathcal{U}^{(k)}(\mathfrak{n}),$$

where $\mathcal{U}^{(k)}(\mathfrak{n})$ is the span of products of k or fewer elements from $\mathfrak{n}$. The symmetrization mapping $\sigma\colon \mathcal{U}(\mathfrak{n}) \to S(\mathfrak{n})$ is a linear isomorphism that preserves the filtrations of the two algebras and induces the isomorphism

$$\mathcal{U}^{(k)}/\mathcal{U}^{(k-1)} \cong S^k$$

given by the Poincaré–Birkhoff–Witt theorem.

Let $\mathcal{P}(\mathfrak{n})$ denote the space of polynomials on $\mathfrak{n}$, and let $\mathcal{P}^{(k)}$ denote the subspace of polynomials of degree at most k. Similarly, denote by $\mathcal{D}(\mathfrak{n})$ the space of constant coefficient differential operators on $\mathfrak{n}$, and by $\mathcal{D}^{(k)}$ the operators of degree at most k. Since $[\mathcal{P}^{(1)}, \mathcal{D}^{(1)}]$ is contained in the scalars, the algebra of polynomial coefficient differential operators on $\mathfrak{n}$, $\mathcal{PD}(\mathfrak{n})$, has a natural filtration

$$\mathcal{PD}(\mathfrak{n}) = \bigoplus_{j,k=1}^{\infty} \mathcal{PD}^{(j,k)}(\mathfrak{n})$$

where $\mathcal{PD}^{(j,k)}(\mathfrak{n})$ is the image of $\mathcal{P}^{(j)}(\mathfrak{n})\otimes\mathcal{D}^{(k)}(\mathfrak{n})$ under the mapping $p\otimes D \to pD$. One has similar bifiltrations, and similar notation, for the algebras of polynomial coefficient differential operators defined on arbitrary subspaces of $\mathfrak{n}$ or $\mathfrak{n}^*$.

Denote by log the inverse of exp. For $L\in\mathcal{U}(\mathfrak{n})$, composition with log defines a differential operator on $\mathfrak{n}$, $\varrho_L\circ\log$, i.e. if f is defined on $\mathfrak{n}$, then $(\varrho_L\circ\log)f(Y) = \varrho_L(f\circ\log)(\exp(Y))$.

Lemma 1.2. *For* $X\in\mathfrak{n}$, $\varrho_X\circ\log f(Y) = \partial_{E(\mathrm{ad}(Y))(X)} f(Y)$, *where* E *is given in* (1.1). *Thus,* $\varrho_X\circ\log\in\mathcal{PD}^{(r-1,1)}(\mathfrak{n})$. *It follows that if* $L\in\mathcal{U}^{(k)}(\mathfrak{n})$, *then* $\varrho_L\circ\log\in\mathcal{PD}^{(k(r-1),k)}(\mathfrak{n})$. *Also,* $\partial_X f(Y) = \varrho_{E(\mathrm{ad}(Y))^{-1}(X)}\circ\log f(Y)$. *It follows that for* $V\in S^{(k)}(\mathfrak{n})$,

$$\partial_V \in \mathcal{P}^{((k-1)(2r-3)+r-1)}(\mathfrak{n})\otimes\varrho\big(\mathcal{U}^{(k)}(\mathfrak{n})\big)\circ\log.$$

Proof.
$$\varrho_X\circ\log f(Y) = \frac{d}{dt} f\big(\log(\exp(Y)\exp(tX))\big)\Big|_{t=0} = \frac{d}{dt} f\big(C(Y, tX)\big)\Big|_{t=0} = \frac{d}{dt} f\big(Y+tE(Y)\,X\big)\Big|_{t=0}.$$

The other observations follow from the commutation relations

$$[\varrho(\mathfrak{n})\circ\log,\ \mathcal{P}^{(k)}(\mathfrak{n})] \subset \mathcal{P}^{(k+r-2)}(\mathfrak{n}).$$

Henceforth in this section, $\mathfrak{h}$, $\mathfrak{k}$ and $\mathfrak{m}$ will denote subspaces of $\mathfrak{n}$ such that the center of $\mathfrak{n}$, $\mathfrak{z}$, is the direct sum of $\mathfrak{h}$ and $\mathfrak{k}$, and such that $n = \mathfrak{m}\oplus\mathfrak{k}\oplus\mathfrak{h}$. Let $\langle\ ,\ \rangle$ be a positive-definite inner product on $\mathfrak{n}$ for which $\mathfrak{m}$, $\mathfrak{k}$ and $\mathfrak{h}$ are mutually orthogonal, and let $\|\cdot\|$ denote the corresponding Euclidean norm. Pick an orthonormal basis $\{X_1, \ldots, X_d\}$ of $\mathfrak{n}$ such that $\{X_h, X_{h+1}, \ldots, X_d\}$ is a basis for $\mathfrak{h}$, $\{X_k, X_{k+1}, \ldots, X_{h-1}\}$ is a basis for $\mathfrak{k}$, and with $\{X_1, \ldots, X_{k-1}\}$ a basis for $\mathfrak{m}$. Let $\{X_1^*, \ldots, X_d^*\}$ be the dual basis in $\mathfrak{n}^*$. There is a unique norm on $\mathfrak{n}^*$ for which $\{X_1^*, \ldots, X_d^*\}$ is an orthonormal set.

By restriction, $\langle\ ,\ \rangle$ defines an inner product on any subspace of $\mathfrak{n}$. If $\mathfrak{p}$ is an ideal in $\mathfrak{n}$, we define an inner product on $\mathfrak{n}/\mathfrak{p}$ by requiring that the projection from $\mathfrak{q}$, the orthogonal complement to $\mathfrak{p}$ in $\mathfrak{n}$, to $\mathfrak{n}/\mathfrak{p}$ be an isometry. In this manner, the inner product on $\mathfrak{n}$ can be used to define inner products on any subquotient of $\mathfrak{n}$.

Given the inner product $\langle\ ,\ \rangle$ on $\mathfrak{n}$ or $\mathfrak{n}^*$, there is a natural extension of $\langle\ ,\ \rangle$ to $S(V)$, again denoted by $\langle\ ,\ \rangle$, for any subspace V of $\mathfrak{n}$ or $\mathfrak{n}^*$. It is obtained by requiring that the homogeneous components of $S^j(V)$ be orthogonal and that, for $X, Y\in V$, it satisfy $\langle X^i, Y^i\rangle = (\langle X, Y\rangle)^i$. By using the symmetrization mapping σ, the inner product on $S(\mathfrak{n})$ pulls back to an inner product on $\mathcal{U}(\mathfrak{n})$, i.e. for $L, L'\in\mathcal{U}(\mathfrak{n})$, set $\langle L, L'\rangle = \langle\sigma(L), \sigma(L')\rangle$.

For $X\in\mathfrak{n}$, $\mathrm{ad}(X)$ is an endomorphism on $\mathfrak{n}$, and hence has an operator norm, $\|\mathrm{ad}(X)\|$. Also, the mapping $X\to\mathrm{ad}(X)$ of $\mathfrak{n}$ into $\mathrm{End}(\mathfrak{n})$ has an operator norm $\|\mathrm{ad}\|$.

Lemma 1.3. *Let* $\mathfrak{p}$ *be an ideal of* $\mathfrak{n}$ *and let* $\mathfrak{q}$ *be the orthogonal complement to* $\mathfrak{p}$ *in* $\mathfrak{n}$. *There is a constant* C *that bounds the function* M *defined on* $\mathfrak{q}\times\mathfrak{p}$ *by*

$$M(X, Y) = (1+\|X\|)(1+\|Y\|)/(1+\|C(X, Y)\|^2).$$

It follows that for any nonnegative integers r, s *there is an integer* t *and a constant* C *such that for* $X\in\mathfrak{q}$ *and* $Y\in\mathfrak{p}$,

$$(1+\|X\|^2)^r(1+\|Y\|^2)^s \leqslant C\big(1+\|C(X, Y)\|^2\big)^t.$$

Proof. Assume there exist $\{X_n\}\subset\mathfrak{q}$ and $\{Y_n\}\subset\mathfrak{p}$ such that $M(X_n, Y_n)\to\infty$. Then $s_n = \|X_n\|\to\infty$ and $t_n = \|Y_n\|\to\infty$. Let $\mathfrak{p}^i$, $i = 1, \ldots, m$, be orthogonal subspaces of $\mathfrak{p}$ such that $\mathfrak{p} = \bigoplus\mathfrak{p}^i$ and

$$[\mathfrak{p}^i, \mathfrak{p}^j] \subset \sum_{k\geqslant i+j}\mathfrak{p}^k.$$

Let $Y_n = \sum Y_n^i$ and $C(X_n, Y_n) = X_n + \sum C^i(X_n, Y_n)$, where Y_n^i and $C^i(X_n, Y_n)$ are elements of $\mathfrak{p}^i$.

Since $(1+\|X_n\|)(1+\|Y_n\|)/(s_n t_n) \to 1$, $\|C(X_n, Y_n)\|/(s_n t_n) \to 0$. Thus $X_n/(s_n t_n)^{1/2} \to 0$, which implies that $t_n/s_n \to \infty$. Now $C^1(X_n, Y_n) = Y_n^1$ and for $i \geq 2$, $C^i(X_n, Y_n)$ is a sum of terms involving Y_n^i and Lie products of X_n, $Y_n^1, \ldots, Y_n^{i-1}$, with constants depending on $\mathfrak{n}$. Since $\|C^i(X_n, Y_n)\|/(s_n t_n)^{1/2} \to 0$ for each i, by induction one sees that $\|Y_n\|/(s_n t_n)^{1/2} \to 0$. This implies that $s_n/t_n \to \infty$, a contradiction.

Similarly, as one can show very easily, for each $r \geq 0$ there is a constant C and an integer s such that

$$(1+\|C(X, Y)\|^2)^r \leq C(1+\|X\|^2)^s(1+\|Y\|^2)^s.$$

2. Schwartz spaces. Although one can define the Schwartz space on N in terms of the Schwartz space on $\mathfrak{n}$ and exp, or even go further, suppress N altogether by introducing a second group structure on $\mathfrak{n}$ via the Campbell–Hausdorff formula, for our purposes it is necessary to make estimates that involve the action of N on Schwartz functions. Thus, it is more convenient to define the Schwartz space explicitly in terms of N.

The Schwartz space on N, denoted by $\mathscr{S}(N)$, is the space of all smooth functions defined on N for which the seminorm

$$\Big(\int_{\mathfrak{n}} (1+\|X\|^2)^{qp}\big|\varrho_L f(\exp(X))\big|^q dX\Big)^{1/q}$$

is finite for each $p \geq 0$, for each $1 \leq q \leq \infty$, and for each $L \in \mathscr{U}(\mathfrak{n})$. The topology on $\mathscr{S}(N)$ is generated by these seminorms.

Recall that $\mathfrak{m}$, $\mathfrak{k}$ and $\mathfrak{h}$ are orthogonal subspaces of $\mathfrak{n}$ such that $\mathfrak{n} = \mathfrak{m} \oplus \mathfrak{k} \oplus \mathfrak{h}$, and $\mathfrak{z}$, the center of $\mathfrak{n}$, is given by $\mathfrak{k} \oplus \mathfrak{h}$. For each $\lambda \in \mathfrak{h}^*$, define the unitary character χ_λ on $H = \exp(\mathfrak{h})$ by

$$\chi_\lambda(\exp(X)) = e^{2\pi i\langle\lambda, X\rangle}.$$

Define $\mathscr{S}(N/H, \chi_\lambda)$ to be the space of smooth functions defined on N that satisfy

$$f(\exp(X+Y)) = \chi_\lambda(\exp(Y)) f(\exp(X)), \quad X \in \mathfrak{n},\ Y \in \mathfrak{h}, \tag{2.1}$$

and for which the seminorm

$$\Big(\int_{\mathfrak{n}/\mathfrak{h}} |(1+\|X\|^2)^{qp} \varrho_L f(\exp(X))|^q dX\Big)^{1/q} \tag{2.2}$$

is finite for each $p \geq 0$, all $1 \leq q \leq \infty$, and all $L \in \mathscr{U}^{(q)}(\mathfrak{n})$. (It should be noted that the measures on $\mathfrak{n}$ and $\mathfrak{n}/\mathfrak{h}$ are the Lebesgue measures induced by the inner products, and that they are carried by the exponential map to Haar measures on the groups N and N/H. It should also be noted that becasue $\mathfrak{h}$ is a central ideal, if f is a smooth function that satisfies (2.1) and $L \in \mathscr{U}(\mathfrak{n})$ then $\varrho_L f$ again satisfies (2.1).)

