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Spectrum of generators of a noncommutative Banach algebra

by

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Abstract. The joint spectrum of generators of a noncommutative Banach algebra with unit has many properties similar to the commutative case. The left and the right joint spectra of generators coincide and the joint spectrum is determined by the values of multiplicative linear functionals (see also [1]). The joint spectrum of generators is always polynomially convex (but possibly empty). It is also shown that a unital Banach algebra A has a nonzero multiplicative linear functional if and only if such a functional exists on every finitely generated subalgebra of A. Finally, we investigate the properties of the function which assigns to each finite collection of elements in a Banach algebra the joint spectrum of these elements in the Banach algebra generated by them.

Throughout this paper the term "Banach algebra" will stand for a unital Banach algebra over the complex field $\boldsymbol{C}$. Let A be a Banach algebra with the unit 1 and let $a_1, \ldots, a_n \in A$. The *left joint spectrum* $\sigma_l^A(a_1, \ldots, a_n)$, simply written as $\sigma_l(a_1, \ldots, a_n)$ if there is no confusion, is defined as the set of those n-tuples $(\lambda_1, \ldots, \lambda_n) \in \boldsymbol{C}^n$ for which the left ideal generated by $a_1-\lambda_1, \ldots, a_n-\lambda_n$ is proper. (We shortly write $a_j-\lambda_j$ instead of $a_j-\lambda_j 1$.) The right joint spectrum $\sigma_r(a_1, \ldots, a_n)$ can be defined analogously. The *joint spectrum* is the union of the left and the right joint spectra,

$$\sigma(a_1, \ldots, a_n) = \sigma_l(a_1, \ldots, a_n) \cup \sigma_r(a_1, \ldots, a_n).$$

For basic properties of σ_l, σ_r, and σ see [2].

It is well known that the joint spectrum of generators of a commutative Banach algebra has an important additional property, namely it is polynomially convex. The aim of this paper is to study the joint spectrum of generators of a noncommutative Banach algebra. We show that it exhibits a lot of properties of the joint spectrum in a commutative Banach algebra.

Denote by $\mathfrak{M}(A)$ the set of all nonzero multiplicative (linear) functionals on a Banach algebra A.

PROPOSITION. *Let $x_1, \ldots, x_n$ be generators of a Banach algebra A. Denote by I_l (respectively I_r) the closed left (right) ideal generated by $x_1, \ldots, x_n$. Let C*

be the closed two-sided ideal generated by the commutators $x_j x_k - x_k x_j$ $(j, k = 1, \ldots, n)$. *Then:*

(1) $I_l = I_r$. *In particular,* $(0, \ldots, 0) \in \sigma_l(x_1, \ldots, x_n)$ *if and only if* $(0, \ldots, 0) \in \sigma_r(x_1, \ldots, x_n)$.

(2) $C \subset I_l$.

(3) *If* $I_l \neq A$, *then there exists* $f \in \mathfrak{M}(A)$ *such that*

$$f(x_1) = \ldots = f(x_n) = 0.$$

Proof. (1) Let $\mathscr{P}$ be the set of all complex polynomials in n noncommutative indeterminates, i.e. the free associative noncommutative algebra with n generators and with unit. Denote by $\mathscr{P}_0$ the subset of $\mathscr{P}$ consisting of polynomials without the constant term. As $x_1, \ldots, x_n$ are generators of A we have

$$A = \{p(x_1, \ldots, x_n)\colon p \in \mathscr{P}\}^-.$$

Further,

$$I_l = \{\sum_{j=1}^{n} p_j(x_1, \ldots, x_n) x_j\colon p_1, \ldots, p_n \in \mathscr{P}\}^- = \{p(x_1, \ldots, x_n)\colon p \in \mathscr{P}_0\}^- = I_r.$$

Since the closure of a proper ideal is proper, we conclude that $(0, \ldots, 0) \in \sigma_l(x_1, \ldots, x_n)$ if and only if $I_l \neq A$ if and only if $I_r \neq A$ if and only if $(0, \ldots, 0) \in \sigma_r(x_1, \ldots, x_n)$.

