COROLLARY 2. *For any* $p \in [1, 2]$ *there exists a basis* $B \subset \boldsymbol{B}_{\Theta_0}$ *such that* $B \in D\big(L(\log^+ L)^p\big)(\boldsymbol{R}^2)$.

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L^p-Multiplier transference induced by representations in Hilbert space

by

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Abstract. Let $(\mathcal{M}, \mu)$ be a measure space, and S a representation of a locally compact abelian group G by measure-preserving transformations of the points of $\mathcal{M}$. Under suitable further hypotheses on μ, G, and S, the Coifman–Weiss Transference Theorem for Multipliers provides a machinery whereby S can be made to transfer "normalized" $L^p(G)$-multiplier transforms, along with their bounds, to $L^p(\mu)$, for all finite p. We show below that multiplier transference can be freed of technical restrictions so that its broader, structurally simpler nature emerges in the following form: whenever an arbitrary locally compact abelian group G has a uniformly bounded strongly continuous representation R in $L^2(\mu)$ (μ an arbitrary measure) such that R has a uniformly bounded $L^p(\mu)$-version for some $p \in (1, \infty)$, then R will transfer continuous $L^p(G)$-multipliers to $L^p(\mu)$. The added generality is illustrated in some elementary examples and in a short proof of the Homomorphism Theorem for Multipliers. An application to generalized analyticity is presented in the last section.

1. Introduction. The Coifman–Weiss theory of transference methods unifies and expands diverse streams of thought in general analysis by transferring operators affiliated with groups, along with their bounds, to spaces in which the groups act (see [6] for an expository account of the theory's nature and lineage). In this article we shall be concerned with a generalization, to a wider context, of the Coifman–Weiss Transference Theorem for Multipliers (see Theorem (2.1) below for this generalization).

Suppose that $(\mathcal{M}, \mu)$ is a measure space, and S is a representation of a locally compact abelian group G by measure-preserving transformations of the points of $\mathcal{M}$. Suppose also that the unitary representation R of G in $L^2(\mathcal{M}, \mu)$ implemented by S is strongly continuous. The Coifman–Weiss Transference Theorem for Multipliers ([5, Theorem 3.7]) provides a method for using R to transfer "normalized" multiplier transforms from $L^p(G)$ to $L^p(\mathcal{M}, \mu)$ without increasing their operator norms. In order to support this multiplier transference method, various technical hypotheses are imposed in [5]: the group G is assumed to be σ-compact, $(\mathcal{M}, \mu)$ is taken to be σ-finite, and joint measurability in $(u, \omega) \in G \times \mathcal{M}$ is implicitly assumed for functions of the form $f(S_u \omega)$ $(f \in L^p(\mathcal{M}, \mu))$. Recently, it was shown in [4, Proposition

2.6] that the σ-finiteness of $(\mathcal{M}, \mu)$ and the joint measurability in (u, ω) can be dropped from the hypotheses. In the Coifman–Weiss context, the σ-compactness of G is used to produce a countable approximate identity for L^1 of the dual group $\hat{G}$ so as to imitate the properties of the classical Fejér kernel. The ensuing analogue on $\hat{G}$ of Cesàro summability is at the core of their definition and subsequent treatment of normalized multipliers. In barest outline, the key property enjoyed by a normalized L^p-multiplier is that it has a suitable perturbation which is the pointwise limit of a sequence of Fourier transforms corresponding to functions in $L^2(G)$ having compact support [5, Lemma 3.5 (2)]. The Transference Theorem for Multipliers is achieved in [5] by starting with this property of normalized multipliers, and then applying the General Transference Result for compactly supported $L^1(G)$-kernels ([5, § 2], [6, Theorem 2.4]) together with the Lebesgue Bounded Convergence Theorem. In Theorem (2.1) below, we show that the σ-compactness restriction on G can be dropped, and that the hypothesized strongly continuous representation R of G in $L^2(\mathcal{M}, \mu)$ need only be uniformly bounded, rather than implemented by measure-preserving transformations. By eliminating all technical restrictions, our theorem simplifies the Transference Theorem for Multipliers, and generalizes it to the more universal context of arbitrary bounded Hilbert space representations for the general locally compact abelian group. Thus, the precise nature of multiplier transference emerges: whenever an arbitrary locally compact abelian group has a uniformly bounded, strongly continuous representation in the L^2-space of a measure μ, and this representation has a uniformly bounded $L^p(\mu)$-version (for some p in the range $1 < p < \infty$), then the representation in $L^2(\mu)$ will transfer continuous $L^p(G)$-multipliers to $L^p(\mu)$.

