STUDIA MATHEMATICA, T. XCIV (1989)

Some remarks on ratio inequalities for continuous martingales

by

NORIHIKO KAZAMAKI and MASATO KIKUCHI (Toyama)

Abstract. We are concerned with various ratio inequalities for martingales. One of the results we will prove is that for every $0 < p < \infty$ and every $0 \leqslant \alpha < \infty$ the ratio inequality

$$E[\langle X\rangle_\infty^p \exp(\alpha \langle X\rangle_\infty^{1/2}/X_\infty^*)] \leqslant C_{\alpha,p} E[\langle X\rangle_\infty^p]$$

holds for all continuous martingales X. This is an improvement of the results given in [4], [5] and [8].

1. Statement of the problem. Let $(\Omega, \mathfrak{F}, (\mathfrak{F}_t), P)$ be a filtered probability space satisfying the usual conditions, and let Q be a second probability measure, equivalent to P on $\mathfrak{F}_\infty$. Suppose, moreover, that the (martingale) Radon–Nikodym density $Z_t = dQ/dP|_{\mathfrak{F}_t}$ is continuous. In this note, we deal only with continuous martingales adapted to the filtration $(\mathfrak{F}_t)$ and, unless otherwise stated, "a martingale" means "a P-martingale". For a martingale X, let $X_t^* = \sup_{s\leqslant t}|X_s|$ and let $\langle X\rangle$ be its associated increasing process. We shall consider in addition the family $(L_t^a)_{t\geqslant 0, a\in \mathbf{R}}$ of its local times. It is shown in [1] that the process $L_t^* = \sup_{a\in\mathbf{R}} L_t^a$ is also continuous and increasing.

Let now Y_1 and Y_2 be any two of the three random variables X_∞^*, $\langle X\rangle_\infty^{1/2}$, L_∞^*, and consider an increasing function Φ from $[0, \infty]$ into $[0, \infty]$. Our object here is to study the problem: *does there exist a constant $C > 0$, depending only on p and Φ, such that the inequality*

$$E_Q[Y_1^p \Phi(Y_1/Y_2)] \leqslant C E_Q[Y_1^p] \quad (0 < p < \infty) \tag{1}$$

holds for all martingales X? Here E_Q denotes expectation with respect to Q. This inequality for the case where $Q = P$ and $\Phi(x) = x^r$ $(r > 0)$ was established in 1982 by Gundy [4] and independently by Yor [8]. Quite recently, we have improved their result to the case where $Q = P$ and $\Phi(x) = \exp(cx)$ for *some* $c > 0$ (see [5]). However, the inequality (1) does not necessarily hold for any Φ even if $Q = P$. We shall first exemplify it. For that, let $B = (B_t, \mathfrak{F}_t)$ be a one-dimensional Brownian motion starting at 0, and we set $X_t = B_{t\wedge 1}$, $\Phi(x) = \exp(x^2/2)$. It is clear that $X_\infty^* \in L^p$ for every $p > 0$, but $\exp(B_1^2/2)$ is not integrable. On the other hand, noticing $\langle X\rangle_\infty = 1$ we find

$$E[\exp(\tfrac{1}{2}B_1^2)] \leqslant e^{1/2} + E[X_\infty^{*p}\Phi(X_\infty^*/\langle X\rangle_\infty^{1/2}):\ X^* > 1],$$

so that $E[X_\infty^{*p}\Phi(X_\infty^*/\langle X\rangle_\infty^{1/2})] = \infty$ for any $p > 0$. This implies that (1) fails if $Y_1 = X_\infty^*$ and $Y_2 = \langle X\rangle_\infty^{1/2}$.

