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Singular integrals on C_p^α

by

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Abstract. Singular integrals are estimated in terms of maximal operators which reflect the smoothness of a function. These inequalities generalize the classical Privalov result of Lipschitz estimates for the conjugate operator and provide regularity inequalities for Poisson's equation in the setting of C_p^α spaces.

The classical result of Privalov states that the conjugate operator is bounded on Lipα for $0 < \alpha < 1$ (see, e.g., [8], Theorem (13.29)). C_p^α spaces form a natural extension of the Lipschitz and Sobolev spaces for fractional α. The norm of C_p^α is defined in terms of local approximation properties and is equivalent to the Lipα norm when $p = \infty$ or to a Sobolev norm if α is an integer. Otherwise, C_p^α is known to be neither a Bessel potential space nor a Besov space and thereby provides an attractive fractional substitute for the Sobolev spaces when approximation properties of a function appear to be an important consideration.

The purpose of this note is to establish the boundedness of singular integral operators on these spaces and as a consequence obtain a regularity estimate for Poisson's equation in terms of the corresponding norms. Similar estimates have appeared in the work of Calderón and Zygmund [3], [4] where a program was developed for local L^p regularity of systems of elliptic operators. In those references estimates for singular integrals are provided in terms of norms incorporating both local regularity via maximal operators and global L^p growth. Our estimates involve only the appropriate maximal operators $f_\alpha^\#$ described below. Additional requirements in [3], [4] are that the singular integrals have C^∞ kernels away from the origin and that $1 < p < \infty$. We establish results for kernels satisfying the standard properties (3)–(5) below and for the range $1 \leqslant p \leqslant \infty$.

Boundedness of singular integral operators has been established for other generalizations of the Lipα spaces. In particular, Taibleson [7] has

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obtained results for generalized Lipschitz spaces (based on moduli of smoothness) which are commonly referred to as Besov spaces in the literature.

The space $C_p^\alpha(\boldsymbol{R}^n)$ is defined in terms of the maximal operator

$$f_{\alpha,q}^{\#}(x) := \sup_{Q \ni x} \{ \inf_{P \in \boldsymbol{P}_{[\alpha]}} |Q|^{-\alpha/n} (|Q|^{-1} \int_Q |f-P|^q)^{1/q} \}. \tag{1}$$

The infimum (for each fixed cube $Q \subset \boldsymbol{R}^n$) is taken over all polynomials of degree no larger than α, and the supremum is then taken over all n-dimensional cubes Q which contain x. If $q = 1$, we denote this maximal function by simply $f_\alpha^{\#}$. The space C_p^α is defined as the Banach space of functions for which the norm

$$\|f\|_{C_p^\alpha} := \|f\|_{L^p} + \|f_\alpha^{\#}\|_{L^p}, \quad 1 \leqslant p \leqslant \infty,$$

is finite. Likewise the seminorm for the homogeneous space $\dot{C}_p^\alpha$ is defined by

$$|f|_{C_p^\alpha} := \|f_\alpha^{\#}\|_{L^p}.$$

For the properties of these spaces and maximal functions, we refer to [5]. In particular, by Theorem 4.3 of [5] there exists $\varepsilon = \varepsilon(p, \alpha, n) > 0$ such that an equivalent norm for C_p^α results if $f_\alpha^{\#}$ is replaced by $f_{\alpha,q}^{\#}$ and q is chosen to satisfy $0 < q < p+\varepsilon$.