An application of Theorem (2.1) to generalized analyticity is given in § 3 (Theorem (3.1)). The full generality of Theorem (2.1), which no longer requires the representation R to be implemented by measure-preserving transformations, is utilized in proving Theorem (3.1). Let K be a compact abelian group with archimedean ordered dual. In [3], Helson's theory of generalized analyticity was extended from $L^2(K)$ to $L^p(K)$, $1 < p < \infty$. It was there shown that each normalized, simply invariant subspace of $L^p(K)$ is, in a canonical fashion, the range of an idempotent operator. In Theorem (3.1) we show that the norm of this idempotent does not exceed that of classical analytic projection in L^p of the unit circle $\mathbf{T}$.

2. Multiplier transference. Let G be a locally compact abelian group with dual group $\hat{G}$, let $(\mathcal{M}, \mu)$ be an arbitrary measure space, and suppose that $u \to R_u$ is a strongly continuous representation of G in $L^2(\mathcal{M}, \mu)$ such that $\sup\{\|R_u\|: u \in G\} < +\infty$. Thus the bounded representation R is automatically similar to a unitary representation of G in $L^2(\mathcal{M}, \mu)$, the similarity being implemented by a bounded selfadjoint operator with strictly positive spectrum [8, Theorem 8.1]. Applying Stone's Theorem for unitary representations, we obtain a unique strongly countably additive regular spectral measure $\mathcal{E}(\cdot)$, defined on the Borel sets of $\hat{G}$ and acting in $L^2(\mathcal{M}, \mu)$, such that

$$R_u = \int_{\hat{G}} \gamma(u)\, d\mathcal{E}(\gamma), \quad \text{for all } u \in G.$$

(2.1) Theorem. *Let G, $(\mathcal{M}, \mu)$, R, and $\mathcal{E}(\cdot)$ be as just described. Suppose that for some p satisfying $1 < p < \infty$ the following conditions hold:*

(i) *For each $u \in G$, R_u can be extended from $L^2(\mathcal{M}, \mu) \cap L^p(\mathcal{M}, \mu)$ to a continuous linear mapping $R_u^{(p)}$ of $L^p(\mathcal{M}, \mu)$ into $L^p(\mathcal{M}, \mu)$.*

(ii) $s_p \equiv \sup\{\|R_u^{(p)}\|: u \in G\} < \infty$.

Then if φ is an $L^p(G)$-multiplier which is continuous on $\hat{G}$, the operator $\int_{\hat{G}} \varphi\, d\mathcal{E}$ extends from $L^2(\mathcal{M}, \mu) \cap L^p(\mathcal{M}, \mu)$ to a bounded linear mapping of $L^p(\mathcal{M}, \mu)$ into $L^p(\mathcal{M}, \mu)$ with norm not exceeding $s_p^2 \|\varphi\|_{M_p(\hat{G})}$, where $\|\varphi\|_{M_p(\hat{G})}$ denotes the p-multiplier norm of φ.

Proof. Since $L^2(\mathcal{M}, \mu) \cap L^p(\mathcal{M}, \mu)$ is dense in $L^p(\mathcal{M}, \mu)$, it is obvious that $u \to R_u^{(p)}$ is a representation of G in $L^p(\mathcal{M}, \mu)$. We next show that $R^{(p)}$ is a weakly continuous representation. Let $v \in G$, and suppose that $\{u_\gamma\}$ is a net in G convergent to v. Fix a function $f \in L^2(\mathcal{M}, \mu) \cap L^p(\mathcal{M}, \mu)$. Since $L^p(\mathcal{M}, \mu)$ is reflexive, we see from (ii) above that $\{R_{u_\gamma}^{(p)} f\}$ has a subnet $\{R_{u_{\gamma_\alpha}}^{(p)} f\}$ weakly convergent in $L^p(\mathcal{M}, \mu)$ to a function F. It follows from the strong continuity of R that $F = R_v f = R_v^{(p)} f$. We infer that $\{R_{u_\gamma}^{(p)} f\}$ converges weakly in $L^p(\mathcal{M}, \mu)$ to $R_v^{(p)} f$. It now follows with the aid of (ii) that the representation $R^{(p)}$ is weakly continuous.

Hence [11_{I}, Theorem (22.8)], $R^{(p)}$ is a strongly continuous representation in $L^p(\mathcal{M}, \mu)$. The Coifman–Weiss General Transference Result for convolution kernels ([5, p. 290] = [6, Theorem 2.4]) now provides that whenever $k \in L^1(G)$, the Bochner integral $\int_G k(u)(R_{-u}^{(p)} f)\, du$, taken for all $f \in L^p(\mathcal{M}, \mu)$, defines a bounded linear mapping $T_k^{(p)}$ of $L^p(\mathcal{M}, \mu)$ into itself such that $\|T_k^{(p)}\| \leqslant s_p^2 \|\hat{k}\|_{M_p(\hat{G})}$ (for the Coifman–Weiss General Transference Result in the generality needed here, see the discussion in [4, § 2]).