(2) Clearly $x_j x_k - x_k x_j \in I_l$ $(j, k = 1, \ldots, n)$. Hence $C \subset I_l$.

(3) Suppose $I_l \neq A$. Then I_l is a closed two-sided ideal of codimension 1. The factor algebra A/I_l is isomorphic to the algebra of complex numbers and the canonical homomorphism $f\colon A \to A/I_l$ is a multiplicative functional satisfying $f(x_1) = \ldots = f(x_n) = 0$. ∎

THEOREM 1. *Let A be a Banach algebra with generators $x_1, \ldots, x_n$. Then:*

(1) $$\sigma_l(x_1, \ldots, x_n) = \sigma_r(x_1, \ldots, x_n) = \sigma(x_1, \ldots, x_n) = \{(f(x_1), \ldots, f(x_n))\colon f \in \mathfrak{M}(A)\}.$$

(2) *Let C be the closed two-sided ideal generated by the commutators $x_j x_k - x_k x_j$ $(j, k = 1, \ldots, n)$. Then $\sigma(x_1, \ldots, x_n) = \emptyset$ if and only if $C = A$. If $C \neq A$, then $\sigma(x_1, \ldots, x_n) = \sigma^{A/C}(\nu x_1, \ldots, \nu x_n)$ where $\nu\colon A \to A/C$ is the canonical homomorphism from A onto the commutative Banach algebra A/C.*

(3) *$\sigma(x_1, \ldots, x_n)$ is polynomially convex.*

Proof. (1) Let $(\lambda_1, \ldots, \lambda_n) \in \mathbf{C}^n$. As $x_1 - \lambda_1, \ldots, x_n - \lambda_n$ are also generators of A, part (1) of the Proposition gives $(\lambda_1, \ldots, \lambda_n) \in \sigma_l(x_1, \ldots, x_n)$ if and only if $(\lambda_1, \ldots, \lambda_n) \in \sigma_r(x_1, \ldots, x_n)$, and so $\sigma_l(x_1, \ldots, x_n) = \sigma_r(x_1, \ldots, x_n) = \sigma(x_1, \ldots, x_n)$.

Let $(\lambda_1, \ldots, \lambda_n) \in \sigma(x_1, \ldots, x_n)$. By part (3) of the Proposition there exists $f \in \mathfrak{M}(A)$ such that $f(x_1) = \lambda_1, \ldots, f(x_n) = \lambda_n$. So

$$\sigma(x_1, \ldots, x_n) \subset \{(f(x_1), \ldots, f(x_n))\colon f \in \mathfrak{M}(A)\}.$$

Conversely, let $f \in \mathfrak{M}(A)$. Then $\ker f$ is a proper two-sided ideal and $x_j - f(x_j) \in \ker f$ $(j = 1, \ldots, n)$. Therefore $(f(x_1), \ldots, f(x_n)) \in \sigma(x_1, \ldots, x_n)$.

(2) Let $f \in \mathfrak{M}(A)$. Then $f(x_j x_k - x_k x_j) = 0$ $(j, k = 1, \ldots, n)$ and so $f(C) = 0$.

If $C = A$, then $\mathfrak{M}(A) = \emptyset$ and $\sigma(x_1, \ldots, x_n) = \emptyset$ by part (1).

Suppose $C \neq A$. Then $\sigma^A(x_1, \ldots, x_n) \supset \sigma^{A/C}(\nu x_1, \ldots, \nu x_n)$. Let $(\lambda_1, \ldots, \lambda_n) \in \sigma^A(x_1, \ldots, x_n)$. Then there exists $f \in \mathfrak{M}(A)$ such that $f(x_j) = \lambda_j$ $(j = 1, \ldots, n)$. Since $f(C) = 0$ there exists $\tilde{f} \in \mathfrak{M}(A/C)$ such that $f = \tilde{f} \circ \nu$, i.e. $\tilde{f}(\nu x_j) = f(x_j) = \lambda_j$ $(j = 1, \ldots, n)$. Therefore $(\lambda_1, \ldots, \lambda_n) \in \sigma^{A/C}(\nu x_1, \ldots, \nu x_n)$.