The space $\mathscr{S}(N/H, \chi_\lambda)$ has the topology generated by the seminorms

given in (2.2). There are three different generating families of these seminorms that we will use. They are denoted by ${}_s\|\ \|_{p,q}$, where $s = 1, 2, \infty$, and p, q are nonnegative integers, and are defined by

$${}_s\|f\|_{p,q} = \Big(\sum \int_{\mathfrak{n}/\mathfrak{h}} (1+\|X\|^2)^{sp}\big|\varrho_L f(\exp(X))\big|^s dX\Big)^{1/s} \tag{2.3}$$

for $s = 1,2$, and where the sum is over an orthonormal basis of $\mathscr{U}^{(q)}(\mathfrak{n}/\mathfrak{h})$, and

$${}_\infty\|f\|_{p,q} = \sup_{X \in \mathfrak{n}/\mathfrak{h}} \sup_{L \in \mathscr{U}^{(q)}(\mathfrak{n}/\mathfrak{h})} (1+\|X\|^2)^p\big|\varrho_L f(\exp(X))\big|/\|L\|. \tag{2.4}$$

The seminorms in (2.3) do, of course, depend on the choice of orthonormal basis, but only up to equivalence.

If V is a subspace of $\mathfrak{n}$ or $\mathfrak{n}^*$, the Schwartz space of V, $\mathscr{S}(V)$, is defined as usual, i.e. $\mathscr{S}(V)$ is the space of all smooth functions defined on V for which the seminorms

$$\Big(\int_V (1+\|X\|^2)^{rp}\,|\partial_Y f(X)|^r dX\Big)^{1/r}$$

are finite, for each $p \geq 0$, $r \geq 1$, and each $Y \in S^{(q)}(V)$, and the topology is generated by these seminorms. Similarly to the above, there are analogous families of seminorms ${}_s\|\ \|_{p,q}$.

The following lemma shows that the exponential mapping induces an isomorphism between $\mathscr{S}(N)$ and $\mathscr{S}(\mathfrak{n})$.

LEMMA 2.5. *For $p \geq 0$ and $q \geq 1$, there is a constant $C_{p,q}$ such that for each $f \in \mathscr{S}(N)$,*

$${}_s\|f\|_{p,q} \leq C_{p,q\,s}\|f \circ \exp\|_{p+q(r-1),q'},$$

and for $f \in \mathscr{S}(\mathfrak{n})$,

$${}_s\|f\|_{p,q} \leq C_{p,q\,s}\|f \circ \log\|_{p+(q-1)(2r-3)+r-1,q}.$$

The proof is an immediate consequence of Lemma 1.2.

If $f \in \mathscr{S}(\mathfrak{n})$ and $p \geq 0$, $q \geq 1$, there exist constants A, B, C, and p_i, q_i, $i = 1, 2, 3$, such that

$${}_\infty\|f\|_{p,q} \leq A\,{}_1\|f\|_{p_1,q_1} \leq B\,{}_2\|f\|_{p_2,q_2} \leq C\,{}_\infty\|f\|_{p_3,q_3}. \tag{2.6}$$

The first inequality is a Sobolev inequality, while the second and third inequalities are established using the Schwarz inequality. Using Lemma 2.5, one can prove analogous inequalities on $\mathscr{S}(N)$. By restricting to $\exp(\mathfrak{m} \oplus \mathfrak{k})$, one gets an isometry from $\mathscr{S}(N/H, \chi_\lambda)$ to $\mathscr{S}(N/H)$. This, combined with the mappings between $\mathscr{S}(\mathfrak{n}/\mathfrak{h})$ and $\mathscr{S}(N/H)$ induced by the exponential, establishes (2.6) for the spaces $\mathscr{S}(N/H, \chi_\lambda)$.

Let $\mathfrak{h}^\perp$ denote the annihilator of $\mathfrak{h}$ in $\mathfrak{n}^*$. For $\lambda \in \mathfrak{h}^*$, $\mathscr{S}(\mathfrak{h}^\perp + \lambda)$ is the space of functions f defined on $\mathfrak{h}^\perp + \lambda$ such that the function f_λ given by $f_\lambda(\eta)$

$=f(\eta+\lambda)$ is in $\mathscr{S}(\mathfrak{h}^{\perp})$. The seminorms on $\mathscr{S}(\mathfrak{h}^{\perp})$ are pulled back to $\mathscr{S}(\mathfrak{h}^{\perp}+\lambda)$ by the mapping $f\to f_{\lambda}$ and generate the topology.

For $f\in\mathscr{S}(N)$, define $\hat{f}$ on $\mathfrak{n}^*$ by

$$\hat{f}(\xi)=\int_{\mathfrak{n}} f(\exp(X))\,e^{-2\pi i\langle\xi,X\rangle}\,dX.$$

Since, by Lemma 2.5, the mapping $f\to f\circ\exp$ is an isomorphism from $\mathscr{S}(N)$ to $\mathscr{S}(\mathfrak{n})$, the usual theory establishes that $f\to\hat{f}$ is an isomorphism from $\mathscr{S}(N)$ to $\mathscr{S}(\mathfrak{n}^*)$. Likewise, for $F\in\mathscr{S}(\mathfrak{n}^*)$, $\hat{F}\circ\log\in\mathscr{S}(N)$, where, for $X\in\mathfrak{n}$,

$$\hat{F}(X)=\int_{\mathfrak{n}^*} F(\xi)\,e^{-2\pi i\langle\xi,X\rangle}\,d\xi.$$

Note that for $F\in\mathscr{S}(\mathfrak{n}^*)$

$$((\hat{F}\circ\log)\circ\exp)^{\wedge}(\xi)=F(-\xi)=\check{F}(\xi), \tag{2.7}$$

while for $f\in\mathscr{S}(N)$

$$(\hat{f})^{\wedge}\circ\log(\exp(X))=f(\exp(-X))=\check{f}(\exp(X)). \tag{2.8}$$

Let $\mathscr{S}^*(N)$ denote the dual space of $\mathscr{S}(N)$. For $D\in\mathscr{S}^*(N)$, $\hat{D}$ is defined in $\mathscr{S}^*(\mathfrak{n}^*)$ by $\langle\hat{D},F\rangle=\langle D,\hat{F}\circ\log\rangle$. Similarly, for $D\in\mathscr{S}^*(\mathfrak{n}^*)$, $\hat{D}$ is defined on $\mathscr{S}(N)$ by $\langle\hat{D},f\rangle=\langle D,\hat{f}\rangle$. From (2.7) and (2.8) one has $(\hat{D})^{\wedge}=\check{D}$, where $\langle\check{D},f\rangle=\langle D,\check{f}\rangle$.

The spaces $\mathscr{S}(N/H,\chi_{\lambda})$ and $\mathscr{S}(\mathfrak{h}^{\perp}+\lambda)$ may be considered as subspaces of $\mathscr{S}^*(N)$ and $\mathscr{S}^*(\mathfrak{n}^*)$ respectively, and thus the above definition of Fourier transform applies. We have

LEMMA 2.9. *For $f\in\mathscr{S}(N/H,\chi_{\lambda})$, the distribution $\hat{f}$ is absolutely continuous with respect to the Lebesgue measure on $\mathfrak{h}^{\perp}+\lambda$, and has density given by*

$$\hat{f}(\xi+\lambda)=\int_{\mathfrak{n}/\mathfrak{h}} f(\exp(Y))\,e^{-2\pi i\langle\xi+\lambda,Y\rangle}\,dY. \tag{2.10}$$

For $F\in\mathscr{S}(\mathfrak{h}^{\perp}+\lambda)$,

$$\hat{F}(\exp(X))=\int_{\mathfrak{h}^{\perp}} F(\xi+\lambda)\,e^{-2\pi i\langle\xi+\lambda,X\rangle}\,d\xi. \tag{2.11}$$

It follows that

$$\{\hat{f}\colon f\in\mathscr{S}(N/H,\chi_{\lambda})\}=\mathscr{S}(\mathfrak{h}^{\perp}+\lambda), \tag{2.12}$$

$$\{\hat{F}\colon F\in\mathscr{S}(\mathfrak{h}^{\perp}+\lambda)\}=\mathscr{S}(N/H,\chi_{-\lambda}). \tag{2.13}$$

Proof. Let $f\in\mathscr{S}(N/H,\chi_{\lambda})$ and $\varphi\in\mathscr{S}(\mathfrak{n})$. Then

$$\langle\hat{f},\varphi\rangle=\int_{\mathfrak{n}} f(\exp(X))\,\hat{\varphi}(\exp(X))\,dX$$
$$=\int_{\mathfrak{n}}\int_{\mathfrak{n}^*} f(\exp(X))\,\varphi(\xi)\,e^{-2\pi i\langle\xi,X\rangle}\,d\xi\,dX$$

$$=\int_{\mathfrak{m}\oplus\mathfrak{k}}\int_{\mathfrak{h}}\int_{(\mathfrak{m}\oplus\mathfrak{k})^*}\int_{\mathfrak{h}^*} f(\exp(Y+Z))\,\varphi(\eta+\mu)\,e^{-2\pi i(\langle\eta,Y\rangle+\langle\mu,Z\rangle)}\,d\mu\,d\eta\,dZ\,dY$$
$$=\int_{\mathfrak{h}^{\perp}}\cdots\int_{\mathfrak{h}^*} f(\exp(Y))\,\varphi(\eta+\mu)\,e^{-2\pi i(\langle\eta,Y\rangle+\langle\mu-\lambda,Z\rangle)}\,d\mu\,d\eta\,dZ\,dY$$
$$=\int_{\mathfrak{h}^{\perp}}\Big\{\int_{\mathfrak{h}^*} f(\exp(Y))\,e^{-2\pi i\langle\eta,Y\rangle}\,dY\Big\}\,\varphi(\eta+\lambda)\,d\eta.$$

This establishes (2.10). Since $f\circ\exp|_{\mathfrak{m}\oplus\mathfrak{k}}\in\mathscr{S}(\mathfrak{m}\oplus\mathfrak{k})$, it follows that $\hat{f}\in\mathscr{S}(\mathfrak{h}^{\perp}+\lambda)$.

The proof of 2.11 is immediate from the definitions, as well as the fact that $\hat{F}\in\mathscr{S}(N/H,\chi_{-\lambda})$. This and the previous inclusion establish the equalities (2.12) and (2.13).

For $\lambda\in\mathfrak{h}^*$, define $P_{\lambda}\colon\mathscr{S}(N)\to\mathscr{S}(N/H,\chi_{\lambda})$ by

$$P_{\lambda}f(\exp(X))=\int_{\mathfrak{h}} f(\exp(X+Y))\,e^{-2\pi i\langle\lambda,Y\rangle}\,dY.$$

Let $f\in\mathscr{S}(N/H,\chi_{\lambda})$ and define $f_1\in\mathscr{S}(\mathfrak{m}\oplus\mathfrak{k})$ by $f_1(X)=f(\exp(X))$ for $X\in\mathfrak{m}\oplus\mathfrak{k}$. Let $f_2\in\mathscr{S}(\mathfrak{h})$ such that $\hat{f}_2(\lambda)=1$. Then $\tilde{f}=f_1\otimes f_2\circ\log\in\mathscr{S}(N)$ and $P_{\lambda}\tilde{f}=f$. Note that if $f_n\to f$ in $\mathscr{S}(N/H,\chi_{\lambda})$, this construction yields a sequence $\tilde{f}_n\to\tilde{f}$ in $\mathscr{S}(N)$ such that $f=P_{\lambda}\tilde{f}=\lim P_{\lambda}\tilde{f}_n$.

Let $\mathscr{S}^*(N/H,\chi_{\lambda})$ denote the dual space of $\mathscr{S}(N/H,\chi_{\lambda})$. Then P_{λ}^*, the adjoint of P_{λ}, is a continuous injection of $\mathscr{S}^*(N/H,\chi_{\lambda})$ into $\mathscr{S}^*(N)$. Furthermore, the range of P_{λ}^* is the annihilator of the kernel of P_{λ}.

For $D\in\mathscr{S}^*(N)$, $f\in\mathscr{S}(N)$, and $x\in N$, define $l_x f(y)=f(x^{-1}y)$, and $\langle l_x D,f\rangle=\langle D,l_x f\rangle$. Let $\mathscr{S}_{\lambda}^*(N)=\{D\in\mathscr{S}^*(N)\colon l_x D=\chi_{\lambda}(x)D$ for $x\in H\}$.

LEMMA 2.14. $P_{\lambda}^*(\mathscr{S}^*(N/H,\chi_{\lambda}))=\mathscr{S}_{\lambda}^*(N)$.

Proof. It is easy to see that for $D\in\mathscr{S}^*(N/H,\chi_{\lambda})$, $P_{\lambda}^*(D)\in\mathscr{S}_{\lambda}^*(N)$. For the other inclusion, we recall some facts from the Schwartz theory on $\boldsymbol{R}^n$. Specifically, if $D\in\mathscr{S}^*(\mathfrak{n})$ and $f\in\mathscr{S}(\mathfrak{n})$ then $D*f$ is defined on $\mathfrak{n}$ by $D*f(X)=\langle D,l_X\check{f}\rangle$, where $l_X\check{f}(Y)=f(X-Y)$. $D*f$ is again a tempered distribution on $\mathfrak{n}$.