We next consider the special case which, from the standpoint of transference methods, is the crux of the matter. Assume that $\varphi \equiv \hat{k}$ for some $k \in L^1(G)$. Let $f \in L^p(\mathcal{M}, \mu) \cap L^2(\mathcal{M}, \mu)$, and $g \in L^q(\mathcal{M}, \mu) \cap L^2(\mathcal{M}, \mu)$, where $p^{-1} + q^{-1} = 1$. Then for the operator $T_k^{(p)}$ defined above we have with the aid of Fubini's Theorem

$$\langle T_k^{(p)} f, g \rangle = \int_{\hat{G}} \varphi(\gamma)\, d\langle \mathcal{E}(\gamma) f, g \rangle.$$

Hence

$$\left|\left\langle \int_{\hat{G}} \varphi(\gamma)\, d\mathscr{E}(\gamma) f, g \right\rangle\right| \leqslant s_p^2 \|\varphi\|_{M_p(\hat{G})} \|f\|_{L^p(\mathscr{M},\mu)} \|g\|_{L^q(\mathscr{M},\mu)}.$$

The desired conclusion for the special case when $\varphi \equiv \hat{k}$ is now apparent. In order to place the proof of Theorem (2.1) under the control of this special case, we shall require the following scholium ([11_{II}, Theorems (28.52) and (33.12)]).

(2.2) Scholium. *Let G be a locally compact abelian group. The algebra $L^1(G)$ has an approximate identity $\{h_j\}_{j \in J}$ such that:*

(i) *Each $h_j \geqslant 0$.*

(ii) *$\int_G h_j(u)\, du = 1$ for all $j \in J$.*

(iii) *For each $j \in J$, $\hat{h}_j$, the Fourier transform of h_j, is compactly supported, and $\hat{h}_j \geqslant 0$.*

(iv) *$\{\hat{h}_j\}$ converges uniformly to the constant function 1 on each compact subset of $\hat{G}$.*

(v) *For each open neighborhood W of 0, the net $\{\int_{G \setminus W} h_j(u)\, du\}$ converges to 0.*

Suppose next that the desired conclusion is known to hold when $\varphi \in M_p(\hat{G})$ is continuous with compact support. In the general case, pick $\{h_j\}$ as in Scholium (2.2). For $j \in J$, let $\varphi_j \equiv \varphi \hat{h}_j$. Then an obvious application of Plancherel's Theorem shows that $\|\varphi_j\|_{M_p(\hat{G})} \leqslant \|\varphi\|_{M_p(\hat{G})}$ for all j. For $j \in J$, let $\Phi_j^{(2)}$ be the bounded operator on $L^2(\mathscr{M}, \mu)$ given by $\Phi_j^{(2)} = \int_{\hat{G}} \varphi_j\, d\mathscr{E}$. By our supposition, $\Phi_j^{(2)}$ can be extended from $L^2(\mathscr{M}, \mu) \cap L^p(\mathscr{M}, \mu)$ to an operator $\Phi_j^{(p)}$ on $L^p(\mathscr{M}, \mu)$ such that

$$\|\Phi_j^{(p)}\| \leqslant s_p^2 \|\varphi\|_{M_p(\hat{G})}. \tag{2.3}$$

The regularity of the spectral measure $\mathscr{E}(\cdot)$ ensures that $L^2(\mathscr{M}, \mu)$ is the closure of $\bigcup \{\mathscr{E}(C) L^2(\mathscr{M}, \mu)$: C is a compact subset of $\hat{G}\}$. With the aid of this fact and (2.2) (iv), it is easily seen that $\{\Phi_j^{(2)}\}$ converges to $\int_{\hat{G}} \varphi\, d\mathscr{E}$ in the strong operator topology of $\mathscr{B}(L^2(\mathscr{M}, \mu))$. Since $L^p(\mathscr{M}, \mu)$ is reflexive, it follows from (2.3) that $\{\Phi_j^{(p)}\}$ has a subnet $\{\Phi_{j_\alpha}^{(p)}\}$ convergent in the weak operator topology of $\mathscr{B}(L^p(\mathscr{M}, \mu))$ to an operator $Q^{(p)}$ such that

$$\|Q^{(p)}\| \leqslant s_p^2 \|\varphi\|_{M_p(\hat{G})}. \tag{2.4}$$

For $f \in L^2(\mathscr{M}, \mu) \cap L^p(\mathscr{M}, \mu)$ and $g \in L^2(\mathscr{M}, \mu) \cap L^q(\mathscr{M}, \mu)$, we have that $\langle \Phi_{j_\alpha}^{(p)} f, g\rangle = \langle \Phi_{j_\alpha}^{(2)} f, g\rangle$ converges to $\langle Q^{(p)} f, g\rangle = \left\langle \int_{\hat{G}} \varphi\, d\mathscr{E} f, g\right\rangle$. This together with (2.4) gives the desired conclusion.