(3) The joint spectrum $\sigma^A(x_1, \ldots, x_n)$ is either empty or $\sigma^A(x_1, \ldots, x_n) = \sigma^{A/C}(\nu x_1, \ldots, \nu x_n)$. As A/C is a commutative Banach algebra with generators $\nu x_1, \ldots, \nu x_n$ we conclude that $\sigma^A(x_1, \ldots, x_n)$ is polynomially convex (see [6], 17.10). ∎

Remarks. 1. The equality $\sigma(x_1, \ldots, x_n) = \{(f(x_1), \ldots, f(x_n))\colon f \in \mathfrak{M}(A)\}$ was proved in [1], Proposition 2. We have proved it in a different way.

2. The formula $\sigma^A(x_1, \ldots, x_n) = \sigma^{A/C}(\nu x_1, \ldots, \nu x_n)$ can also be deduced from [1], Remark 3, p. 218 and Proposition 2, but our proof is simpler.

3. If we introduce the joint spectrum of infinite families of elements of a Banach algebra A, then the results of Theorem 1 remain true.

Let $a_1, \ldots, a_n$ be elements of a Banach algebra A. The *left approximate point spectrum* $\tau_l(a_1, \ldots, a_n)$ is defined as the set of those n-tuples $(\lambda_1, \ldots, \lambda_n) \in \mathbf{C}^n$ for which

$$\inf\{\sum_{j=1}^{n} \|(a_j - \lambda_j) z\|\colon z \in A,\ \|z\| = 1\} = 0.$$

The right approximate point spectrum $\tau_r(a_1, \ldots, a_n)$ can be defined in a similar manner (see [2] or [3]).

Let A be a Banach algebra with generators $x_1, \ldots, x_n$ and let $(\lambda_1, \ldots, \lambda_n) \in \tau_l(x_1, \ldots, x_n)$. Denote by I the closed two-sided ideal generated by $x_1 - \lambda_1, \ldots, x_n - \lambda_n$. Clearly I consists of joint left topological divisors of zero, i.e. there is a net (z_α) of elements in A such that $\|z_\alpha\| = 1$ for all α and $y z_\alpha \to 0$ for every $y \in I$. Thus

$$\tau_l(x_1, \ldots, x_n) = \{(f(x_1), \ldots, f(x_n)) \colon f \in \mathfrak{M}(A),\ \ker f \text{ consists of joint left topological divisors of zero}\}.$$

On the other hand, the following two examples show that the left and the right approximate point spectra of $x_1, \ldots, x_n$ may differ and they may be empty even if $\sigma(x_1, \ldots, x_n) \neq \emptyset$ (cf. also [3], Example 1).

EXAMPLE 1. Let V be the free (noncommutative) semigroup with generators x, y and with the unit 1. Define a norm on V by

$$\|x^{n_1} y^{m_1} \ldots x^{n_k} y^{m_k}\|_V = \frac{1}{n_1! \, m_1! \ldots n_k! \, m_k!}$$

$(k \geq 1,\ m_1, n_2, m_2, \ldots, m_{k-1}, n_k \geq 1,\ n_1, m_k \geq 0)$. Clearly,

$$\|v_1 v_2\|_V \leq \|v_1\|_V \|v_2\|_V \quad (v_1, v_2 \in V).$$

Let A be the l^1 algebra over V, i.e.

$$A = \{a = \sum_{v \in V} \alpha_v v \colon \alpha_v \in \boldsymbol{C}\ (v \in V),\ \|a\| = \sum_{v \in V} |\alpha_v| \, \|v\|_V < \infty\}.$$

Then A is a Banach algebra with generators x, y. It is easy to check that $\sigma(x, y) = \{(0, 0)\} \neq \emptyset$ while $\tau_l(x, y) = \tau_r(x, y) = \emptyset$. To see this, it is sufficient to prove that $\|zx\| + \|zy\| \geq \|z\|$ $(z \in A)$ (and analogously $\|xz\| + \|yz\| \geq \|z\|$).