Suppose now that $D\in\mathscr{S}_{\lambda}^*(N)$ and define $D\circ\exp\in\mathscr{S}^*(\mathfrak{n})$ by $\langle D\circ\exp,f\rangle=\langle D,f\circ\exp\rangle$. One easily checks that $((D\circ\exp)*f)\circ\log\in\mathscr{S}_{\lambda}^*(N)$. Thus, if $g\in\ker(P_{\lambda})$

$$\langle(D\circ\exp)*f,g\circ\exp\rangle=\int_{\mathfrak{n}/\mathfrak{h}}\int_{\mathfrak{h}}\langle D\circ\exp,l_{X+Y}(f\circ\exp)^{\check{}}\rangle g(\exp(X+Y))\,dY\,dX$$
$$=\int_{\mathfrak{n}/\mathfrak{h}}\langle D\circ\exp,l_X(f\circ\exp)^{\check{}}\rangle\int_{\mathfrak{h}} g(\exp(X+Y))\,\chi_{\lambda}(\exp(Y))\,dY\,dX=0.$$

Since f was arbitrary in $\mathscr{S}(N)$, this implies that $\langle D,g\rangle=0$.

For $\lambda\in\mathfrak{h}^*$, let $R_{\lambda}\colon\mathscr{S}(\mathfrak{n}^*)\to\mathscr{S}(\mathfrak{h}^{\perp}+\lambda)$ be the restriction mapping. Then

R_λ^*, the adjoint of R_λ, is an injection of $\mathcal{S}^*(\mathfrak{h}^\perp+\lambda)$ into the tempered distributions supported on $\mathfrak{h}^\perp+\lambda$, and without normal derivatives.

LEMMA 2.15. *For $D\in\mathcal{S}^*(N/H,\chi_\lambda)$, $(P_\lambda^* D)^\wedge = R_{-\lambda}^*\hat{D}$, where $\hat{D}$ is defined by the adjoint of the Fourier transform mapping $\mathcal{S}(\mathfrak{h}^\perp-\lambda)\to\mathcal{S}(N/H,\chi_\lambda)$.*

Proof. Let $F\in\mathcal{S}(\mathfrak{n}^*)$ and $X\in\mathfrak{m}\oplus\mathfrak{k}$. Then

$$\begin{aligned} P_\lambda(\hat{F}\circ\log)(\exp(X)) &= \int_{\mathfrak{h}} \hat{F}(X+Y)\,e^{-2\pi i\langle\lambda,Y\rangle}\,dY\\ &= \int_{\mathfrak{h}}\int_{\mathfrak{h}^\perp}\int_{\mathfrak{h}^*} F(\eta+\mu)\,e^{-2\pi i(\langle\eta,X\rangle+\langle\lambda+\mu,Y\rangle)}\,d\mu\,d\eta\,dY\\ &= \int_{\mathfrak{h}^\perp} F(\eta-\lambda)\,e^{-2\pi i\langle\eta,X\rangle}\,d\eta = (R_{-\lambda}F)^\wedge(X). \end{aligned}$$

An element $D\in\mathcal{S}^*(N/H,\chi_\lambda)$ is said to be *bi-invariant* if $\langle D, l_x f\rangle = \langle D, r_{x^{-1}} f\rangle$ for all $x\in N$ and $f\in\mathcal{S}(N/H,\chi_\lambda)$, where $r_x f(y) = f(yx)$. An element $D\in\mathcal{S}^*(\mathfrak{h}^\perp+\lambda)$ is said to be Ad*-*invariant* if for all $f\in\mathcal{S}(\mathfrak{h}^\perp+\lambda)$, $\langle D, f\circ\mathrm{Ad}^*\rangle = \langle D, f\rangle$.

LEMMA 2.16. *$D\in\mathcal{S}^*(N/H,\chi_\lambda)$ is bi-invariant if, and only if, $\hat{D}\in\mathcal{S}^*(\mathfrak{h}^\perp-\lambda)$ is Ad*-invariant.*

Proof. Let $f\in\mathcal{S}(N/H,\chi_\lambda)$ and $x\in N$. Then

$$\begin{aligned} (l_x r_x f)^\wedge(\xi-\lambda) &= \int_{\mathfrak{n}/\mathfrak{h}} f(x^{-1}\exp(Y)\,x)\,e^{-2\pi i\langle\xi-\lambda,Y\rangle}\,dY\\ &= \int_{\mathfrak{n}/\mathfrak{h}} f(\exp(\mathrm{Ad}\,x(Y)))\,e^{-2\pi i\langle\xi-\lambda,Y\rangle}\,dY\\ &= \int_{\mathfrak{n}/\mathfrak{h}} f(\exp(Y))\,e^{-2\pi i\langle\mathrm{Ad}^*x(\xi-\lambda),Y\rangle}\,dY\\ &= \hat{f}(\mathrm{Ad}^*x(\xi-\lambda)). \end{aligned}$$

The following lemmas are needed in the proof of Theorem A.

LEMMA 2.17. *Let f be a smooth function on N and suppose that for each pair of nonnegative integers p,q there are constants $l=l_{p,q}$ and $C_{p,q}$ such that for $X\in\mathfrak{m}\oplus\mathfrak{k}$, $Y\in\mathfrak{h}$, and $L\in\mathcal{U}^{(q)}(\mathfrak{n})$,*

(i) $$|\varrho_L f(\exp(X+Y))| \leqslant C_{p,q}\|L\|(1+\|X\|^2)^l/(1+\|Y\|^2)^p,$$

(ii) $$\int_{\mathfrak{h}^*} {}_\infty\|P_\lambda\varrho_L f\|_{p,0}\,d\lambda \leqslant C_{p,q}\|L\|.$$

Then $f\in\mathcal{S}(N)$.

Proof. We will show that ${}_\infty\|f\|_{p,q}$ is finite for all $p,q\geqslant 0$.

Note that by (i), for each fixed X and L, the mapping $Y\to\varrho_L f(\exp(X+Y))\in\mathcal{S}(\mathfrak{h})$. Thus the usual Fourier inversion gives

$$\varrho_L f(\exp(X+Y)) = \int_{\mathfrak{h}^*} P_\lambda\varrho_L f(\exp(X+Y))\,d\lambda.$$

Thus,

$$\begin{aligned} (1+\|X\|^2)^p|\varrho_L f(\exp(X+Y))| &\leqslant \int_{\mathfrak{h}^*}(1+\|X\|^2)^p|P_\lambda\varrho_L f(\exp(X+Y))|\,d\lambda\\ &\leqslant \int_{\mathfrak{h}^*} {}_\infty\|P_\lambda\varrho_L f\|_{p,0}\,d\lambda \leqslant C_{p,q}\|L\|. \end{aligned}$$

Therefore,

$$\begin{aligned} &(1+\|X+Y\|^2)^p|\varrho_L f(\exp(X+Y))|^2\\ &\quad\leqslant (1+\|X\|^2)^p(1+\|Y\|^2)^p|\varrho_L f(\exp(X+Y))|\\ &\quad\leqslant C_{p,q}\|L\|(1+\|X\|^2)^{p+l}|\varrho_L f(\exp(X+Y))| \leqslant C_{p,q}C_{p+l,q}\|L\|^2. \end{aligned}$$

LEMMA 2.18. *Given integers $p,q\geqslant 0$, there exist integers $p',q'\geqslant 0$ and a constant C such that for all $f\in\mathcal{S}(N)$,*

$$\int_{\mathfrak{h}^*} {}_s\|P_\lambda f\|_{p,q}\,d\lambda \leqslant C\,{}_s\|f\|_{p',q'}.$$

Proof. For $s=1$,

$$\int_{\mathfrak{h}^*} {}_1\|P_\lambda f\|_{p,q}\,d\lambda = \int_{\mathfrak{h}^*}\sum\int_{\mathfrak{n}/\mathfrak{h}}(1+\|X\|^2)^p|\varrho_L P_\lambda f(\exp(X))|\,dX\,d\lambda$$

(the sum is over an orthonormal basis of $\mathcal{U}^{(q)}(\mathfrak{n}/\mathfrak{h})$)

$$\begin{aligned} &\leqslant \int_{\mathfrak{h}^*}\sum\int_{\mathfrak{n}/\mathfrak{h}}\int_{\mathfrak{h}}(1+\|X\|^2)^p(1+\|\lambda\|^2)^{-k}|(1-\Delta)^k\varrho_L f(\exp(X+Y))|\,dY\,dX\,d\lambda\\ &\leqslant C\,{}_1\|f\|_{p+2,q+2k}, \end{aligned}$$

for k sufficiently large.

3. Convolution. Recall that for a function f defined on N and for x and y in N, $l_x f(y) = f(x^{-1}y)$ and $\check{f}(x) = f(x^{-1})$. Thus, if $f\in\mathcal{S}(N/H,\chi_\lambda)$, $\check{f}\in\mathcal{S}(N/H,\chi_{-\lambda})$. For $D\in\mathcal{S}^*(N/H,\chi_{-\lambda})$ and $f\in\mathcal{S}(N/H,\chi_\lambda)$ the function $D*f$ is defined on N by

$$D*f(x) = \langle D, l_x\check{f}\rangle.$$

Note that if $\mathfrak{h}=\{0\}$ and $D=g\in\mathcal{S}(N)$, this definition agrees with the usual one, i.e.

$$g*f(x) = \langle g, l_x\check{f}\rangle = \int_{\mathfrak{n}} g(\exp(Y))\,f(\exp(-Y)\,x)\,dY.$$

Since convolution (on the left) by D commutes with right translation, $D*f$ will be a smooth function on N for each $D\in\mathcal{S}^*(N/H,\chi_{-\lambda})$ and $f\in\mathcal{S}(N/H,\chi_\lambda)$. Thus to show that $D*f$ is again in $\mathcal{S}(N/H,\chi_\lambda)$, it suffices to show that $D*f$ is rapidly decreasing. More precisely, we have

LEMMA 3.1. *Let $D\in\mathcal{S}^*(N/H,\chi_{-\lambda})$ and suppose that for each nonnegative integer p there exist a constant C_p and nonnegative integers p', q' such that*

$$_s\|D*f\|_{p,0}\leqslant C_p\,{}_s\|f\|_{p',q'},$$

for each $f\in\mathcal{S}(N/H,\chi_\lambda)$. Then for each q there exist a constant C_q, independent of f, and nonnegative integers p'', q'' such that

$$_s\|D*f\|_{p,q}\leqslant C_p C_q\,{}_s\|f\|_{p'',q''}.$$

Proof. We give the proof for $s=2$:

$$_2\|D*f\|_{p,q}=\big(\sum\big(_2\|\varrho_L(D*f)\|_{p,0}\big)^2\big)^{1/2},$$

the sum being over an orthonormal basis for $\mathcal{U}^{(q)}(\mathfrak{n})$. Thus

$$_2\|D*f\|_{p,q}\leqslant\big(C_p^2\sum{}_2\|\varrho_L f\|^2_{p',q'}\big)^{1/2},$$

$$_2\|\varrho_L f\|_{p',q'}=\big(\sum{}_2\|\varrho_{L'}\varrho_L f\|^2_{p',0}\big)^{1/2}\leqslant C_q C_p\,{}_2\|f\|_{p',q+q'},$$

where

$$C_q=\sup\{\|L'L\|/(\|L'\|\,\|L\|):\ L\in\mathcal{U}^{(q)}(\mathfrak{n}),\ L'\in\mathcal{U}^{(q')}(\mathfrak{n})\}.$$

Let $\mathfrak{p}$ be an ideal in $\mathfrak{n}$ that contains $\mathfrak{h}$, and let $P=\exp(\mathfrak{p})$. Let R: $\mathcal{S}(N/H,\chi_\lambda)\to\mathcal{S}(P/H,\chi_\lambda)$ be the restriction mapping, and denote its adjoint by R^*. The following lemma shows that if $D\in\mathcal{S}^*(P/H,\chi_{-\lambda})$ that convolves $\mathcal{S}(P/H,\chi_\lambda)$ into itself, then R^*D convolves $\mathcal{S}(N/H,\chi_\lambda)$ into itself.

It is convenient for the proof to work on the group level. In particular, the exponential mapping carries the Lebesgue measures on $\mathfrak{n}$ and $\mathfrak{h}$ onto Haar measures on N and H respectively. There exist Haar measures on N/H and P/H such that for $f\in L^1(N)$ and $g\in L^1(N/H)$,

$$\int_N f(x)\,dx=\int_{N/H}\int_H f(xy)\,dy\,dx,\qquad\int_{N/H}g(x)\,dx=\int_{P\backslash N}\int_{P/H}g(yz)\,dz\,dy.$$

We identify the Lie algebra of $P\backslash N$ with the orthogonal, direct sum complement of $\mathfrak{p}$ in $\mathfrak{n}$, and the Lie algebra of P/H with $\mathfrak{p}\cap\mathfrak{m}\oplus\mathfrak{k}$. (Note that the right and left cosets of H coincide.)