So it suffices to prove Theorem (2.1) under the additional hypothesis (which we now adopt) that φ has compact support. Pick an approximate identity $\{m_j\}$ for $L^1(\hat{G})$ satisfying (with respect to $\hat{G}$) the conditions of Scholium (2.2). For each j, let ψ_j be the convolution $m_j * \varphi$. Thus

$$\|\psi_j\|_{M_p(\hat{G})} \leqslant \|\varphi\|_{M_p(\hat{G})}. \tag{2.5}$$

Moreover, $\psi_j \in L^1(\hat{G})$, ψ_j is continuous on $\hat{G}$, and (by (2.2) (iii)) $\hat{\psi}_j = \hat{m}_j \hat{\varphi} \in L^1(G)$. By Fourier inversion ([11_{II}, (31.44) (c)]), each ψ_j is the Fourier transform of a function in $L^1(G)$. It is easy to see from the uniform continuity of φ together with (2.2) (v) in the present setting that $\{\psi_j\}$ converges to φ uniformly on $\hat{G}$.

Let $f \in L^2(\mathscr{M}, \mu) \cap L^p(\mathscr{M}, \mu)$ and $g \in L^2(\mathscr{M}, \mu) \cap L^q(\mathscr{M}, \mu)$. We showed earlier that the conclusion of the theorem holds in the case when φ is a Fourier transform. Applying this fact to ψ_j, we have

$$\left|\int_{\hat{G}} \psi_j(\gamma)\, d\langle \mathscr{E}(\gamma) f, g\rangle\right| \leqslant s_p^2 \|\psi_j\|_{M_p(\hat{G})} \|f\|_{L^p(\mathscr{M},\mu)} \|g\|_{L^q(\mathscr{M},\mu)}.$$

Hence by (2.5) and the uniform convergence of $\{\psi_j\}$ to φ on $\hat{G}$,

$$\left|\int_{\hat{G}} \varphi(\gamma)\, d\langle \mathscr{E}(\gamma) f, g\rangle\right| \leqslant s_p^2 \|\varphi\|_{M_p(\hat{G})} \|f\|_{L^p(\mathscr{M},\mu)} \|g\|_{L^q(\mathscr{M},\mu)}.$$

This completes the proof of Theorem (2.1).

We next show how the continuity requirement on φ in Theorem (2.1) can be weakened. We shall continue to denote the space of $L^p(G)$-multipliers by $M_p(\hat{G})$.

Definition. Let G be a locally compact abelian group with dual group $\hat{G}$. We denote by $\Omega(\hat{G})$ the set of all complex-valued, bounded, Borel measurable functions on $\hat{G}$ with the following property: for each compact subset K of $\hat{G}$ there is a sequence $\{u_n\}_{n=1}^{\infty}$ (depending on φ and K) such that $\{u_n\}_{n=1}^{\infty} \subseteq L^1(\hat{G})$, $\|u_n\|_{L^1(\hat{G})} \leqslant 1$ for all n, and $u_n * \varphi \to \varphi$ pointwise on K ($*$ denotes convolution on $\hat{G}$).

(Note that the existence of a sequence $\{u_n\}$ with the stated properties is independent of the choice of Haar measure in $\hat{G}$).

With the aid of Scholium (2.2) together with uniform continuity of continuous functions on compact sets, it is not difficult to see that every bounded, continuous, complex-valued function on $\hat{G}$ belongs to $\Omega(\hat{G})$. Hence the following result generalizes Theorem (2.1).

Theorem (2.1) bis. *The statement of Theorem (2.1) remains valid if we replace "φ is an $L^p(G)$-multiplier which is continuous on $\hat{G}$" by "$\varphi \in M_p(\hat{G}) \cap \Omega(\hat{G})$".*

Proof. Let q be the index conjugate to p, and suppose that $f \in L^p(\mathscr{M}, \mu) \cap L^2(\mathscr{M}, \mu)$, $g \in L^q(\mathscr{M}, \mu) \cap L^2(\mathscr{M}, \mu)$. Define the regular Borel measure ν by putting $\nu = \langle \mathscr{E}(\cdot) f, g\rangle$. Given $\varepsilon > 0$, we have, by regularity of ν, a compact subset F of $\hat{G}$ such that $|\nu|(\hat{G} \setminus F) < \varepsilon$. By [$11_{\mathrm{II}}$, Theorem