We consider principal-value convolution operators

$$Kf(x) := \text{p.v.} \int f(x-y)\, k(y)\, dy, \tag{2}$$

where the kernel k satisfies the following properties (see Ch. II of [6]):

$$|k(y)| \leqslant c/|y|^n, \quad y \neq 0, \tag{3}$$

$$|\nabla k(y)| \leqslant c/|y|^{n+1}, \quad y \neq 0, \tag{4}$$

$$\int_{r<|y|<R} k(y)\, dy = 0 \quad \text{if } 0 < r < R < \infty. \tag{5}$$

It is well known that these singular integral operators are bounded on $L^p(\boldsymbol{R}^n)$ if $1 < p < \infty$, but only satisfy weak type conditions at the endpoints $p = 1$ (see [6], § 2 of Ch. II) and $p = \infty$ (see [2]). In particular, if f belongs to $L^1(\boldsymbol{R}^n)$, then Kf exists almost everywhere and belongs to weak-L^1. These kernels are patterned after the Riesz transforms $R_j f$ whose kernels are given by

$$k_j(y) = \frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}} \frac{y_j}{|y|^{n+1}}, \quad 1 \leqslant j \leqslant n,$$

where $y = (y_1, \ldots, y_n)$. If g is a C^2 function with compact support, it follows (see p. 59 of [6]) from the Fourier transforms that

$$D^\mu g = -R^\mu(\Delta g)$$

if $|\mu| = 2$, $D^\mu = (\partial/\partial x_1)^{\mu_1} \ldots (\partial/\partial x_n)^{\mu_n}$, $R^\mu = R_1^{\mu_1} \ldots R_n^{\mu_n}$, and Δ denotes the Laplacian. This last identity indicates the importance of singular integrals in studying the differentiability properties of functions.

In order to facilitate certain estimates we also consider a related maximal operator (see [3])

$$N_q^\alpha(f, x) := \inf_{P \in \boldsymbol{P}_{(\alpha)}} \{ \sup_{r>0} [r^{-\alpha-n/q} (\int_{|y-x| \leqslant r} |f(y)-P(y)|^q\, dy)^{1/q}] \},$$

where (α) is the largest integer less than α. By Theorem 5.3 of [5] (when α is nonintegral) there are positive constants c_1, c_2 such that

$$c_1 f_{\alpha,q}^{\#}(x) \leqslant N_q^\alpha(f, x) \leqslant c_2 f_{\alpha,q}^{\#}(x) \tag{6}$$

holds for all x and all locally L^q functions f.

The first lemma may be considered as a "reverse Hardy inequality" and is implicit in the calculations carried out in [4].

Lemma 1. *Suppose ψ is nonnegative on $(0, \infty)$ and $\Psi(t) = \int_0^t \psi(s)\, ds$. If there exist positive constants A, γ such that*

$$\Psi(t) \leqslant At^\gamma \tag{7}$$

holds for all $t > 0$, then it follows that

$$\int_0^t \psi(s) s^{-\beta}\, ds \leqslant cAt^{\gamma-\beta} \quad \text{if } \beta < \gamma, \tag{8}$$

$$\int_t^\infty \psi(s) s^{-\beta}\, ds \leqslant cAt^{\gamma-\beta} \quad \text{if } \beta > \gamma. \tag{9}$$

The constant c in these inequalities may be taken as $(\gamma+|\beta|)/|\gamma-\beta|$.

Proof. We establish inequality (8). The proof of inequality (9) is similar. Inequality (8) is trivial if $\beta \leqslant 0$ so we may assume that $\beta > 0$. In this case, using integration by parts and the estimate (7) we obtain

$$\int_0^t \psi(s) s^{-\beta}\, ds = \int_0^t s^{-\beta}\, d\Psi(s) \leqslant \Psi(t) t^{-\beta} + \beta \int_0^t \Psi(s) s^{-\beta-1}\, ds$$

$$\leqslant \left(1 + \frac{\beta}{\gamma-\beta}\right) At^{\gamma-\beta}. \quad \blacksquare$$

Define L^1+L^∞ to be the set of all $f = g+b$ such that $g \in L^1$ and $b \in L^\infty$. For each such f, define

$$\tilde{K}f(x) := \lim_{\varepsilon \to 0+} \int [k_\varepsilon(x-y) - k_1(-y)] f(y)\, dy,$$

where

$$k_\varepsilon(y) := k(y)\, \chi_{\{|y| \geq \varepsilon\}};$$

then $\tilde{K}f$ is well defined (see Section 5.6 of [1] for details of the necessary techniques). Note that when f belongs to L^p (p finite), $\tilde{K}f$ and Kf differ only by a constant.