LEMMA 3.2. *Let $D\in\mathcal{S}^*(P/H,\chi_{-\lambda})$ and suppose that for each nonnegative integer p there exist a constant C_p and nonnegative integers p',q' such that for each $f\in\mathcal{S}(P/H,\chi_\lambda)$,*

$$_s\|D*f\|_{p,0}\leqslant C_p\,{}_s\|f\|_{p',q'}.$$

Then there exist a constant C'_p and nonnegative integers p'', q'' such that for each $f\in\mathcal{S}(N/H,\chi_\lambda)$,

$$_s\|R^*D*f\|_{p,0}\leqslant C'_p\,{}_s\|f\|_{p'',q''}.$$

Proof. We give the proof for $s=2$:

$$\int_{N/H}|R^*D*f(x)|^2(1+\|\log(x)\|^2)^{2p}\,dx$$
$$=\int_{P\backslash N}\int_{P/H}|R^*D*f(zy)|^2(1+\|\log(zy)\|^2)^{2p}\,dz\,dy$$
$$\leqslant C\int_{P\backslash N}\int_{P/H}|\langle D,R(l_{zy}\check f)\rangle|^2(1+\|\log(z)\|^2)^{2r}(1+\|\log(y)\|^2)^{2r}\,dz\,dy$$
(for some $r=r(p)$, and C independent of f and D)
$$\leqslant C\int_{P\backslash N}{}_2\|D*R(l_y\check f)\|^2_{r,0}(1+\|\log(y)\|^2)^{2r}\,dy$$
$$\leqslant CC_p\int_{P\backslash N}{}_2\|R(l_y\check f)\|^2_{p',q'}(1+\|\log(y)\|^2)^{2r}\,dy$$
$$=CC_p\int_{P\backslash N}\sum{}_2\|\varrho_L(l_y\check f)\|^2_{p',0}(1+\|\log(y)\|^2)^{r}\,dy$$
(the sum is over an orthonormal basis of $\mathcal{U}^{(q')}(\mathfrak{p}/\mathfrak{h})$)
$$=CC_p\sum\int_{P\backslash N}\int_{P/H}|\varrho_L f(zy)|^2(1+\|\log(z)\|^2)^{2p'}(1+\|\log(y)\|^2)^{2r}\,dz\,dy$$
(by Lemma 1.2)
$$\leqslant C'C_p\sum\int_{P\backslash N}\int_{P/H}|\varrho_L f(zy)|^2(1+\|\log(zy)\|^2)^{2p''}\,dz\,dy$$
$$=C'C_p\,{}_2\|f\|^2_{p'',q'}.$$

LEMMA 3.3. *Suppose that $D\in\mathcal{S}^*(N/H,\chi_{-\lambda})$ such that $(P^*_{-\lambda}D)\hat{}$ is a smooth function on $\mathfrak{h}^\perp+\lambda$, all of whose derivatives have polynomial bounds. Then the function $Y\to D*f(\exp(X+Y))\in\mathcal{S}(\mathfrak{l})$. More specifically, suppose that for each integer $j\geqslant 0$ there is an integer l and a constant $C_j(D)$ such that for $\eta\in S^{(j)}(\mathfrak{h}^\perp)$,*

$$|\partial_\eta\hat D(v+\lambda)|\leqslant C_j(D)\,\|\eta\|\,(1+\|v\|^2)^l,\qquad v\in\mathfrak{h}^\perp.$$

Then, given a nonnegative integer p, an $X\in\mathfrak{n}$, and an $f\in\mathcal{S}(N/H,\chi_\lambda)$, there is a constant $C_p(X)$ and positive integers l, p', q', independent of f, such that

(i) $$(1+\|Y\|^2)^p|D*f(\exp(X+Y))|\leqslant C_p(X)\,C_{2p}(D)\,{}_2\|f\|_{p',q'},$$

(ii) $$C_p(X)\leqslant C(1+\|X\|^2)^l.$$

Proof. Let $F = (P^*_{-\lambda} D)\hat{}$ and let $g \in \mathscr{S}(N)$ such that $P_\lambda g = f$. Then $D * f(x) = \langle D, l_x((P_\lambda g)\check{})\rangle = \langle P^*_\lambda D, l_x \check{g}\rangle = \langle F, (l_x \check{g})\hat{}\rangle$. If $Y \in \mathfrak{k}$ and $h \in \mathscr{S}(N)$ then $(l_{\exp(Y)} \check{h})\hat{}(\xi) = e^{2\pi i\langle \xi, Y\rangle}(\check{h})\hat{}(\xi)$. Thus, setting

$$\Delta = \sum_{j=k}^{h-1} \frac{1}{4\pi^2} \partial^2_{X_j^*}, \tag{3.4}$$

one has $(1-\Delta)^p (l_{\exp(Y)} \check{h})\hat{} = (1+\|Y\|^2)^p (l_{\exp(Y)} \check{h})\hat{}$. For positive integer k, define φ_k on $\mathfrak{h}^\perp + \lambda$ by $\varphi_k(\xi+\lambda) = (1+\|\xi+\lambda\|^2)^k$. Then, for $X \in \mathfrak{n}$ and $Y \in \mathfrak{k}$,

$$\begin{aligned}
&(1+\|Y\|^2)^p |D * f(\exp(X+Y))| \\
&= \Big|\int_{\mathfrak{h}^\perp} F(\xi+\lambda)(1-\Delta)^p (l_{\exp(X+Y)} \check{g} \circ \exp)\hat{}(\xi+\lambda)\, d\xi\Big| \\
&= \Big|\int_{\mathfrak{h}^\perp} \varphi_k^{-1}(\xi+\lambda)(1-\Delta)^p F(\xi+\lambda)(l_{\exp(X+Y)} \check{g} \circ \exp)\hat{}(\xi+\lambda)\, \varphi_k(\xi+\lambda)\, d\xi\Big| \\
&= \Big|\int_{\mathfrak{n}} (\varphi_k^{-1}(1-\Delta)^p F)\hat{}(W) ((1-\hat{\Delta})^k l_{\exp(X+Y)} \check{g} \circ \exp)(W)\, dW\Big| \\
&\qquad\qquad \Big(\text{where } \hat{\Delta} = \sum_{j=1}^{d} \frac{1}{4\pi^2} \partial^2_{X_j}\Big) \\
&= \Big|\int_{\mathfrak{m}\oplus\mathfrak{k}} (\varphi_k^{-1}(1-\Delta)^p F)\hat{}(W)\, P_\lambda((1-\hat{\Delta})^k l_{\exp(X+Y)} \check{g})(\exp(W))\, dW\Big| \\
&\leqslant \|\varphi_k^{-1}(1-\Delta)^p F\|_{L^2(\mathfrak{h}^\perp+\lambda)} \|(1-\hat{\Delta})^k l_{\exp(X+Y)} P_\lambda \check{g} \circ \exp\|_{L^2(\mathfrak{n}/\mathfrak{h})} \\
&\qquad\qquad (\text{for } k \text{ sufficiently large}) \\
&\leqslant C_{2p}(D) \|(1-\hat{\Delta})^k l_{\exp(X+Y)} P_\lambda \check{g} \circ \exp\|_{L^2(\mathfrak{n}/\mathfrak{h})}.
\end{aligned}$$

By Lemma 1.2, $\partial_X (f \circ \exp)(Y) = (\varrho_{E(\mathrm{ad}(Y))^{-1}X} f) \circ \exp(Y)$. Thus,

$$\begin{aligned}
(1-\hat{\Delta})^k l_{\exp(X+Y)} P_\lambda \check{g}((\exp(W)) \\
= l_{\exp(X+Y)} \Big(1 - \sum_{j=1}^{d} \varrho^2_{E(\mathrm{ad}(W))^{-1}X_j}\Big)^k (P_\lambda \check{g})(\exp(W)).
\end{aligned} \tag{3.5}$$

Since Y is in the center of $\mathfrak{n}$, the norm of (3.5) is independent of Y. However, the norm does depend on the Jacobian of the mapping $W \to C(X, W)$, which has a polynomial bound.

LEMMA 3.6. *Let $D \in \mathscr{S}^*(N/H, \chi_{-\lambda})$ that satisfies the hypothesis of Lemma 3.3. Then for each $f \in \mathscr{S}(N/H, \chi_\lambda)$,*

$$D * f(\exp(X)) = \int_{\mathfrak{k}^*} D_\nu * f_\nu(\exp(X))\, d\nu,$$

where for $\nu \in \mathfrak{k}^*$, $\chi_{\nu+\lambda}$ is the character on $Z = \exp(\mathfrak{z})$ given by

$$\chi_{\nu+\lambda}(\exp(X+Y)) = e^{2\pi i(\langle \nu, X\rangle + \langle \lambda, Y\rangle)}$$

for $X \in \mathfrak{k}$ and $Y \in \mathfrak{h}$, $f_\nu \in \mathscr{S}(N/Z, \chi_{\nu+\lambda})$ such that

$$(f_\nu \circ \exp)\hat{} = (f \circ \exp)\hat{}|_{\mathfrak{z}^\perp+\nu+\lambda},$$

and $D_\nu \in \mathscr{S}^*(N/Z, \chi_{-\nu-\lambda})$ such that $\hat{D}_\nu = \hat{D}|_{\mathfrak{z}^\perp+\nu+\lambda}$.

Proof. Lemma 3.3 shows that for each $X \in \mathfrak{n}$, $Y \to D * f(\exp(X+Y)) \in \mathscr{S}(\mathfrak{k})$. Thus, the usual Fourier inversion gives

$$\begin{aligned}
D * f(\exp(X)) &= \int_{\mathfrak{k}^*}\int_{\mathfrak{k}} D * f(\exp(X+Y))\, e^{2\pi i\langle \nu, Y\rangle}\, dY\, d\nu \\
&= \int_{\mathfrak{k}^*}\int_{\mathfrak{k}}\int_{\mathfrak{h}^\perp} \hat{D}(\xi+\lambda)(l_{\exp(X+Y)} f \circ \exp)\hat{}(\xi+\lambda)\, e^{2\pi i\langle \nu, Y\rangle}\, d\xi\, dY\, d\nu \\
&\qquad (\text{writing } \xi = \eta+\mu \text{ according to the decomposition } \mathfrak{h}^\perp = \mathfrak{z}^\perp \oplus \mathfrak{k}^*) \\
&= \int_{\mathfrak{k}^*}\int_{\mathfrak{z}^\perp}\int_{\mathfrak{k}}\int_{\mathfrak{k}^*} \hat{D}(\eta+\mu+\lambda)(l_{\exp(X)} f \circ \exp)\hat{}(\eta+\mu+\lambda) e^{2\pi i\langle \nu-\mu, Y\rangle}\, d\mu\, dY\, d\eta\, d\nu \\
&= \int_{\mathfrak{k}^*}\int_{\mathfrak{z}^\perp} \hat{D}(\eta+\nu+\lambda)(l_{\exp(X)} f \circ \exp)\hat{}(\eta+\nu+\lambda)\, d\eta\, d\nu = \int_{\mathfrak{k}^*} D_\nu * f_\nu(\exp(X))\, d\nu.
\end{aligned}$$

Recall that for $X \in \mathfrak{n}$, $\mathrm{Ad}(\exp(X))$ is the endomorphism of $\mathfrak{n}$ given by $\mathrm{Ad}(\exp(X)) = \exp(\mathrm{ad}(X))$. Ad^* denotes the contragredient of Ad acting on $\mathfrak{n}^*$. One can easily check that for each $\lambda \in \mathfrak{h}^*$ and $X \in \mathfrak{n}$, $\mathrm{Ad}^*(\exp(X))(\mathfrak{h}^\perp + \lambda) = \mathfrak{h}^\perp + \lambda$.

THEOREM 3.7. *Let $D \in \mathscr{S}^*(N/Z, \chi_{-\lambda})$ such that $(P^*_{-\lambda} D)\hat{}$ is a smooth function on $\mathfrak{z}^\perp + \lambda$, all of whose derivatives have polynomial bounds. Assume further that $(P^*_{-\lambda} D)\hat{}$ is Ad^*-invariant, i.e. is constant on the Ad^*-orbits in $\mathfrak{z}^\perp + \lambda$. Then for $f \in \mathscr{S}(N/Z, \chi_\lambda)$, $D * f \in \mathscr{S}(N/Z, \chi_\lambda)$. More precisely, suppose that for each nonnegative integer j there is an integer l and a constant $C_j(D)$ such that*

$$|\partial_\eta (P^*_{-\lambda} D)\hat{}(\nu+\lambda)| \leqslant C_j(D)\|\eta\|(1+\|\nu\|^2)^l, \quad \nu \in \mathfrak{z}^\perp,$$

for all $\eta \in S^{(j)}(\mathfrak{z}^\perp)$. Then for each integer $p \geqslant 0$ there exist integers j, p', and q', and a constant C_p such that for each $f \in \mathscr{S}(N/Z, \chi_\lambda)$,

$${}_s\|D * f\|_{p,0} \leqslant C_p C_j(D)\, {}_s\|f\|_{p',q'}.$$

Proof. The proof, with $s = \infty$, is by induction on the dimension of $\mathfrak{n}$. If $\dim(\mathfrak{n}) = 1$ or 2, the theorem is trivial.