(31.37)], there is $w \in L^1(G)$ such that $\hat{w} = 1$ on F, $\hat{w}$ has compact support, and $\|w\|_{L^1(G)} < 1+\varepsilon$. Let K be the support of $\hat{w}$, and let $\|\varphi\|_u$ denote $\sup\{|\varphi(\gamma)|: \gamma \in \hat{G}\}$. We have

$$\Big|\int_{\hat{G}} \varphi(\gamma)\, d\langle \mathscr{E}(\gamma) f, g\rangle\Big| \leqslant \Big|\int_K \varphi\hat{w}\, dv\Big| + \Big|\int_{\hat{G}\setminus F} \varphi\{1-\hat{w}\}\, dv\Big| \leqslant \Big|\int_K \varphi\hat{w}\, dv\Big| + \|\varphi\|_u (2+\varepsilon)\varepsilon.$$

Since $\varphi \in \Omega(\hat{G})$, we can choose a sequence $\{u_n\}$ corresponding to φ and the compact set K as in the definition of $\Omega(\hat{G})$. Put $\varphi_n \equiv u_n * \varphi$, for each n. Thus, $\|\varphi_n\|_u \leqslant \|\varphi\|_u$, $\varphi_n \to \varphi$ pointwise on K, $\|\varphi_n\|_{M_p(\hat{G})} \leqslant \|\varphi\|_{M_p(\hat{G})}$, and φ_n is continuous on $\hat{G}$. Hence for each n, $\varphi_n \hat{w}$ is continuous,

$$\|\varphi_n \hat{w}\|_{M_p(\hat{G})} \leqslant \|w\|_{L^1(G)} \|\varphi_n\|_{M_p(\hat{G})} \leqslant (1+\varepsilon)\|\varphi\|_{M_p(\hat{G})}.$$

Applying Theorem (2.1) to $\varphi_n \hat{w}$, we obtain and

$$\Big|\int_K \varphi_n \hat{w}\, dv\Big| = \Big|\int_{\hat{G}} \varphi_n(\gamma)\hat{w}(\gamma)\, d\langle \mathscr{E}(\gamma) f, g\rangle\Big| \leqslant s_p^2 \|\varphi_n \hat{w}\|_{M_p(\hat{G})} \|f\|_{L^p(\mu)} \|g\|_{L^q(\mu)} \leqslant s_p^2 (1+\varepsilon)\|\varphi\|_{M_p(\hat{G})} \|f\|_{L^p(\mu)} \|g\|_{L^q(\mu)}.$$

Using this and bounded convergence on K, we see that

$$\Big|\int_K \varphi\hat{w}\, dv\Big| \leqslant s_p^2 (1+\varepsilon)\|\varphi\|_{M_p(\hat{G})} \|f\|_{L^p(\mu)} \|g\|_{L^q(\mu)}.$$

Applying this to the inequality immediately following the definition of K, we get

$$\Big|\int_{\hat{G}} \varphi(\gamma)\, d\langle \mathscr{E}(\gamma) f, g\rangle\Big| \leqslant s_p^2 (1+\varepsilon)\|\varphi\|_{M_p(\hat{G})} \|f\|_{L^p(\mu)} \|g\|_{L^q(\mu)} + \|\varphi\|_u (2+\varepsilon)\varepsilon.$$

Letting $\varepsilon \to 0^+$ now gives the desired conclusion.

As an illustration, we next show how Theorem (2.1) affords a short proof of the following well-known result.

(2.6) Homomorphism Theorem for Multipliers ([9, Theorem B.2.1]). *Let ϱ be a continuous homomorphism of the locally compact abelian group Γ_1 into the locally compact abelian group Γ_2. Suppose that $1 < p < \infty$, $\varphi \in M_p(\Gamma_2)$, and φ is continuous on Γ_2. Then the composition $\varphi \circ \varrho$ belongs to $M_p(\Gamma_1)$, and*

$$\|\varphi \circ \varrho\|_{M_p(\Gamma_1)} \leqslant \|\varphi\|_{M_p(\Gamma_2)}.$$

Proof. Straightforward reasoning ([9, pp. 184, 185]) based on Scholium (2.2) shows that we can further assume without loss of generality that φ is compactly supported. (In particular, φ is Baire measurable in the sense of [10].) Let $\hat{G}_j = \Gamma_j$, $j = 1, 2$, and let $\hat{\varrho}$: $G_2 \to G_1$ be the dual homomorphism of ϱ. For $u \in G_2$, let R_u be translation by $\hat{\varrho}(u)$ on $L^2(G_1)$. Thus R is a strongly

continuous unitary representation of G_2 in $L^2(G_1)$. It is easy to see that the regular spectral measure $\mathscr{E}(\cdot)$ of R satisfies the following whenever B is a Baire subset of Γ_2: $\mathscr{E}(B)$ is the $L^2(G_1)$-multiplier transformation corresponding to the characteristic function of $\varrho^{-1}(B)$.