LEMMA 2. *Suppose k satisfies the kernel conditions (3)-(5) and let $0 < \alpha < 1$. If $\tilde{K}$ is the associated singular integral operator and $1 < q < \infty$, then there exists a constant c, depending only on α and q, such that for each f belonging to L^1+L^∞,*

$$(\tilde{K}f)^{\#}_{\alpha,q}(x) \leq c f^{\#}_{\alpha,q}(x). \tag{10}$$

Consequently, if f belongs to L^p $(1 \leq p < \infty)$, then

$$(Kf)^{\#}_{\alpha,q}(x) \leq c f^{\#}_{\alpha,q}(x).$$

Proof. Fix $x = x_0$ and let A denote $f^{\#}_{\alpha,q}(x_0)$. We may assume that A is finite. Let $r > 0$, then by Hölder's inequality and the relation (6), if

$$h_r(y) := [f(x_0+y) - f(x_0)]\, \chi_{\{|y| \leq r\}}, \tag{11}$$

we have from the definition of $f^{\#}_{\alpha,q}(x_0)$ the estimate

$$\|h_r\|_{L^1} \leq c r^{n(1-1/q)} \|h_r\|_{L^q} \leq c A r^{\alpha+n}. \tag{12}$$

Now by property (5) of k we may write for $|z| \leq r$,

$$\begin{aligned}\tilde{K}f(x_0+z) - \tilde{K}f(x_0) = Kh_{2r}(z) &+ \int_{|y| \geq 2r} [f(x_0+y) - f(x_0)][k(z-y) - k(-y)]\, dy \\ &- \text{p.v.} \int_{|y| \leq 2r} [f(x_0+y) - f(x_0)]\, k(-y)\, dy.\end{aligned}$$

But by properties (4) and (3), we then obtain

$$\begin{aligned}|\tilde{K}f(x_0+z) - \tilde{K}f(x_0)| \leq |Kh_{2r}(z)| &+ c|z| \int_{|y| \geq 2r} |f(x_0+y) - f(x_0)|\, |y|^{-n-1}\, dy \\ &+ c \int_{|y| \leq 2r} |f(x_0+y) - f(x_0)|\, |y|^{-n}\, dy.\end{aligned} \tag{13}$$

But K is bounded on $L^q(R^n)$ and so inequality (12) implies

$$\Big(\int_{|z| \leq r} |Kh_{2r}|^q\Big)^{1/q} \leq c\|h_{2r}\|_{L^q} \leq c A r^{\alpha+n/q}, \qquad r > 0. \tag{14}$$

In addition, by inequality (9) of Lemma 1 we obtain

$$\int_{|y| \geq 2r} |f(x_0+y) - f(x_0)|\, |y|^{-n-1}\, dy \leq c A r^{\alpha-1} \tag{15}$$

by changing to polar coordinates and setting $\beta = n+1$, $\gamma = \alpha+n$, and $\psi(t) = t^{n-1} \int_{|\omega|=1} |f(x_0+t\omega) - f(x_0)|\, d\omega$. That the condition (7) of the lemma is satisfied follows from inequality (12). Similarly, inequality (8) implies

$$\int_{|y| \leq 2r} |f(x_0+y) - f(x_0)|\, |y|^{-n}\, dy \leq c A r^{\alpha} \tag{16}$$

by letting $\beta = n$. If we apply an L^q norm to inequality (13) over the ball $|z| \leq r$, then the inequalities (14)–(16) imply that

$$\Big(r^{-n} \int_{|z| \leq r} |\tilde{K}f(x_0+z) - \tilde{K}f(x_0)|^q\, dz\Big)^{1/q} \leq c A r^{\alpha} \tag{17}$$

since $\big(\int_{|z| \leq r} |z|^q\, dz\big)^{1/q} \leq c r^{1+n/q}$. Dividing by r^α and taking supremum over all $r > 0$ gives the desired estimate (10). ■