Assume that $\dim(\mathfrak{n}) \geqslant 3$, and that $\dim(\mathfrak{z}) = 1$. In this case $(P^*_{-\lambda} D)\hat{}$ is constant on the cosets in $\mathfrak{h}^\perp + \lambda$ of the annihilator of the centralizer of the second center of $\mathfrak{n}$, which we denote by $\mathfrak{p}^\perp$.

To see this pick $X, Y, Z \in \mathfrak{n}$ so that $\mathfrak{z}$ is the span of Z, $[X, Y] = Z$, and $\mathfrak{n} = \mathfrak{n}_0 \oplus \mathfrak{n}_1$, where $\mathfrak{n}_0$ is the span of X, and $\mathfrak{n}_1$ is the centralizer of Y. Fix a $\xi \in \mathfrak{h}^\perp + \lambda$, with $\langle \xi, Z\rangle \neq 0$, and let $\mathfrak{r}_\xi$ denote the radical of ξ. If $W \in \mathfrak{r}_\xi$,

$V \in \mathfrak{z}^{(2)}$, the second center of $\mathfrak{n}$, and $X' \in \mathfrak{n}$, then

$$\begin{aligned}[\mathrm{Ad}(\exp(X'))W, V] &= [W+[X', W]+(1/2)[X', [X', W]]+ \ldots, V] \\ &= [W, V] = \alpha Z.\end{aligned}$$

Thus, $\langle \xi, [\Omega, \vartheta] \rangle = \alpha \langle \xi, Z \rangle$, which implies that $\alpha = 0$. Therefore, $\mathrm{Ad}(\exp(X))\mathfrak{r}_\xi \subset \mathfrak{c}(\mathfrak{z}^{(2)})$. Hence, $\mathfrak{p}^\perp \subset (\mathrm{span}\{\mathrm{Ad}(N)\mathfrak{r}_\xi\})^\perp$ for all $\xi \in \mathfrak{h}^\perp + \lambda$ with $\langle \xi, Z \rangle \neq 0$. Since this latter subspace is contained in the Ad*-orbits of ξ (cf. [CGP], Theorem 4.1), the claim follows by continuity of $(P^*_{-\lambda} D)\hat{}$.

Let $\mathfrak{p} = \mathfrak{c}(\mathfrak{z}^{(2)})$. Since $(P^*_{-\lambda} D)\hat{}$ is constant on the cosets of $\mathfrak{p}^\perp$, one sees, by partial Fourier transform, that $\mathrm{supp}(D) \subset P = \exp(\mathfrak{p})$. Then, by Lemma 3.2, it suffices to show that for $g \in \mathcal{S}(P/Z, \chi_\lambda)$ and $D \in \mathcal{S}^*(P/Z, \chi_{-\lambda})$ the desired estimates hold, which is the case if $\mathfrak{z}$ is the center of $\mathfrak{p}$.

Thus suppose that $\mathfrak{z}$ is not the center of $\mathfrak{p}$, and pick subspaces $\mathfrak{p}_1$, $\mathfrak{z}_1$ of $\mathfrak{p}$ such that $\mathfrak{z}_1 \oplus \mathfrak{z}$ is the center of $\mathfrak{p}$, and with $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{z}_1 \oplus \mathfrak{z}$. Then for $X \in \mathfrak{p}_1$, $Y \in \mathfrak{z}_1$ and positive integer p,

$$\begin{aligned}
&(1+\|X\|^2)^p(1+\|Y\|^2)^p|D*g(\exp(X+Y))|^2 \\
&\leqslant C(1+\|X\|^2)^{p+l}|D*g(\exp(X+Y))| \quad \text{(by Lemma 3.3)} \\
&\leqslant C\int_{\mathfrak{z}_1^*}(1+\|X\|^2)^{p+l}|D_\nu * g_\nu(\exp(X))|\,d\nu \quad \text{(by Lemma 3.6)} \\
&\leqslant C\int_{\mathfrak{z}_1^*}(1+\|X\|^2)^{p+l}|D_\nu *(1-\Delta)^k g_\nu(\exp(X))|(1+\|\nu\|^2)^{-k}\,d\nu \quad (\Delta \text{ as in } (3.4)) \\
&\leqslant C\int_{\mathfrak{z}_1^*} {}_\infty\|D_\nu *(1-\Delta)^k g_\nu\|_{p+l}(1+\|\nu\|^2)^{-k}\,d\nu \quad \text{(by induction hypothesis)} \\
&\leqslant C' C_{p+l}\int_{\mathfrak{z}_1^*} C_j(D_\nu)\,{}_1\|(1-\Delta)^k g_\nu\|_{p',q'}(1+\|\nu\|^2)^{-k}\,d\nu \leqslant C'' C_j(D)\,{}_1\|g\|_{p'',q''}.
\end{aligned}$$

The last inequality uses Lemma 2.18 and the fact that $C_j(D_\nu) \leqslant C_j(D)$. The conclusion then follows from the remarks following Lemma 2.5.

Suppose that $\dim(\mathfrak{z}) \geqslant 2$. Fix $\lambda \in \mathfrak{z}^*$ and let $\mathfrak{h} = \ker(\lambda)$ and $H = \exp(\mathfrak{h})$ Then χ_λ is constant on the H cosets, and so defines a character $\bar{\chi}_\lambda$ on Z/H Similarly, for $f \in \mathcal{S}(N/Z, \chi_\lambda)$ there is a corresponding $\bar{f} \in \mathcal{S}((N/H)/(Z/H), \bar{\chi}_\lambda$ It easily follows that $D*f = \bar{D}*\bar{f}$, where $\bar{D} \in \mathcal{S}^*((N/H)/(Z/H), \bar{\chi}_\lambda)$ such tha $(\bar{D})\hat{} = \hat{D}|_{\mathfrak{h}^\perp}$. Thus, if $\mathfrak{z}/\mathfrak{h}$ is the center of $\mathfrak{n}/\mathfrak{h}$, the estimates follow from the induction assumption. If the center of $\mathfrak{n}/\mathfrak{h}$ is larger than $\mathfrak{z}/\mathfrak{h}$, one can again use Lemma 3.6 as above to complete the proof.

THEOREM 3.8. *Let $D \in \mathcal{S}^*(N)$ such that $\hat{D}$ is smooth, Ad*-invariant, and such that for each integer $j \geqslant 0$ there exist an integer l and a constant $C_j(D)$ such that*

$$|\partial_\eta \hat{D}(\nu)| \leqslant C_j(D)\|\eta\|(1+\|\nu\|^2)^l, \qquad \nu \in \mathfrak{n}^*,$$

for all $\eta \in S^{(j)}(\mathfrak{n}^*)$. Then for each integer $p \geqslant 0$ there exist integers j, p', q', and a constant C_p such that for each $f \in \mathcal{S}(N)$,

$${}_s\|D*f\|_{p,0} \leqslant C_p C_j(D)\,{}_s\|f\|_{p',q'}.$$

Proof.

$$\begin{aligned}
{}_\infty\|D*f\|^2_{p,0} &= \sup_{X\in\mathfrak{m}}\sup_{Y\in\mathfrak{z}}(1+\|X\|^2)^{2p}(1+\|Y\|^2)^{2p}|D*f(\exp(X+Y))|^2 \\
&\leqslant \sup_{X\in\mathfrak{m}} CC_{4p}(D)\,{}_2\|f\|_{p'',q''}\int_{\mathfrak{z}^*}(1+\|X\|^2)^{2p+2l}|D_\nu * f_\nu(\exp(X))|\,d\nu \\
&\leqslant CC_{4p}(D)\,{}_2\|f\|_{p'',q''}\int_{\mathfrak{z}^*}(1+\|\nu\|^2)^{-k}\,{}_\infty\|D_\nu *(1-\Delta)^k f_\nu\|_{2p+2l,0}\,d\nu \\
&\leqslant C_p C_j^2(D)\,{}_\infty\|f\|^2_{p',q'}
\end{aligned}$$

for some j, p', q', C_p, and $C_j(D)$ that are each independent of f.

In light of Theorem 3.8 we consider the space $\mathrm{PB}^\infty_N(\mathfrak{n}^*)$ of all smooth Ad*N-invariant functions defined on $\mathfrak{n}^*$, all of whose derivatives have polynomial bounds. We define seminorms $\nu_{i,j}$ on $\mathrm{PB}^\infty_N(\mathfrak{n}^*)$ by

$$\nu_{i,j}(\theta) = \sup_{\xi\in\mathfrak{n}^*}\sup_{V\in S^{(j)}(\mathfrak{n}^*)}\frac{|\partial_V\theta(\xi)|}{\|V\|(1+\|\xi\|^2)^i}.$$

A sequence $\{\theta_n\}$ converges to θ in $\mathrm{PB}^\infty_N(\mathfrak{n}^*)$ if for each j and all i sufficiently large, $\nu_{i,j}(\theta_n-\theta) \to 0$. Given $\theta \in \mathrm{PB}^\infty_N(\mathfrak{n}^*)$, we let D_θ denote the element of $\mathcal{S}^*(N)$ such that $(D_\theta)\hat{} = \theta$. Theorem 3.8 may be rephrased as

THEOREM 3.8′. *The mapping $\mathrm{PB}^\infty_N(\mathfrak{n}^*) \times \mathcal{S}(N) \to \mathcal{S}(N)$ given by $(\theta, f) \to D_\theta * f$ is jointly continuous.*

Let $\mathcal{MS}(N/H, \chi_\lambda)$ denote the space of all bi-invariant distributions D in $\mathcal{S}^*(N/H, \chi_{-\lambda})$ such that $D*f \in \mathcal{S}(N/H, \chi_\lambda)$ for each $f \in \mathcal{S}(N/H, \chi_\lambda)$, topologized so that $D_n \to 0$ in $\mathcal{MS}(N/H, \chi_\lambda)$ if $D_n * f \to 0$ in $\mathcal{S}(N/H, \chi_\lambda)$.

THEOREM 3.9. *The mapping $\theta \to D_\theta$ is a homeomorphism of $\mathrm{PB}^\infty_N(\mathfrak{n}^*)$ onto $\mathcal{MS}(N)$.*

Remark. The fact that this mapping is also an algebra homomorphism is proved in Corollary 4.4.

Proof. It remains only to show that the Fourier transform of each bi-invariant Schwartz multiplier is in $\mathrm{PB}^\infty_N(\mathfrak{n}^*)$. We must first show that for $D \in \mathcal{MS}(N/Z, \chi_\lambda)$, $\hat{D} \in \mathrm{PB}^\infty_N(\mathfrak{z}^\perp + \lambda)$. For this, note that if $D \in \mathcal{MS}(N/H, \chi_\lambda)$ and $\nu \in \mathfrak{k}^*$, there is a $D_\nu \in \mathcal{MS}(N/Z, \chi_{\nu+\lambda})$ such that $D_\nu *(P_\nu f) = P_\nu(D*f)$ for each $f \in \mathcal{S}(N/H, \chi_\lambda)$, where P_ν: $\mathcal{S}(N/H, \chi_\lambda) \to \mathcal{S}(N/Z, \chi_{\nu+\lambda})$ is defined by

$$P_\nu f(x) = \int_{\mathfrak{k}} f(x\exp(Y))e^{-2\pi i\langle \nu, Y\rangle}\,dY,$$

and $\chi_{\nu+\lambda}$ is the character defined on $Z = \exp(\mathfrak{z}) = \exp(\mathfrak{k}\oplus\mathfrak{h})$ by $\chi_{\nu+\lambda}(\exp(Y+W)) = e^{2\pi i(\langle\nu,Y\rangle+\langle\lambda,W\rangle)}$. To see that D_ν is well defined, let $\{\varphi_n\} \subset \mathscr{S}(N/H, \chi_\lambda)$ be an approximate identity. Then for $f \in \mathscr{S}(N/H, \chi_\lambda)$, $D*f = \lim_{n\to\infty}(D*\varphi_n)*f$. Thus, if $F \in \mathscr{S}(N/Z, \chi_{\nu+\lambda})$ and $F = P_\nu f = P_\nu g$, then

$$\langle D_\nu, (p_\nu f)^{\check{}}\rangle = P_\nu(D*f)(e) = \lim_{n\to\infty}(D*\varphi_n)*P_\nu f(e) = \lim_{n\to\infty}(D*\varphi_n)*P_\nu g(e).$$

To see that D_ν is continuous on $\mathscr{S}(N/Z, \chi_{\nu+\lambda})$ it suffices to note that if $F_n \to F$ in $\mathscr{S}(N/Z, \chi_{\nu+\lambda})$, then one can construct a sequence $f_n \to f$ in $\mathscr{S}(N/H, \chi_\lambda)$ such that $F_n = P_\nu f_n$ and $F = P_\nu f$. It follows that $\langle D_\nu, \check{F}_n\rangle = P_\nu(D*f_n)(e) \to P_\nu(D*f)(e) = \langle D_\nu, \check{F}\rangle$. This also shows that $D_\nu * F_n \to D*F$ in $\mathscr{S}(N/Z, \chi_{\nu+\lambda})$.