Use of this fact in conjunction with Theorem (2.1) readily completes the proof of (2.6).

Remarks. (a) Theorem (2.1) and Theorem (2.1) bis are valid in the case $p = 1$ also. Since $L^1(\mathscr{M}, \mu)$ is not reflexive, the extension of Theorem (2.1) to the case $p = 1$ requires suitable modifications in the proof given above. We omit the details for expository reasons. Note that the extension of Theorem (2.1) to the case $p = 1$ makes the foregoing proofs of Theorem (2.1) bis and Theorem (2.6) valid for $p = 1$ without alteration of methods. In particular, the resulting extension of Theorem (2.6) recovers the case $p = 1$ of [9, Theorem B.2.1]. In what follows, we shall not consider further the case $p = 1$ of Theorems (2.1), (2.1) bis, and (2.6.).

(b) Theorem (2.6) obviously contains, as special cases, well-known results of de Leeuw [7, Proposition 3.2] and Saeki [13, Corollary 4.6 *et seq.*] concerning restrictions of continuous multipliers.

(c) For another interesting multiplier homomorphism theorem, see [12, Théorème 2].

(2.7) *Some comments and examples concerning Theorem* (2.1). Suppose that G, $(\mathscr{M}, \mu)$, and R are as described at the outset of § 2. Let $\mathscr{I}_R$ denote the subset of $\boldsymbol{R}$ consisting of all p in the interval $(1, +\infty)$ such that conditions (i) and (ii) of Theorem (2.1) hold. Obviously, $2 \in \mathscr{I}_R$. It follows readily from the M. Riesz Convexity Theorem [14, Theorem V.1.3] that $\mathscr{I}_R$ is an interval in $(1, +\infty)$.

If the representation R is unitary, then obvious use of the relation $R_{-u} = (R_u)^*$ shows that if the index $p \in \mathscr{I}_R$, then so does the conjugate index p', and (in the notation of Theorem (2.1)) $s_{p'} = s_p$. Hence in the special case of R a strongly continuous unitary representation in $L^2(\mathscr{M}, \mu)$, either $\mathscr{I}_R = [p_0, p_0']$ for some $p_0 \in (1, 2]$ or $\mathscr{I}_R = (p_0, p_0')$ for some $p_0 \in [1, 2)$. In the general setting, wherein R is assumed to be bounded rather than unitary, the interval $\mathscr{I}_R$ need not have such a special form.

We now give three simple examples ((2.8)–(2.10)) to illustrate how the form of $\mathscr{I}_R$ can vary with the choice of the representation R. In each of these examples, R is not implemented by a group of measure-preserving transformations of the underlying measure space $\mathscr{M}$, and this fact serves to illustrate with basic tools one direction in which Theorem (2.1) generalizes the original Transference Theorem for Multipliers ([5, Theorem 3.7]). In § 3 we shall consider a full-fledged application of this more general aspect of Theorem (2.1).

(2.8) Example. *Each R_u is unitary, $\mathscr{I}_R = (1, +\infty)$, and $s_p > 1$ for $p \neq 2$.*

Take $\mathscr{M}$ to be the additive group $\boldsymbol{Z}$ of integers, and μ to be counting measure. Take G to be the additive group $\boldsymbol{R}$ of real numbers, and let T: $L^2(\boldsymbol{Z}) \to L^2(\boldsymbol{Z})$ be the multiplier transform corresponding to ψ: $\boldsymbol{T} \to [0, \pi)$, where $\psi(e^{it}) = t/2$ for $0 \leqslant t < 2\pi$. For $u \in \boldsymbol{R}$ put $R_u = e^{iuT}$. Since T is selfadjoint, R_u is unitary. By, for instance, Stechkin's Theorem [9, Theorem 6.4.4], $\psi \in M_p(\boldsymbol{T})$ for $1 < p < \infty$. Denoting the corresponding multiplier transform of ψ on $L^p(\boldsymbol{Z})$ by T_p, we see that condition (i) of Theorem (2.1) is satisfied for each $p \in (1, +\infty)$, with $R_u^{(p)} = \exp(iuT_p)$ for $u \in \boldsymbol{R}$. Moreover, $R_2^{(p)}$ is translation by 1 on $L^p(\boldsymbol{Z})$, and so $\|R_{2n}^{(p)}\| = 1$ for all $n \in \boldsymbol{Z}$, whence $s_p < +\infty$, and $\mathscr{I}_R = (1, \infty)$.