Let $z \in A$. We can write $z = \alpha + z_1 x + z_2 y$ for some z_1, z_2 in A and $\alpha \in \boldsymbol{C}$, $\|z\| = |\alpha| + \|z_1 x\| + \|z_2 y\|$. Then

$$\|zx\| = \|\alpha x + z_1 x^2 + z_2 yx\| = \|\alpha x\| + \|z_1 x^2\| + \|z_2 yx\| \geq |\alpha| + \|z_2 y\|.$$

Similarly $\|zy\| \geq |\alpha| + \|z_1 x\|$. This gives together

$$\|zx\| + \|zy\| \geq 2|\alpha| + \|z_1 x\| + \|z_2 y\| \geq \|z\|.$$

EXAMPLE 2. Let V be as in Example 1. Now the norm on V will be given by

$$\|x^{n_1} y^{m_1} \ldots x^{n_k} y^{m_k}\|_V = \frac{1}{n_1! \, (m_1+1)! \ldots n_k! \, (m_k+1)!}$$

for $k \geq 1,\ n_1, m_1, \ldots, n_k \geq 1,\ m_k \geq 0$, and

$$\|y^{m_1} x^{n_2} y^{m_2} \ldots x^{n_k} y^{m_k}\|_V = \frac{1}{m_1! \, n_2! \, (m_2+1)! \ldots n_k! \, (m_k+1)!}$$

for $k \geq 1,\ m_1, n_2, \ldots, n_k \geq 1,\ m_k \geq 0$.

Let A be the l^1 algebra over V with this norm. Then A is again a Banach algebra with generators x, y, and $\sigma(x, y) = \{(0, 0)\} \neq \emptyset$. Further,

$$\|xy^n\| = \|yy^n\| = \frac{1}{(n+1)!} = \frac{1}{n+1} \|y^n\|,$$

so $(0, 0) \in \tau_l(x, y)$, and hence $\tau_l(x, y) = \sigma(x, y) \neq \emptyset$. On the other hand, $\tau_r(x, y) = \emptyset$ since $\|zx\| + \|zy\| \geq \|z\|$ $(z \in A)$ as in Example 1.

Let A be a Banach algebra and let $a_1, \ldots, a_n \in A$. Denote by $[a_1, \ldots, a_n]$ the Banach algebra generated by $a_1, \ldots, a_n$ and the unit. It is clear that if A has a multiplicative functional f, then so do all algebras $[a_1, \ldots, a_n]$, where $a_1, \ldots, a_n$ are arbitrary elements of A and $n = 1, 2, \ldots$ (e.g. the restriction of f to $[a_1, \ldots, a_n]$). Now we shall show the converse of this fact.

THEOREM 2. *If every finitely generated subalgebra of a Banach algebra A has a multiplicative functional, then A itself has such a functional.*

First we shall prove the following:

LEMMA. *If a function $f\colon A \to \boldsymbol{C}$ is such that*

$$(f(a), f(b), f(c)) \in \sigma^{[a,b,c]}(a, b, c)$$

for arbitrary elements a, b, c in a Banach algebra A, then it is linear and multiplicative.