We now show that if $D \in \mathscr{M}\mathscr{S}(N/Z, \chi_\lambda)$, $\hat{D} \in \mathrm{PB}_N^\infty(\mathfrak{z}^\perp+\lambda)$. The proof is by induction on $\dim(\mathfrak{n})$. If $\dim(\mathfrak{n}) \leqslant 2$, the result is trivial. Thus, assume $\dim(\mathfrak{n}) \geqslant 3$.

Suppose first that $\dim(\mathfrak{z}) = 1$. Pick X, Y, Z, $\mathfrak{n}_0$, and $\mathfrak{n}_1$ as in the beginning of the proof of Theorem 3.7. We denote by (t, W) the group element $\exp(tX+W)$, where $t \in \boldsymbol{R}$, $W \in \mathfrak{n}_1$. For $x \in N$, denote by f^x the function defined on N by $f^x(y) = f(x^{-1}yx)$. Note that $f^{\exp(sY)}(t, W) = f(t, W+stZ)$. Thus, using a partition of unity in the t-direction, one can show that if $f \in \mathscr{S}(N)$ such that $f(0, W) = 0$ for all $W \in \mathfrak{n}_1$, then $\langle D, \varrho_Z f\rangle = 0$. Hence, $D = D_1 + D_0$, where $D_1 \in \mathscr{M}\mathscr{S}(N_1)$ $(N_1 = \exp(\mathfrak{n}_1))$ and $D_0 \in \mathscr{M}\mathscr{S}(N)$ with $\varrho_Z D_0 = 0$. However, since $D_0 * f \in \mathscr{S}(N)$ for each $f \in \mathscr{S}(N)$, $D_0 = 0$. If $\mathfrak{z}$ is the center of $\mathfrak{n}_1$, the induction hypothesis yields that $\hat{D}_1 \in \mathrm{PB}_{N_1}^\infty(\mathfrak{z}^\perp+\lambda)$. The Ad*-invariance of $\hat{D}$ shows that $\hat{D} \in \mathrm{PB}_N^\infty(\mathfrak{z}^\perp+\lambda)$. Suppose therefore that $\mathfrak{z}$ is not the center of $\mathfrak{n}_1$. Let $\mathfrak{z}_1$ be a subspace of $\mathfrak{n}_1$ such that $\mathfrak{z}_1 \oplus \mathfrak{z}$ is the center of $\mathfrak{n}_1$. The mapping defined on $\mathfrak{z}_1$ by $Y \to D*f(x\exp(Y)) \in \mathscr{S}(\mathfrak{z}_1)$. Thus, by Fourier inversion,

$$D*f(x) = \int_{\mathfrak{z}_1^*} P_\nu(D*f)(x)\,d\nu = \int_{\mathfrak{z}_1^*} D_\nu * P_\nu f(x)\,d\nu,$$

where $D_\nu \in \mathscr{M}\mathscr{S}(N_1/Z_1 Z, \chi_{\nu+\lambda})$. By the induction assumption, $\hat{D}_\nu \in \mathrm{PB}_{N_1}^\infty((\mathfrak{z}_1+\mathfrak{z})^\perp+\nu+\lambda)$ for each $\nu \in \mathfrak{z}_1^*$. It follows that $\hat{D}$ is given by the function defined on $\mathfrak{z}^\perp+\lambda$ by $\eta+\nu+\lambda \to \hat{D}_\nu(\eta+\nu+\lambda)$, where $\eta \in (\mathfrak{z}_1+\mathfrak{z})^\perp$ and $\nu \in \mathfrak{z}_1^*$.

Let $\{\varphi_n\} \subset \mathscr{S}(N/Z, \chi_\lambda)$ be an approximate identity. Then $\{P_\nu\varphi_n\}$ is an approximate identity in $\mathscr{S}(N/Z_1 Z, \chi_{\nu+\lambda})$ for each $\nu \in \mathfrak{z}_1^*$. Thus, $(D_\nu * P_\nu\varphi_n) * P_\nu f \to D_\nu * P_\nu f$ in $\mathscr{S}(N/Z_1 Z, \chi_{\nu+\lambda})$. Since $(D_\nu * P_\nu\varphi_n)^\wedge \in \mathscr{S}((\mathfrak{z}_1+\mathfrak{z})^\perp+\nu+\lambda)$ it follows that for each $\eta \in (\mathfrak{z}_1+\mathfrak{z})^\perp$, the mapping defined on $\mathfrak{z}_1^*$ by $\nu \to \hat{D}(\eta+\nu+\lambda)(\hat{f})(\eta+\nu+\lambda) \in \mathscr{S}(\mathfrak{z}_1^*)$. Since $\nu \to (\hat{f})(\eta+\nu+\lambda) \in \mathscr{S}(\mathfrak{z}_1^*)$, $\nu \to \hat{D}(\eta+\nu+\lambda) \in \mathrm{PB}_N^\infty(\mathfrak{z}_1^*)$. This implies that $\hat{D} \in \mathrm{PB}_N^\infty(\mathfrak{z}^\perp+\lambda)$.

Suppose now that $\dim(\mathfrak{z}) \geqslant 2$. Let $\mathfrak{z}_0$ denote the kernel of λ in $\mathfrak{z}$. Then there is a natural identification of $\mathscr{S}(N/Z, \chi_\lambda)$ with $\mathscr{S}((N/Z_0)/(Z/Z_0), \bar{\chi}_\lambda)$, $f \to \bar{f}$, where $Z_0 = \exp(\mathfrak{z}_0)$ and $\bar{\chi}_\lambda$ is the expected character on Z/Z_0. Thus, given $D \in \mathscr{S}(N/Z, \chi_{-\lambda})$, there is a $\bar{D} \in \mathscr{S}^*((N/Z_0)/(Z/Z_0), \bar{\chi}_{-\lambda})$ such that $D*f = \bar{D}*\bar{f}$. By the induction hypothesis, $(\bar{D})^\wedge \in \mathrm{PB}_{N/Z_0}^\infty((\mathfrak{z}/\mathfrak{z}_0)^\perp+\lambda)$. Since $(\mathfrak{z}/\mathfrak{z}_0)^\perp+\lambda$ in $(\mathfrak{n}/\mathfrak{z}_0)^*$ is identified with $\mathfrak{z}^\perp+\lambda$ in $\mathfrak{n}^*$, $\hat{D} \in \mathrm{PB}_N^\infty(\mathfrak{z}^\perp+\lambda)$.

Suppose now that $D \in \mathscr{M}\mathscr{S}(N)$ and let $f \in \mathscr{S}(N)$. Then

$$D*f(x) = \int_{\mathfrak{z}^*} P_\lambda(D*f)(x)\,d\lambda = \int_{\mathfrak{z}^*} D_\lambda * P_\lambda f(x)\,d\lambda.$$

By the previous argument, $\hat{D}_\lambda \in \mathrm{PB}_N^\infty(\mathfrak{z}^\perp+\lambda)$. Repeating the argument used in the paragraph second above, one concludes that $\hat{D} \in \mathrm{PB}_N^\infty(\mathfrak{n}^*)$.

For the next corollary we need the following: $\mathfrak{n}$ is said to be *stratified* if there exist subspaces $\mathfrak{n}_i$ of $\mathfrak{n}$, $1 \leqslant i \leqslant k$, such that $\mathfrak{n} = \mathfrak{n}_1 \oplus \ldots \oplus \mathfrak{n}_k$, $[\mathfrak{n}_i, \mathfrak{n}_j] \subset \mathfrak{n}_{i+j}$, $1 \leqslant i, j \leqslant k$, and such that $\mathfrak{n}_1$ generates $\mathfrak{n}$ as a Lie algebra. Define a family of automorphisms on $\mathfrak{n}$, called *dilations*, and denoted by δ_t, $t > 0$, by setting $\delta_t X_i = t^i X_i$ for $X_i \in \mathfrak{n}_i$ and extending linearly. Define δ_t^* on $\mathfrak{n}^*$ by $\langle\delta_t^*\xi, X\rangle = \langle\xi, \delta_t X\rangle$, and for $\theta \in \mathrm{PB}_N^\infty(\mathfrak{n}^*)$, set $\theta_t(\xi) = \theta(\delta_t^*\xi)$. Note that $\theta_t \in \mathrm{PB}_N^\infty(\mathfrak{n}^*)$ for all $t > 0$.

COROLLARY 3.10. *Suppose* $\theta \in \mathrm{PB}_N^\infty(\mathfrak{n}^*)$ *with* $\theta(0) = 1$. *Then for* $f \in \mathscr{S}(N)$,

$$f = \mathscr{S}(N)\text{-}\lim_{t\to\infty} D_{\theta_t} * f.$$

Proof. Since $D_{\theta_t - 1} * f = D_{\theta_t} * f - f$, it suffices to show that if $\theta(0) = 0$, then $\theta_t \to 0$ in $\mathrm{PB}_N^\infty(\mathfrak{n}^*)$. This follows immediately from the observation that for $X_i \in \mathfrak{n}_{j_i}$,

$$\partial_{X_i^*}(\theta_t) = t^{j_i}(\partial_{X_i^*}\theta)\circ\delta_t^*.$$

4. Representations and bi-invariant Schwartz multipliers. Given $\xi \in \mathfrak{n}^*$, a *polarization* of ξ is a subalgebra $\mathfrak{m}$ of $\mathfrak{n}$ of maximum dimension with $[\mathfrak{m}, \mathfrak{m}] \subset \ker(\xi)$. (Note that a polarization will always contain $\mathfrak{z}$, the center of $\mathfrak{n}$.) Given such a polarization, a unitary character Ψ_ξ is defined on $M = \exp(\mathfrak{m})$ by

$$\Psi_\xi(\exp(X)) = e^{-2\pi i\langle\xi, X\rangle}. \tag{4.1}$$

If $\mathfrak{n}'$ is a subalgebra of $\mathfrak{n}$ that contains $\mathfrak{m}$, and $f \in \mathscr{S}(N')$, where $N' = \exp(\mathfrak{n}')$, we define $Q_\xi(f)$ on N' by

$$Q_\xi(f)(n') = \int_{\mathfrak{m}} f(n'\exp(X))\,\Psi_\xi(\exp(X))\,dX.$$

The following theorem is a generalization of a result proved in [J].

THEOREM 4.2. *Let $\mathfrak{m}$ be a polarization of $\xi\in\mathfrak{n}^*$, and let $D\in\mathcal{S}^*(N)$ such that $\hat{D}\in \mathrm{PB}_N^\infty(\mathfrak{n}^*)$. Then for each $f\in\mathcal{S}(N)$,*

$$Q_\xi(D*f)=\hat{D}(\xi)\,Q_\xi(f).$$

Proof. The proof is by induction on $\dim(\mathfrak{n})$. If $\mathfrak{n}$ is Abelian, $\mathfrak{m}=\mathfrak{n}$, and for $f\in\mathcal{S}(N)$, $Q_\xi(f)=\hat{f}(\xi)$. Thus, $Q_\xi(D*f)=(D*f)\hat{\ }(\xi)=\hat{D}(\xi)\hat{f}(\xi)=\hat{D}(\xi)\,Q_\xi(f)$.

Assume that $\dim(\mathfrak{z})=1<\dim(\mathfrak{n})$. Let $\mathfrak{n}_1$ be a Kirillov subalgebra of $\mathfrak{n}$, i.e. there exist elements $X_0,Y_0,Z_0\in\mathfrak{n}$ such that $\mathfrak{z}$ is spanned by Z_0 which is equal to $[X_0,Y_0]$, $\mathfrak{n}_1$ is the centralizer of Y_0, and $\mathfrak{n}=\boldsymbol{R}X_0\oplus\mathfrak{n}_1$. Let X_0^* be the element of $\mathfrak{n}^*$ that is dual to X_0 (with respect to $\mathfrak{n}_1$). Then $\mathrm{Ad}^*(\exp(tY_0))(\eta)=\eta+t\langle\eta,Z_0\rangle X_0^*$. Thus, if $\langle\eta,Z_0\rangle\neq 0$, $\hat{D}(\eta+tX_0^*)=\hat{D}(\eta)$ for all $t\in\boldsymbol{R}$. By continuity of $\hat{D}$, this holds for all $\eta\in\mathfrak{n}^*$.