Suppose that $1 < p < \infty$, $p \neq 2$. If s_p were equal to 1, then $\exp(iuT_p)$ would be an isometry for all $u \in \boldsymbol{R}$. Thus T_p would be a hermitian operator (in the sense of Lumer and Vidav [8, Chapter 4]) on $L^p(\boldsymbol{Z})$. Since $p \neq 2$, the hermitian operators on $L^p(\boldsymbol{Z})$ are well known and easily characterized concretely. In fact, T_p would be forced to be coordinatewise multiplication by a fixed sequence of real numbers $\{\alpha_n\} \in L^\infty(\boldsymbol{Z})$. This is clearly absurd. For, if $\{\delta_n\}_{n=-\infty}^{\infty}$ denotes the natural basis of $L^p(\boldsymbol{Z})$, we would then have $T_p(\delta_0) = \alpha_0 \delta_0$. Upon taking Fourier transforms on both sides of this equation and using the definition of T_p, we get $\psi = \alpha_0$ almost everywhere on $\boldsymbol{T}$. Thus, $s_p > 1$ for $p \neq 2$.

It follows that in the present context R cannot be implemented by a group of measure-preserving transformations of $\mathscr{M}$. In fact, the following simple argument shows that if $u \in \boldsymbol{R}$, and R_u is expressed as composition with a permutation of $\boldsymbol{Z}$, then u is an even integer. For, if R_u is so expressible, then $R_u(\delta_0) = \delta_N$ for some $N \in \boldsymbol{Z}$. Taking Fourier transforms, we have $e^{iu\psi(z)} = z^{-N}$ for almost all $z \in \boldsymbol{T}$. Hence $e^{iut/2} = e^{-iNt}$ for all $t \in [0, 2\pi)$, and so $u = -2N$.

(2.9) Example. *$\mathscr{I}_R$ is an arbitrary closed, bounded interval situated in $(1, +\infty)$ and containing the real number 2.*

We remark at the outset that simple variants of the reasoning used in this example would provide representations R which allow $\mathscr{I}_R$ to be an arbitrarily prescribed bounded subinterval of $(1, +\infty)$ containing 2. Notice also the general fact that whenever a representation R in $L^2(\mu)$ has $\mathscr{I}_R$ different from $(1, +\infty)$, then R cannot be implemented by measure-preserving transformations.

For the example in (2.9), we take $\mathscr{M}$ to be $[0, 1]$, μ to be Lebesgue measure, and G to be $\boldsymbol{R}$. Fix p_0, q_0 with $1 < p_0 \leqslant 2 \leqslant q_0 < +\infty$. We shall construct a representation R of G in $L^2(\mu)$ such that $\mathscr{I}_R = [p_0, q_0]$.

Let $F(t) = t^{-1}(\log t)^{-2}$ for $0 < t \leqslant e^{-1}$. Extend F by making it constant on $[e^{-1}, 1]$. Put $f_0 \equiv F^{1/q_0}$, $g_0 \equiv F^{1/p'_0}$. Thus, $f_0 \in L^{q_0}(\mu)$, $f_0 \notin L^r(\mu)$ for $r > q_0$, $g_0 \in L^{p'_0}(\mu)$, $g_0 \notin L^r(\mu)$ for $r > p'_0$. Since $f_0 \in L^{p_0}(\mu)$, $0 < \int_{[0,1]} f_0 g_0 \, d\mu < +\infty$. We divide f_0 and g_0 by the same positive real constant so as to obtain corresponding functions f_1 and g_1 with $\int_{[0,1]} f_1 g_1 \, d\mu = 1$. For $p \in [p_0, q_0]$, $f_1 \in L^p(\mu)$ and $g_1 \in L^{p'}(\mu)$.

Define the bounded idempotent operator E_p on $L^p(\mu)$ by putting

$$E_p f = \Big(\int_{[0,1]} f g_1 \, d\mu\Big) f_1, \quad \text{for } f \in L^p(\mu).$$

Define the representation R of $\boldsymbol{R}$ in $L^2(\mu)$ by putting

$$R_u = e^{iu} E_2 + (I - E_2), \quad \text{for } u \in \boldsymbol{R}.$$

Taking account of the $L^r(\mu)$-spaces excluding f_1, and the spaces excluding g_1, it is easy to see that if E_2 is L^r-bounded on $L^2(\mu) \cap L^r(\mu)$, then $r \in [p_0, q_0]$. Hence $\mathscr{I}_R \subseteq [p_0, q_0]$. On the other hand, if $p \in [p_0, q_0]$, then for each $u \in \boldsymbol{R}$, we have $R_u^{(p)} = e^{iu} E_p + (I - E_p)$. Thus $[p_0, q_0] = \mathscr{I}_R$.

(2.10) Example. *A unitary representation R with $\mathscr{I}_R = [p_0, p'_0]$, where p_0 is an arbitrarily prescribed number in the interval $(1, 2]$.*

For this example, we need only take q_0 in Example (2.9) to be p'_0. Then, in the notation of (2.9), $f_1 = g_1$, and so E_2 is selfadjoint, and R is a unitary representation.