Proof. Take arbitrary a, b in A. By assumption we get

$$(f(a), f(b), f(ab)) \in \sigma^{[a,b,ab]}(a, b, ab),$$

which, in view of Theorem 1 (cf. also [1], Proposition 2), implies that there exists a multiplicative functional f_0 on the Banach algebra $[a, b]$ such that

$$f_0(a) = f(a), \quad f_0(b) = f(b), \quad f_0(ab) = f(ab).$$

Thus we have

$$f(ab) = f_0(ab) = f_0(a) f_0(b) = f(a) f(b),$$

which means that f is multiplicative. Similarly we can prove that it is linear. ∎

Proof of Theorem 2. Denote by $B(0, \|a\|_s)$ the closed ball in $\boldsymbol{C}$ centered at zero with radius $\|a\|_s$ (= the spectral radius of a) and let $K = \prod_{a \in A} B(0, \|a\|_s)$. Then K is a compact set with respect to the product topology. Further, for each n-tuple $(a_1, \ldots, a_n)$ of elements in A, we write

$$K_0(a_1, \ldots, a_n) = \{(\lambda_a)_{a \in A} \in K \colon \text{there exists } f \in \mathfrak{M}[a_1, \ldots, a_n] \text{ such that } f(a_j) = \lambda_{a_j} \text{ for } j = 1, \ldots, n\}.$$

By our assumption $K_0(a_1, \ldots, a_n)$ is nonempty. Moreover, it is obvious that $K_0(a_1, \ldots, a_n)$ is compact and

$$K_0(a_1, \ldots, a_n, b_1, \ldots, b_m) \subset K_0(a_1, \ldots, a_n) \cap K_0(b_1, \ldots, b_m)$$

for arbitrary elements $a_1, \ldots, a_n, b_1, \ldots, b_m$ in A and any positive integers n, m. Hence the family $\{K_0(a_1, \ldots, a_n)\}$, where $\{a_1, \ldots, a_n\}$ runs through all finite subsets of A, has the finite intersection property. Therefore its intersection is nonempty. If $(\lambda_a)_{a\in A}$ belongs to this intersection, then the function $f: A \to C$ defined by the formula $f(a) = \lambda_a$ is, in view of the Lemma, a multiplicative functional. ∎

Remark. It would be interesting to know an answer to the following:

QUESTION. Does the assertion of Theorem 2 hold true if we only assume that every finitely generated subalgebra $[a_1, \ldots, a_n]$ of A has a multiplicative functional, where $1 \leqslant n \leqslant N$, and $N \geqslant 2$ is fixed?

In the final part of the paper we shall examine the properties of the function

$$(a_1, \ldots, a_n) \mapsto \sigma^{[a_1,\ldots,a_n]}(a_1, \ldots, a_n)$$

which assigns to each n-tuple of elements in a Banach algebra A the joint spectrum of these elements in the Banach algebra generated by them. To simplify notation we shall write in the sequel $\hat{\sigma}(a_1, \ldots, a_n)$ instead of $\sigma^{[a_1,\ldots,a_n]}(a_1, \ldots, a_n)$.

The set $\hat{\sigma}(a_1, \ldots, a_n)$ is compact but possibly empty. In view of Theorem 1 it is always a polynomially convex subset of C^n. But it need not be equal to the polynomially convex hull of the joint spectrum $\sigma^A(a_1, \ldots, a_n)$ (see [3], Example 1).

Theorems 1 and 2 have the following noteworthy consequences:

COROLLARY 1. *If $\hat{\sigma}(a_1, \ldots, a_n)$ is nonempty for an arbitrary n-tuple $(a_1, \ldots, a_n)$ of elements in a Banach algebra A with $n = 1, 2, \ldots$, then A has a multiplicative functional.*

COROLLARY 2. *If all $\hat{\sigma}(a_1, \ldots, a_n)$ $(a_1, \ldots, a_n \in A;\ n = 1, 2, \ldots)$ are nonempty, then so are all $\sigma(a_1, \ldots, a_n)$ (also $\sigma_l(a_1, \ldots, a_n)$ and $\sigma_r(a_1, \ldots, a_n)$).*

Remark. Examples 1 and 2 show that we cannot replace σ_l (respectively σ_r) by τ_l (τ_r) in Corollary 2.