In order to establish estimates for $p = 1$, consider the following maximal operator:

$$M_\sigma g(x) := \sup_{Q \ni x} \Big(|Q|^{-1} \int_Q |g(y)|^\sigma\, dy\Big)^{1/\sigma}, \tag{18}$$

where the supremum ranges over all cubes Q which contain x. It is clear that $(M_\sigma g)^\sigma = M(|g|^\sigma)$, where M is the Hardy–Littlewood maximal operator (see p. 4 of [6]). Since M is a bounded operator on L^q if $1 < q \leq \infty$, it follows immediately that

$$\|M_\sigma g\|_{L^p} \leq c\|g\|_{L^p}, \qquad 0 < \sigma < p \leq \infty, \tag{19}$$

where c depends only on p and σ. Let $\tilde{K}$ be the singular integral defined on L^1+L^∞ whose kernel satisfies the conditions (3)–(5). Then $\tilde{K}$ satisfies the following estimate.

THEOREM 1. *Suppose $0 < \alpha < 1$. Then there exist constants $c > 0$ and $0 < \sigma < 1$, which depend only on α and n, such that for all f belonging to L^1+L^∞,*

$$(\tilde{K}f)^{\#}_{\alpha}(x) \leq c M_\sigma(f^{\#}_{\alpha})(x), \tag{20}$$

where M_σ is the Hardy–Littlewood operator given in (18).

Proof. By Hölder's inequality and Theorem 4.3 of [5] the maximal functions $g^{\#}_{\alpha,q}$ and $g^{\#}_{\alpha}$ may be compared by the inequalities

$$g^{\#}_{\alpha} \leq g^{\#}_{\alpha,q} \leq c M_\sigma(g^{\#}_{\alpha}) \tag{21}$$

if $1/\sigma = 1/q+\alpha/n$. Pick $1 < q < \infty$ for α so that $0 < \sigma < 1$. Applying inequality (21) to inequality (10), we obtain the estimate

$$(\tilde{K}f)_\alpha^\# \leqslant (\tilde{K}f)_{\alpha,q}^\# \leqslant cf_{\alpha,q}^\# \leqslant cM_\sigma(f_\alpha^\#)$$

as desired. ■

COROLLARY 1. *Suppose $\alpha > 0$ is nonintegral and $1 \leqslant p \leqslant \infty$. The singular integral operator $\tilde{K}$ satisfies the estimate*

$$|\tilde{K}f|_{C_p^\alpha} \leqslant c\,|f|_{C_p^\alpha}. \tag{22}$$

Proof. If $0 < \alpha < 1$, the estimate (22) follows directly from (20) and the mapping property (19) of M_σ. If $\alpha > 1$ is nonintegral, then let $j = [\alpha]$. Note that even though $\tilde{K}$ is not translation-invariant, we still have for $|\mu| = j$

$$\Delta_z D^\mu \tilde{K}f(x_0) = D^\mu \Delta_z \tilde{K}f(x_0) = \Delta_z \tilde{K}[D^\mu f](x_0)$$

since D_μ and the difference operator Δ_z commute. The second equation follows by using the decomposition in the equation following inequality (12) in Lemma 2. The proof is completed by an application of the appropriate "reduction theorem" (see Theorem 6.7 of [5]). ■