3. Classical analytic projection and bounds in generalized analyticity. In [3] an L^p counterpart $(1 < p < \infty)$ of Helson's invariant subspace theory was described for compact abelian groups K with archimedean ordered duals. In particular, each normalized simply invariant subspace of $L^p(K)$ was shown to be the range of a canonically corresponding bounded idempotent operator defined on $L^p(K)$. In this section, we shall show that the norm of this corresponding idempotent operator on $L^p(K)$ does not exceed that of classical analytic projection in L^p of the unit circle $\boldsymbol{T}$. In order to keep the discussion brief, we shall omit detailed accounts of the terminology and results in [3].

Let Γ be a subgroup of the additive real line $\boldsymbol{R}$ such that Γ is dense in $\boldsymbol{R}$ with respect to the natural topology. Endow Γ with its discrete topology and with the natural order of $\boldsymbol{R}$. Let K be the dual group of Γ. Fix p in the range $1 < p < \infty$. For each normalized, simply invariant subspace M of $L^p(K)$ there is a biuniquely associated cocycle A on K. The cocycle A gives rise to a strongly continuous one-parameter group of isometries $\{U_t\}$ on $L^p(K)$ defined by

$$(U_t f)(x) = A(t, x) f(x+t), \quad \text{for } f \in L^p(K),\ t \in \boldsymbol{R},\ x \in K.$$

Application to the group $\{U_t\}$ of the Generalized Stone's Theorem for UMD Spaces ([2, Theorem (5.5)]) now provides a uniquely determined spectral family of projections $E(\cdot)\colon \boldsymbol{R} \to \mathscr{B}(L^p(K))$ such that for each $t \in \boldsymbol{R}$, $\int_{-a}^{a} e^{i\lambda t}\, dE(\lambda)$ tends to U_t in the strong operator topology as $a \to +\infty$. The invariant subspace M is recovered from its cocycle A by the relation $M = \{I - E(0^-)\} L^p(K)$, where $E(0^-)$ denotes the strong limit of $E(\lambda)$ as $\lambda \to 0^-$ ([3, Theorem (3.3)]). In terms of this notation, we have the following theorem.

(3.1) THEOREM. *Let M be a normalized, simply invariant subspace of $L^p(K)$, where $1 < p < \infty$, and let C_p denote the norm of the classical Riesz projection of $L^p(\boldsymbol{T})$ onto the Hardy space $H^p(\boldsymbol{T})$. Then the projection operator $\{I - E(0^-)\} \in \mathscr{B}(L^p(K))$ described above satisfies*

$$\|I - E(0^-)\| \leqslant C_p.$$

Proof. We remark at the outset that C_p coincides with the $L^p(\boldsymbol{R})$-multiplier norm for the characteristic function of $\{\lambda \in \boldsymbol{R}\colon \lambda \geqslant 0\}$ ([1, Corollary (3.13)]). Let the strongly continuous one-parameter unitary group $\{V_t\}$ on $L^2(K)$ be defined by

$$(V_t f)(x) = A(t, x) f(x+t), \quad \text{for } f \in L^2(K),\ t \in \boldsymbol{R},\ x \in K.$$

Let $\mathscr{E}(\cdot)$ denote the spectral measure of $\{V_t\}$. It follows from [3, Lemma (3.1)] that for $f \in L^p(K) \cap L^2(K)$,

$$\mathscr{E}[0, +\infty) f = \{I - E(0^-)\} f. \tag{3.2}$$

Fix $f \in L^p(K) \cap L^2(K)$. For each positive integer n, let $\Psi_n \geqslant 0$ be a continuous function on $\boldsymbol{R}$ supported on the interval $[-n^{-1}, 0]$ such that $\int_{\boldsymbol{R}} \Psi_n(t)\, dt = 1$. Denote by χ the characteristic function relative to $\boldsymbol{R}$ of $\{\lambda \in \boldsymbol{R}\colon \lambda \geqslant 0\}$. It is easy to see that for each $t \in \boldsymbol{R}$, $(\Psi_n * \chi)(t) \to \chi(t)$ as $n \to +\infty$. Apply Theorem (2.1) bis by taking $\mathscr{M} = K$, $R = \{V_t\}$, ad $\varphi = \chi$. This shows that

$$\|\mathscr{E}[0, +\infty) f\|_{L^p(K)} \leqslant C_p \|f\|_{L^p(K)}.$$

Combining this fact with (3.2) completes the proof of Theorem (3.1).

Remarks. The one-parameter group $\{V_t\}$ to which we have applied Theorem (2.1) bis in the course of proving Theorem (3.1) is not, in general, implemented by measure-preserving transformations of K. For, $V_t(1) = A(t, \cdot)$ on K, and so, except in special instances, $V_t(1) \neq 1$.

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