Let $\mathfrak{n}_1^*=(\boldsymbol{R}X_0)^\perp$, $\mathfrak{n}_0^*=\boldsymbol{R}X_0^*$, and $\mathfrak{n}_0=\boldsymbol{R}X_0$. Then, for $m\in M$ and $n\in N$,

$$Q_\xi(D*f)(nm)$$
$$=\int_{\mathfrak{n}^*}\hat{D}(\eta)\int_{\mathfrak{n}}\int_{\mathfrak{m}} l_{n-1}f\big(m\exp(Y)\exp(X)\big)e^{-2\pi i(\langle\eta,X\rangle+\langle\xi,Y\rangle)}\,dY\,dX\,d\eta$$
$$=\int_{\mathfrak{n}_0^*}\int_{\mathfrak{n}_1^*}\hat{D}(\eta_0+\eta_1)\int_{\mathfrak{n}_0}\int_{\mathfrak{n}_1}\int_{\mathfrak{m}} l_{n-1}f\big(m\exp(Y)\exp(X_0+X_1)\big)\times e^{-2\pi i(\langle\eta_0,X_0\rangle+\langle\eta_1,X_1\rangle+\langle\xi,Y\rangle)}\,dY\,dX_1\,dX_0\,d\eta_1\,d\eta_0$$
$$=\int_{\mathfrak{n}_1^*}\hat{D}(\eta_1)\int_{\mathfrak{n}_1}\int_{\mathfrak{m}} l_{n-1}f\big(m\exp(Y)\exp(X_1)\big)e^{-2\pi i(\langle\eta_1,X_1\rangle+\langle\xi,Y\rangle)}\,dY\,dX_1\,d\eta_1$$
$$=Q_\xi\big(\bar{D}*(l_{n-1}f)^-\big)(m),$$

where $\bar{D}$ is the element of $\mathcal{S}^*(N_1)$ whose Fourier transform agrees with $\hat{D}$ on $\mathfrak{n}_1^*$, and $(l_{n-1}f)^-$ is the restriction of $l_{n-1}f$ to N_1. Since $\hat{D}$ is constant on $\mathfrak{n}_0^*$ cosets, one may assume that $\xi\in\mathfrak{n}_1^*$. Also note that since $M\subset N_1$, $Q_\xi\big((l_{n-1}f)^-\big)(m)=Q_\xi(f)(nm)$. Thus, by the induction assumption, $Q_\xi(D*f)=\hat{D}(\xi)\,Q_\xi(f)$.

Suppose now that $\dim(\mathfrak{z})\geqslant 2$. Given $\xi\in\mathfrak{n}^*$, pick a basis $\{X_d, X_{d-1},\ldots,X_0\}$ of $\mathfrak{n}$ so that $\{X_1,X_0\}\subset\mathfrak{z}$ and $\langle\xi,X_0\rangle=0$. Let $\mathfrak{n}_1=\mathrm{span}\{X_d,\ldots,X_1\}$, $\mathfrak{n}_0=\boldsymbol{R}X_0$, and let $\mathfrak{m}_1$ be a subspace of $\mathfrak{m}$ so that $\mathfrak{m}=\mathfrak{m}_1\oplus\mathfrak{n}_0$. Let $\{X_d^*,\ldots,X_0^*\}$ be the dual basis in $\mathfrak{n}^*$, with corresponding $\mathfrak{n}_1^*$ and $\mathfrak{n}_0^*$. Finally, assume that for fixed $\eta_1\in\mathfrak{n}_1^*$, $\eta_0\to\hat{D}(\eta_1+\eta_0)\in\mathcal{S}(\mathfrak{n}_0^*)$. For $U_1\in\mathfrak{n}_1$, $U_0\in\mathfrak{n}_0$,

(4.3) $$Q_\xi(D*f)\big(\exp(U_1+U_0)\big)$$
$$=\int_{\mathfrak{n}_1^*}\int_{\mathfrak{n}_0^*}\hat{D}(\eta_1+\eta_0)\int_{\mathfrak{n}_1}\int_{\mathfrak{n}_0}\int_{\mathfrak{m}_1}\int_{\mathfrak{n}_0} f\big(\exp(U_1+U_0)\exp(V_1+V_0)\times\exp(T_1+T_0)\big)e^{-2\pi i(\langle\eta_1,V_1\rangle+\langle\eta_0,V_0\rangle+\langle\xi,T_1\rangle)}\,dT_0\,dT_1\,dV_0\,dV_1\,d\eta_0\,d\eta_1$$

$$=\int_{\mathfrak{n}_1^*}\int_{\mathfrak{n}_0^*}\hat{D}(\eta_1+\eta_0)\int_{\mathfrak{n}_1}\int_{\mathfrak{n}_0}\int_{\mathfrak{m}_1}\bar{f}\big(\exp(U_1)\exp(V_1)\exp(T_1)\big)\times e^{-2\pi i(\langle\eta_1,V_1\rangle+\langle\eta_0,V_0\rangle+\langle\xi,T_1\rangle)}\,dT_1\,dV_0\,dV_1\,d\eta_0\,d\eta_1$$
$$=\int_{\mathfrak{n}_1^*}\int_{\mathfrak{n}_1}\int_{\mathfrak{m}_1}\hat{D}(\eta_1)\bar{f}\big(\exp(U_1)\exp(V_1)\exp(T_1)\big)\times e^{-2\pi i(\langle\eta_1,V_1\rangle+\langle\xi,T_1\rangle)}\,dT_1\,dV_1\,d\eta_1,$$

where $\bar{f}\in\mathcal{S}(N/\exp(\mathfrak{n}_0))$ given by

$$\bar{f}(n)=\int_{\mathfrak{n}_0} f\big(n\exp(T_0)\big)\,dT_0.$$

Since $\langle\xi,X_0\rangle=0$, $\xi\in\mathfrak{n}_1^*$ which is identified with $(\mathfrak{n}/\mathfrak{n}_0)^*$. Also, $\mathfrak{m}/\mathfrak{n}_0$ is a polarization of ξ. Thus, identifying $\mathfrak{m}_1$ with $\mathfrak{m}/\mathfrak{n}_0$ (as vector spaces) we have

$$Q_\xi(D*f)=\bar{Q}_\xi(\bar{D}*\bar{f})=(\bar{D})\hat{\ }(\xi)\,\bar{Q}_\xi(\bar{f})=\hat{D}(\xi)\,Q_\xi(f),$$

where $\bar{D}\in\mathcal{S}^*(N_1)$ whose Fourier transform agrees with $\hat{D}$ on $\mathfrak{n}_1^*$, and $\bar{Q}_\xi$ is defined on $\mathcal{S}(N/\exp(\mathfrak{n}_0))$ by

$$\bar{Q}_\xi(g)(n)=\int_{\mathfrak{m}/\mathfrak{n}_0} g\big(n\exp(X)\big)e^{-2\pi i\langle\xi,X\rangle}\,dX.$$

Finally, the assumption that $\eta_0\to\hat{D}(\eta_1+\eta_0)\in\mathcal{S}(\mathfrak{n}_0^*)$ was required for the Fourier inversion used in the third equality in (4.3). For more general D, approximate $\hat{D}$ in $\mathrm{PB}_N^\infty(\mathfrak{n}^*)$ by $\{\theta_n\}\subset \mathrm{PB}_N^\infty(\mathfrak{n}^*)$ with support of θ_n contained in the slabs $\{\eta\colon |\langle\eta,X_0\rangle|<n\}$. Then, by Theorem 3.8′,

$$Q_\xi(D*f)=\lim_{n\to\infty}Q_\xi(D_{\theta_n}*f)=\lim_{n\to\infty}\theta_n(\xi)\,Q_\xi(f)=\hat{D}(\xi)\,Q_\xi(f).$$

Let $\xi\in\mathfrak{n}^*$, and let π_ξ be the irreducible unitary representation of N corresponding to the Ad^*N-orbit of ξ. π_ξ can be realized as left translation on $L^2(N/M,\Psi_\xi)$, the space of all measurable functions F defined on N with $F(nm)=\Psi_\xi(m)F(n)$ for $n\in N$, $m\in M$, and having

$$\int_{N/M}|F(n)|^2\,dn<\infty,$$

where $\mathfrak{m}=\log(M)$ is a polarization of ξ and Ψ_ξ is the character defined in (4.1).

COROLLARY 4.3. *Let $\theta\in \mathrm{PB}_N^\infty(\mathfrak{n}^*)$ and $f\in\mathcal{S}(N)$. For $\xi\in\mathfrak{n}^*$,*

$$\pi_\xi(D_\theta*f)=\theta(\xi)\,\pi_\xi(f).$$

Proof. First note that $Q_\xi(\mathcal{S}(N))$ is a dense subspace of $L^2(N/M,\Psi_\xi)$. Given $f,g\in\mathcal{S}(N)$, $\theta\in \mathrm{PB}_N^\infty(\mathfrak{n}^*)$,

$$\pi_\xi(D_\theta*f)\,Q_\xi(g)=Q_\xi(D_\theta*f*g)=\theta(\xi)\,Q_\xi(f*g)=\theta(\xi)\,\pi_\xi(f)\,Q_\xi(g).$$

COROLLARY 4.4. *The mapping $\theta \to D_\theta$ is an algebra homomorphism from $\mathrm{PB}_N^\infty(\mathfrak{n}^*)$ to $\mathcal{M}\mathcal{S}(N)$.*

Proof. Given $\theta, \varphi \in \mathrm{PB}_N^\infty(\mathfrak{n}^*)$, $f \in \mathcal{S}(N)$, and $\xi \in \mathfrak{n}^*$,

$$\pi_\xi(D_{\theta\varphi} * f) = \theta(\xi)\varphi(\xi)\pi_\xi(f) = \theta(\xi)\pi_\xi(D_\varphi * f) = \pi_\xi(D_\theta * D_\varphi * f).$$

Thus, $D_{\theta\varphi} * f = D_\theta * D_\varphi * f$. Since f was arbitrary in $\mathcal{S}(N)$, $D_{\theta\varphi} = E_{D_0} \circ E_{D_\varphi}$. Linearity is obvious.

By the Plancherel Theorem for N, there is a measure on $\mathfrak{n}^*/\mathrm{Ad}^*$, Ω, such that for $f \in \mathcal{S}(N)$,

$$f(e) = \int_{\mathfrak{n}^*/\mathrm{Ad}^*} \mathrm{Tr}(\pi_\xi(f))\, d\Omega(\xi).$$

It follows that

$$\|f\|_{L^2}^2 = \int_{\mathfrak{n}^*/\mathrm{Ad}^*} \mathrm{Tr}(\pi_\xi(f)\pi_\xi(f)^*)\, d\Omega(\xi).$$

Thus, we have

COROLLARY 4.5. *If $\theta \in \mathrm{PB}_N^\infty(\mathfrak{n}^*)$ is bounded, then $\|D_\theta * f\| \leqslant \|\theta\|_\infty \|f\|_{L^2}$, i.e. E_{D_0} extends to a bounded operator on $L^2(N)$.*

5. Applications. The first application concerns eigenfunction expansion for certain left-invariant differential operators on N. For this, we require that N be stratified, i.e. $\mathfrak{n}$ has a direct sum decomposition, $\mathfrak{n} = \mathfrak{n}_1 \oplus \ldots \oplus \mathfrak{n}_k$, such that $[\mathfrak{n}_i, \mathfrak{n}_j] \subset \mathfrak{n}_{i+j}$, $1 \leqslant i, j \leqslant k$, and such that $\mathfrak{n}_1$ generates $\mathfrak{n}$. A one parameter family of dilations $\{\delta_t\}_{t>0}$ is defined on $\mathfrak{n}$ by setting $\delta_t X = t^i X$ for $X \in \mathfrak{n}_i$, and extending linearly. Let $Q = k(k+1)/2$. For a function f defined on N, set

$$f_t(\exp(X)) = t^{-Q} f(\exp(\delta_{t^{-1}} X)),$$

and for a function θ defined on $\mathfrak{n}^*$, set $\theta_t(\xi) = \theta(\delta_t^* \xi)$, where $\langle \delta_t^* \xi, X\rangle = \langle \xi, \delta_t X\rangle$.