Corollary 1 does not hold if α is an integer and p is infinite (see p. 121 of [8]). On the other hand, if $1 < p < \infty$ and $\alpha = j$ is an integer, then the statement of Corollary 1 is well known for the Sobolev spaces W_p^j. This classical result can also be included in the framework of Corollary 1 if in the definition (1) of $f_{\alpha,q}^\#$ one adjusts the degree of the polynomials in the infimum to be $j-1$ instead of j. The resulting maximal function is called $f_{\alpha,q}^{\flat}$ and the corresponding spaces based on this maximal function are denoted by $\mathscr{C}_p^\alpha$ (see [5] for details). The spaces C_p^α and $\mathscr{C}_p^\alpha$ differ when α is an integer but coincide otherwise. In the case that α is an integer, j say, $\mathscr{C}_p^\alpha$ is the Sobolev space W_p^j for $1 < p \leqslant \infty$ (see, e.g., Theorem 6.2 of [5]). Together with the fact that K is bounded on L^p $(1 < p < \infty)$, this discussion verifies that for all $\alpha > 0$ and $1 < p < \infty$,

$$\|Kf\|_{\mathscr{C}_p^\alpha} \leqslant c\,\|f\|_{\mathscr{C}_p^\alpha}. \tag{23}$$

For the special elliptic operator Δ we have the analogous version of the regularity results (on $\boldsymbol{R}^n$) of Calderón [3] and Calderón–Zygmund [4].

COROLLARY 2. *Suppose $\alpha > 0$ is nonintegral and $1 \leqslant p \leqslant \infty$. If f belongs to C_p^α, then there exists F in $C_p^{\alpha+2}$ such that*

$$\Delta F = f, \tag{24}$$

$$|F|_{C_p^{\alpha+2}} \leqslant c\,\|f\|_{C_p^\alpha}. \tag{25}$$

Proof. First assume $0 < \alpha < 1$ and let f belong to C_p^α. As mentioned before, the expression $\|f_{\alpha,q}^\#\|_{L^p}+\|f\|_{L^p}$ is equivalent to the norm of f in C_p^α if q is chosen to satisfy

$$1 < q < p+\varepsilon. \tag{26}$$

We begin by establishing the existence of F so that $\Delta F = f$. Let φ be a nonnegative C_0^∞ function which is one on $|x| \leqslant 1$ and vanishes for $|x| \geqslant 2$. For $j = 1, 2, 3, \ldots$, we set

$$f_j(x) = f(x)\,\varphi(x/j). \tag{27}$$

Then f_j has compact support and satisfies

$$\sup_j \|f_j\|_{C_p^\alpha}. \tag{28}$$

Since f_j belongs to L^q, the proof of Lemma 8 of [3] shows that there exists a locally L^q function F_j such that $\Delta F_j = f_j$ and

$$D^\mu F_j = -R^\mu(\Delta F_j) = -R^\mu(f_j) \qquad (|\mu| = 2). \tag{29}$$

Now if we apply the "reduction" inequality for qth powers (established in a similar manner to that for inequality (6.13) of [5] but for q replacing 1), equation (29), and Lemma 2 in that order, we obtain the inequalities

$$(F_j)_{\alpha+2,q}^\# \leqslant c \sum_{|\mu|=2} (D^\mu F_j)_{\alpha,q}^\# = c \sum_{|\mu|=2} (R^\mu f_j)_{\alpha,q}^\# \leqslant c\,(f_j)_{\alpha,q}^\#.$$

But as we have noticed $(f_j)_{\alpha,q}^\# \leqslant c(f_{\alpha,q}^\#+|f|)$ holds, so estimation of the last term on the right of the preceding inequality gives

$$(F_j)_{\alpha+2,q}^\# \leqslant c\,(f_{\alpha,q}^\#+|f|). \tag{30}$$

This shows that F_j has Peano derivatives of at least second order. Suppose now that x_0 is selected so that $f_{\alpha,q}^\#(x_0) < \infty$. If $P_j^{(i)}$ denotes the Taylor polynomial of F_j of order i about x_0 (with the appropriate Peano derivatives, guaranteed above, as coefficients), then we define

$$\tilde{F}_j(x) := F_j(x) - P_j^{(2)}(x)$$

and note that

$$\Delta\tilde{F}_j(x) = f_j(x) - f_j(x_0). \tag{31}$$

Let $B(r)$ denote the ball of radius r about x_0. Then by the definition of $\tilde{F}_j$ and the maximal operator, we have for each $r \geqslant 1$,

$$\|\tilde{F}_j\|_{L^q(B(r))} \leqslant c r^{\alpha+2+n/q}(F_j)_{\alpha+2,q}^\#(x_0). \tag{32}$$