Now observe that the joint spectrum of generators $\hat{\sigma}$ has the so-called "one-way spectral mapping property", i.e.

$$(*) \qquad p\hat{\sigma}(a_1, \ldots, a_n) \subset \hat{\sigma}(p(a_1, \ldots, a_n)),$$

where $p = (p_1, \ldots, p_m)$ is an arbitrary m-tuple of polynomials in n (noncommuting) indeterminates and $a_1, \ldots, a_n$ are arbitrary elements of A.

Indeed, we have

$$(**) \qquad p\hat{\sigma}(a_1, \ldots, a_n) \subset \sigma^{[a_1,\ldots,a_n]}(p(a_1, \ldots, a_n)).$$

Moreover, since $p(a_1, \ldots, a_n) \in [a_1, \ldots, a_n]^m$, we get $[p(a_1, \ldots, a_n)] \subset [a_1, \ldots, a_n]$, which in turn implies

$$(***) \qquad \sigma^{[a_1,\ldots,a_n]}(p(a_1, \ldots, a_n)) \subset \hat{\sigma}(p(a_1, \ldots, a_n)).$$

(**) and (***) together give (*).

If we replace the inclusion by the equality in (*), then we get the spectral mapping property of the joint spectrum of generators $\hat{\sigma}$. But easy examples show (for instance, Harte's example in [2], p. 93) that $\hat{\sigma}$ need not have this property.

In investigations of joint spectra an important role is played by the projection property which is the particular case of the spectral mapping property. Namely, we say that $\hat{\sigma}$ satisfies the *projection property* on a Banach algebra A if for arbitrary elements $a_1, \ldots, a_{n+m}$ in A and arbitrary positive integers n, m

$$P_n^{n+m}\hat{\sigma}(a_1, \ldots, a_{n+m}) = \hat{\sigma}(a_1, \ldots, a_n),$$

where P_n^{n+m} is the canonical projection from C^{n+m} onto C^n which sends $(\lambda_1, \ldots, \lambda_{n+m})$ to $(\lambda_1, \ldots, \lambda_n)$.

We shall now show that if $\hat{\sigma}$ satisfies the projection property, then it must also have the spectral mapping property. Moreover, we shall give some other conditions equivalent to the projection property of $\hat{\sigma}$.

The symbol rad A denotes the (Jacobson) radical of a Banach algebra A.

THEOREM 3. *Let A be a Banach algebra. The following conditions are equivalent:*

(1) *$\hat{\sigma}$ has the projection property on A.*

(2) *$\hat{\sigma}(a_1, \ldots, a_n) = \{(f(a_1), \ldots, f(a_n)): f \in \mathfrak{M}(A)\}$ for all $a_1, \ldots, a_n \in A$, $n = 1, 2, \ldots$*

(3) *$\hat{\sigma}$ has the spectral mapping property on A.*

(4) *$\hat{\sigma}(p(a_1, \ldots, a_n)) \subset p\hat{\sigma}(a_1, \ldots, a_n)$ for every m-tuple $p = (p_1, \ldots, p_m)$ of polynomials in n variables and each n-tuple $(a_1, \ldots, a_n) \in A^n$, $n, m = 1, 2, \ldots$*

(5) *The algebra $A/\mathrm{rad}\,A$ is commutative and every element of A has a totally disconnected spectrum (i.e. A is of type ES, see [4] and [5]).*

Proof. (1) ⇒ (2). Take an arbitrary n-tuple $(a_1, \ldots, a_n)$ of elements in A. Then it is evident that

$$\{(f(a_1), \ldots, f(a_n)): f \in \mathfrak{M}(A)\} \subset \hat{\sigma}(a_1, \ldots, a_n).$$

To establish the converse inclusion fix an arbitrary n-tuple $(\lambda_1, \ldots, \lambda_n) \in \hat{\sigma}(a_1, \ldots, a_n)$. We have to show that there exists a multiplicative functional f on A such that $f(a_j) = \lambda_j$ for $j = 1, \ldots, n$.