A left-invariant differential operator L (we drop the distinction between L and ϱ_L) is said to be a *Rockland operator* (of degree γ) if $L(f \circ \delta_t) = t^\gamma Lf \circ \delta_t$, and if for every nontrivial irreducible unitary representation π of N, $\pi(L)$ is injective on the space of C^∞-vectors. In virtue of Helffer and Nourrigat [HN], a positive Rockland operator L is hypoelliptic, and thus is essentially selfadjoint on $C_c^\infty(N)$ in $L^2(N)$ by a theorem of Nelson and Stinespring [NS]. Thus, the closure of $-L$ is the infinitesimal generator of a convolution semigroup $\{P_t\}_{t>0}$. Folland and Stein [FS] have shown that $\{P_t\} \subset \mathcal{S}(N)$.

Let $\mathcal{A}$ denote the closed subalgebra of $L^1(N)$ spanned by $\{P_t\}_{t>0}$. It follows from the homogeneity of L that $\mathcal{A}$ is closed under the mapping $f \to f_t$. Consequently, the Gelfand space of $\mathcal{A}$ can be identified with $\boldsymbol{R}^+$ in

such a way that given a spectral resolution of L,

$$Lf = \int_0^\infty \lambda\, dE(\lambda) f, \quad f \in C_c^\infty(N),$$

and $k \in \mathcal{A}$, one has

$$k * f = \int_0^\infty \hat{k}(\lambda)\, dE(\lambda) f, \quad f \in L^2(N),$$

where $\hat{k}$ is the Gelfand transform of k. Also, if L is homogeneous of degree γ, then $(k_t)\hat{}(\lambda) = \hat{k}(t^\gamma \lambda)$. It easily follows that if $\int k(x)\,dx = 1$, then for $f \in L^p(N)$,

$$L^p\text{-}\lim_{t\to\infty} k_t * f = f.$$

Thus, if $k \in \mathcal{A}$ and $\int k(x)\,dx = 1$,

$$f = L^p\text{-}\lim_{t\to\infty} \int_0^\infty \hat{k}(t\lambda)\, dE(\lambda) f, \quad f \in L^p(N).$$

In [H], Hulanicki gave conditions on a function K defined on $\boldsymbol{R}^+$ that are sufficient to imply that $K = \hat{k}$ for some $k \in \mathcal{A}$.

Suppose now that N contains a discrete cocompact subgroup Γ. Then L is naturally defined as a differential operator on $\Gamma \backslash N$. Since $\{P_t\} \subset \mathcal{S}(N)$ ($L^1(N)$ is sufficient), it follows that the spectrum of L, $\sigma(L)$, is a discrete subset of $\boldsymbol{R}^+$ with finite multiplicities. One looks for summability kernels K, defined on $\boldsymbol{R}^+$, so that for $F \in L^+(\Gamma \backslash N)$, $1 \leqslant p \leqslant \infty$,

$$F = L^p\text{-}\lim_{t\to\infty} \sum_{\lambda\in\sigma(L)} K(t\lambda) F_\lambda,$$

where F_λ is the projection of F onto the eigensubspace corresponding to λ.

By the Kirillov theory, the irreducible unitary representations of N, $\hat{N}$, can be identified with the Ad^*-orbits in $\mathfrak{n}^*$. Let π_ξ denote the representation corresponding to the orbit of $\xi \in \mathfrak{n}^*$. There is a discrete subspace $(\Gamma \backslash N)\hat{} \subset \hat{N}$ such that

$$L^2(\Gamma \backslash N) = \bigoplus \sum_{\pi_\xi \in (\Gamma\backslash N)\hat{}} \mathcal{H}_\xi,$$

where right translation on $\mathcal{H}_\xi$ is a finite multiple of π_ξ (cf. [R]). Thus, $\sigma(L)$ is the union of $\sigma(\pi_\xi(L))$, $\pi_\xi \in (\Gamma \backslash N)\hat{}$, not counting multiplicities. Therefore, there is a natural identification of $\sigma(L)$ with a subset of $\mathfrak{n}^* \times \boldsymbol{R}^+$, and so one looks for summability kernels on this set.

Let $\theta \in \mathrm{PB}_N^*(\mathfrak{n}^*)$ with $\theta(0) = 1$. It follows from Theorem 3.8′ that there is a function s: $\boldsymbol{R}^+ \to \boldsymbol{R}^+$ such that

$$\lim_{t\to 0} \|D_{\theta_{s(t)}} * k_t - k_t\| = 0.$$

Combining this with Corollary 4.3 gives

THEOREM 5.1. *If $\theta \in PB_N^\infty(\mathfrak{n}^*)$ with $\theta(0) = 1$, and if $k \in \mathcal{A} \cap \mathcal{S}(N)$, then there is an s: $R^+ \to R^+$ such that for $F \in L^p(\Gamma \backslash N)$, $1 \leqslant p \leqslant \infty$,*

$$F = L^p\text{-}\lim_{t \to 0} \sum \theta(\delta^*_{s(t)} \xi) \hat{k}(t\lambda) F_\lambda,$$

where the sum is over $\sigma(L)$ in $\mathfrak{n}^* \times R^+$.

A second application concerns local solvability. A left-invariant differential operator L on N is said to be *locally solvable* if there is an open set $U \subset N$ such that $C_c^\infty(U) \subset L(C^\infty(U))$, i.e. if for each $f \in C_c^\infty(U)$ there is a $u \in C^\infty(U)$ such that $Lu = f$.

Let $o(\xi)$ denote the Ad*-orbit in $\mathfrak{n}^*$ that contains ξ and, having fixed a norm on $\mathfrak{n}^*$, set $|o(\xi)| = \inf\{\|\xi'\|: \xi' \in o(\xi)\}$. There is a linear subspace $V \subset \mathfrak{n}^*$ and a Zariski open subset $V_0 \subset V$ such that the elements in V_0 parametrize an open dense set of orbits in $\mathfrak{n}^*$. Representations corresponding to elements of V_0 are said to be *in general position.*

The Plancherel measure Ω is supported on V_0, and in fact is absolutely continuous with respect to the Lebesgue measure on V with density given by a rational function.

THEOREM 5.2. *Let L be a left-invariant differential operator on N.*

(i) *Suppose that for each $\xi \in V_0$, $\pi_\xi(L)$ has a bounded right inverse A_ξ, that $\xi \to A_\xi$ is measurable, and that the norm of A_ξ is bounded by a polynomial in $|o(\xi)|$.*

(ii) *Suppose that N contains a discrete cocompact subgroup Γ, that for each $\xi \in V_0$ for which $\pi_\xi \in (\Gamma \backslash N)^\wedge$, $\pi_\xi(L)$ has a bounded right inverse A_ξ, and that the norm of A_ξ is bounded by a polynomial in $|o(\xi)|$.*

If either (i) *or* (ii) *holds then L is locally solvable.*

Remarks. The theorem with condition (i) is (essentially) a theorem due to Corwin [C], and our proof is an adaptation of his proof. The theorem with condition (ii) was proved by Corwin and Greenleaf [CG] with the additional assumption that all the representations in general position were induced from a common normal subgroup.

Proof. (i) Let Z be a bi-invariant differential operator on N such that $\pi_\xi(Z) = 0$ for each representation π_ξ not in general position. Since Z has a fundamental solution (cf. [R]), it suffices to show that for some U and each $f \in C_c^\infty(U)$ there is a $g \in C^\infty(U)$ satisfying $Lg = Zf$.

Given f, by [DM] there exist $g_i, h_i \in C_c^\infty(U)$, $i = 1, \ldots, k$, such that

$$f = \sum_{i=1}^k g_i * h_i.$$

Let j be a positive integer sufficiently large so that

$$\int_V \max\{1, \|A_\xi\|\}/(1+|o(\xi)|)^j \, d\xi < \infty,$$

and let φ be a smooth function defined on the complex numbers with values in $[0, 1]$ such that $\varphi(z) = 0$ if $|z| \leqslant 1/2$, and $\varphi(z) = 1$ if $|z| \geqslant 1$. Define θ on V by

$$\theta(\xi) = \varphi\big(\pi_\xi(Z)(1+|o(\xi)|)^j\big)(1+|o(\xi)|)^j + 1.$$

(Note that since Z is bi-invariant, $\xi \to \pi_\xi(Z)$ is an Ad*-invariant polynomial on $\mathfrak{n}^*$.) Then θ and $1/\theta$ have unique extensions to elements of $PB_N^\infty(\mathfrak{n}^*)$. There exist elements $u_i \in L^2(N)$ such that for $v \in L^2(N)$,

$$\langle u_i, v\rangle = \int_{V_0} \langle A_\xi(Q_\xi(D_{1/\theta} * Zh_i)), Q_\xi v\rangle \, d\nu(\xi),$$

where Q_ξ is defined before Theorem 4.2. It follows that $Lu_i = Zh_i$. Thus, if we let

$$u = \sum_{i=1}^k (D_\theta * g_i) * u_i,$$

then

$$Lu = \sum_{i=1}^k (D_\theta * g_i) * Lu_i = \sum_{i=1}^k (D_\theta * g_i) * (D_{1/\theta} * Zh_i) = \sum_{i=1}^k g_i * Zh_i = Zf.$$

The proof of (ii) is similar. For $f \in \mathcal{S}(N)$ we define $\tau f \in L^2(\Gamma \backslash N)$ by

$$\tau f(n) = \sum_{\gamma \in \Gamma} f(\gamma n).$$

Define D'_θ: $\mathcal{S}^*(N) \to \mathcal{S}^*(N)$ by $\langle D'_\theta(D), f\rangle = \langle D, D_\theta * f\rangle$ for $D \in \mathcal{S}^*(N)$ and $f \in \mathcal{S}(N)$. Then, for $f, g \in \mathcal{S}(N)$,

$$\begin{aligned}\langle D'_\theta(\tau f), g\rangle &= \langle \tau f, D_\theta * g\rangle = \int_N \sum f(\gamma n) D_\theta * g(n^{-1})\, dn \\ &= \sum_{\gamma \in \Gamma} \langle l_\gamma f, D_\theta * g\rangle \quad (\text{where } l_\gamma f(n) = f(\gamma n)) \\ &= \sum_{\gamma \in \Gamma} \langle l_\gamma (D_\theta * f), g\rangle = \langle \tau(D_\theta * f), g\rangle.\end{aligned}$$

Thus,

$$D'_\theta(\tau f) = \tau(D_\theta * f). \tag{5.3}$$

It easily follows from (5.2) and (5.3) that for $f, g \in \mathcal{S}(N)$,

$$(D_\theta * f) * \tau g = f * \tau(D_\theta * g). \tag{5.4}$$

Let P_ξ denote the orthogonal projection from $L^2(\Gamma\backslash N)$ to $\mathscr{H}_\xi$. In [J] we proved that

$$D'_\theta(\tau f) = \sum_{\gamma\in(\Gamma\backslash N)^-} \theta(\xi)\, P_\xi(\tau f).$$

In particular then, $P_\xi(D'_\theta(\tau f)) = \theta(\xi)\, P_\xi(\tau f)$.

Let Z be a bi-invariant differential operator on N such that $\pi_\xi(Z) = 0$ for each representation π_ξ not in general position. As noted in [CG], to prove the theorem, it suffices to show that for some U and each $f \in C_c^\infty(U)$ there is a $g \in C^\infty(U)$ satisfying $Lg = Zf$.

Let $U = U^{-1}$ be a neighborhood of the identity in N such that $U^3 \cap \Gamma = \{e\}$, and let $f \in C_c^\infty(U)$. By [DM] there exist $g_i, h_i \in C_c^\infty(U)$, $i = 1, \ldots, k$, such that

$$f = \sum_{i=1}^{k} g_i * h_i.$$

Let j be a positive integer sufficiently large so that

$$\sum_{\pi_\xi\in(\Gamma\backslash N),\xi\in V_0} \max\{1, \|A_\xi\|\}/(1+|o(\xi)|)^j < \infty. \tag{5.5}$$

Define φ and θ as before, and set

$$u_i = \sum_{\pi_\xi\in(\Gamma\backslash N),\xi\in V_0} A_\xi\, P_\xi\big(D'_{1/\theta}(\tau(Zh_i))\big).$$

By splitting the sum into two parts, one containing the terms for which $|\pi_\xi(Z)| < (1+|o(\xi)|)^{-j}$, and using (5.5), one can show that $u_i \in L^2(\Gamma\backslash N)$. Using the same argument as the one used in [CG], one can show that $Lu_i = \tau(D_{1/\theta} * Zh_i)$. Thus, if we let

$$u = \sum_{i=1}^{k} (D_\theta * g_i) * u_i,$$

then, using (5.4) and the fact that $D_\theta * D_{1/\theta} = I$,

$$Lu = \sum_{i=1}^{k} (D_\theta * g_i) * Lu_i = \sum_{i=1}^{k} D_\theta * g_i * \tau(D_{1/\theta} * Zh_i) = Z\Big(\sum_{i=1}^{k} g_i * \tau(h_i)\Big).$$

Since $\operatorname{supp}(g_i * \tau(h_i)) \cap U = \operatorname{supp}(g_i * h_i) \cap U$, $Lu = Zf$ on U.

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