So by inequality (30), $\tilde{F}_j$ is bounded in L^q on bounded sets. This collection is weakly sequentially precompact, so a Cantor diagonalization argument may be used to produce a subsequence $\tilde{F}_{j_m}$ which converges weakly in L^q on all balls $B(r)$ to some function G. Put

$$F(x) := G(x) + \frac{f(x_0)}{2n}|x|^2. \tag{33}$$

Then F is locally in L^q and by taking weak limits in L^q, we obtain from (31)

$$\Delta F = \Delta G + f(x_0) = \lim_{m\to\infty} \Delta \tilde{F}_{j_m} + f(x_0) = f.$$

Next, we show that F satisfies the estimate (25). Since $G_m(x) := \tilde{F}_{j_m}(x) + (f(x_0)/(2n))|x|^2$ converges to F weakly in L^q on bounded sets, the selfadjoint projections (see p. 8 of [5]) $P_{B(r)}(G_m)$ converge to $P_{B(r)}(F)$. Hence

$$(F)^{\#}_{\alpha+2,q} \leqslant \liminf_{m\to\infty} (G_m)^{\#}_{\alpha+2,q} = \liminf_{m\to\infty} (\tilde{F}_{j_m})^{\#}_{\alpha+2,q} \leqslant c\,(f^{\#}_{\alpha,q} + |f|), \tag{34}$$

where the last inequality uses (30) and the fact that $(\tilde{F}_j)^{\#}_{\alpha+2,q} = (F_j)^{\#}_{\alpha+2,q}$. The desired result for $0<\alpha<1$ follows by taking L^p norms.

Finally, we consider the case of general α. In this case f belongs to $C_p^{\alpha-[\alpha]}$ and the first part of the proof may be applied to provide F such that (24) holds. Moreover, as in the case $0<\alpha<1$, we have

$$|F|_{C_p^{\alpha+2}} \leqslant \liminf_{m\to\infty} |\tilde{F}_{j_m}|_{C_p^{\alpha+2}} \leqslant c \liminf_{m\to\infty} \sum_{|\mu|=2} |R^\mu f_{j_m}|_{C_p^\alpha}$$
$$\leqslant c \liminf_{m\to\infty} |f_{j_m}|_{C_p^\alpha} \leqslant c\,\|f\|_{C_p^\alpha},$$

where two applications of Corollary 1 have been made. ∎

Remarks. (a) In Corollaries 1 and 2, the proofs can be considerably simplified in each of the cases $p=1$ and $1<p<\infty$.

(b) By using the extension theorem (Theorem 11.4 of [5]) for C_p^α spaces, the results of Corollary 2 can be established with $\boldsymbol{R}^n$ replaced by any domain Ω (bounded or unbounded) which has a minimally smooth boundary.

(c) As in the extensive work developed in [4] it is also possible, using Riesz transforms, to provide similar estimates for the operator $\Lambda := i\sum_{j=1}^n R_j(\partial f/\partial x_j)$ which plays a central role in the analysis of elliptic partial differential operators. This will be the subject of a future paper.

(d) In many situations the boundedness of operators can be established by verifying boundedness on certain endpoint spaces and using interpolation. Although the C_p^α spaces do have some interpolation properties, they are not stable under interpolation if p is fixed and α is varied. In this case Theorem 8.6 of [5] gives that the resulting interpolation spaces are instead Besov spaces. Hence a corollary of our results is Taibleson's theorem [7] concerning the boundedness of the singular integral K on the Besov spaces $B_p^{\alpha,q}$, for $1\leqslant p, q\leqslant\infty$, $\alpha>0$.

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