Let $K = \prod_{a\in A} B(0, \|a\|_s)$ be as before and let

$$K_1(b_1, \ldots, b_m) = \{(\mu_a)_{a\in A} \in K\colon \mu_{a_j} = \lambda_j \text{ for } j = 1, \ldots, n \text{ and } (\mu_{b_1}, \ldots, \mu_{b_m}) \in \hat{\sigma}(b_1, \ldots, b_m)\}$$

for arbitrary $b_1, \ldots, b_m \in A$ and every positive integer m. The projection property of $\hat{\sigma}$ implies that $K_1(b_1, \ldots, b_m)$ is always nonempty. Further, it is a compact subset of K and

$$K_1(b_1, \ldots, b_m, c_1, \ldots, c_p) \subset K_1(b_1, \ldots, b_m) \cap K_1(c_1, \ldots, c_p)$$

for all $b_1, \ldots, b_m, c_1, \ldots, c_p \in A$ and m, p positive integers. Therefore the family $\{K_1(b_1, \ldots, b_m)\}$ has the finite intersection property. If an element $(\mu_a)_{a\in A}$ belongs to the intersection of this family, then the function $f\colon A \to \boldsymbol{C}$ defined by $f(a) = \mu_a$ is, in view of the Lemma, a multiplicative functional. It is clear that $f(a_j) = \lambda_j$ $(j = 1, \ldots, n)$.

(2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4) are obvious.

(4) $\Rightarrow$ (1) follows by the one-way spectral mapping property of $\hat{\sigma}$.

(2) $\Rightarrow$ (5). Take arbitrary a, b, c in A. Then

$$\sigma((ab-ba)c) \subset \hat{\sigma}((ab-bc)c) = \{f((ab-ba)c)\colon f \in \mathfrak{M}(A)\} = \{0\}.$$

Therefore $ab-ba \in \operatorname{rad} A$, which means that the algebra $A/\operatorname{rad} A$ is commutative. We claim that the algebra A is of type ES. Suppose on the contrary that there exists an element $a_0 \in A$ such that $\sigma(a_0)$ contains a continuum (consisting of more than one point). Then reasoning as in the proof of Lemma 1 in [4] we get an element $b \in A$ with $0 \notin \sigma(b)$ and $0 \in \hat{\sigma}(b)$. But this is impossible since by our assumption $\hat{\sigma}(b) = \{f(b)\colon f \in \mathfrak{M}(A)\} = \sigma(b)$.

(5) $\Rightarrow$ (2). As $A/\operatorname{rad} A$ is commutative and

$$\hat{\sigma}(a_1 + \operatorname{rad} A, \ldots, a_n + \operatorname{rad} A) = \hat{\sigma}(a_1, \ldots, a_n),$$

and to every $f \in \mathfrak{M}(A)$ there corresponds $\tilde{f} \in \mathfrak{M}(A/\operatorname{rad} A)$ such that $\tilde{f}(a + \operatorname{rad} A) = f(a)$, we can assume, without loss of generality, that A itself is commutative. Then

$$\hat{\sigma}(a_1, \ldots, a_n) = \{(f(a_1), \ldots, f(a_n))\colon f \in \mathfrak{M}([a_1, \ldots, a_n])\}.$$

But the algebra A is of type ES which means (see [4]) that every multiplicative functional on $[a_1, \ldots, a_n]$ has an extension to a multiplicative functional on the whole algebra A. Thus

$$\hat{\sigma}(a_1, \ldots, a_n) = \{(f(a_1), \ldots, f(a_n))\colon f \in \mathfrak{M}(A)\}$$

and we are done. ■

Remark. Each of conditions (1)–(5) of Theorem 3 implies

(6) $\sigma(a_1, \ldots, a_n) = \hat{\sigma}(a_1, \ldots, a_n)$ for an arbitrary finite subset $\{a_1, \ldots, a_n\}$ of A.

The converse is not true. It can be proved that the algebra M_2 of all 2×2 complex matrices has property (6) while obviously it satisfies none of (1)–(5).

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