In the same way, we can give an example such that (1) fails if $Y_1 = \langle X\rangle_\infty^{1/2}$ and $Y_2 = X_\infty^*$. To see it, consider this time the martingale X defined by $X_t = B_{t\wedge\tau}$ where $\tau = \inf\{t:\ |B_t| = 1\}$. It is clear that $X_\infty^* = 1$ and $\langle X\rangle_\infty = \tau$. From the Burkholder–Davis–Gundy inequality it follows immediately that $\langle X\rangle_\infty \in L^p$ for any $p > 0$. Let now $\Phi(x) = \exp(\pi^2 x^2/8)$. Then we find

$$E[\exp(\pi^2\tau/8)] \leqslant \exp(\pi^2/8) + E[\langle X\rangle_\infty^p\Phi(\langle X\rangle_\infty^{1/2}/X_\infty^*):\ \langle X\rangle_\infty > 1].$$

Since the expectation on the left-hand side is infinite, we have

$$E[\langle X\rangle_\infty^p\Phi(\langle X_\infty^{1/2}/X_\infty^*)] = \infty.$$

2. A ratio inequality for increasing processes. First of all, let $M_t = \int_0^t Z_s^{-1}dZ_s$ where $dQ = Z_\infty dP$ as is already mentioned. Later we shall assume that $M \in \mathrm{BMO}$. Recall that a uniformly integrable martingale X is said to be in the class BMO if

$$\sup_T \|E[|X_\infty - X_T|\mid \mathfrak{F}_T]\|_\infty < \infty,$$

where the supremum is taken over all stopping times T.

For convenience' sake, let us denote by $C_{\lambda,\eta}$ or $C(\lambda,\eta)$ a positive constant depending only on the indexed parameters λ and η. Note that $C_{\lambda,\eta}$ is not necessarily the same from line to line.

Consider now two right-continuous increasing processes U and V such that $U_0 = V_0 = 0$. The essential result of this note is the following.

THEOREM 1. *If the martingale M belongs to the class* BMO *and if there is a constant $\varkappa > 0$ such that*

$$E[U_\infty^\sigma - U_{T-}^\sigma \mid \mathfrak{F}_T] \leqslant \varkappa E[V_{\sigma-}\mid \mathfrak{F}_T] \tag{2}$$

for any stopping times σ and T, then the ratio inequality

$$E_Q[U_\infty^p\exp(\alpha U_\infty/V_\infty)] \leqslant C(\varkappa,\alpha,p)E_Q[U_\infty^p] \quad (0 < p < \infty) \tag{3}$$

holds for some $\alpha > 0$.

Moreover, if $0 \leqslant \beta < 1$, then we have

$$E_Q[U_\infty^p\exp\{\alpha(U_\infty/V_\infty)^\beta\}] \leqslant C(\varkappa,\alpha,\beta,p)E_Q[U_\infty^p] \quad (0 < p < \infty) \tag{4}$$

for every $\alpha \geqslant 0$.

Here U^σ denotes the process $(U_{t\wedge\sigma})$ as usual. Three lemmas are needed for the proof of this theorem.

LEMMA 1. *Let A be a right-continuous increasing process satisfying $E[A_\infty - A_{T-}\mid \mathfrak{F}_T] \leqslant c$ for all stopping times T, with a constant $c > 0$. Then for $0 \leqslant \alpha < 1/c$ the inequality*

$$E[\exp\{\alpha(A_\infty - A_{T-})\}\mid \mathfrak{F}_T] \leqslant \frac{1}{1-\alpha c}$$

holds for all stopping times T.

For the proof, see [2].

It was proved in [3] by Doléans-Dade and Meyer that if $M \in \mathrm{BMO}$, then the "reverse Hölder inequality"

$$E[Z_\infty^r \mid \mathfrak{F}_T] \leqslant C_r Z_T^r \tag{5}$$

holds for all stopping times T. Let now $1/r + 1/s = 1$. This given, we can easily prove the following result.

LEMMA 2. *Suppose that $M \in \mathrm{BMO}$. If A is a right-continuous increasing process such that $E[A_\infty - A_{T-}\mid \mathfrak{F}_T] \leqslant c$ for all stopping times T, with a constant $c > 0$, then for $0 \leqslant \alpha < 1/(sc)$ we have*

$$E_Q[\exp\{\alpha(A_\infty - A_{T-})\}\mid \mathfrak{F}_T] \leqslant C_r^{1/r}(1-\alpha sc)^{-1/s},$$

where T is an arbitrary stopping time and C_r is the same constant as in (5).

Proof. Applying the definition of conditional expectation and the Hölder inequality with exponents r and s we have

$$\begin{aligned} &E_Q[\exp\{\alpha(A_\infty - A_{T-})\}\mid \mathfrak{F}_T] \\ &\qquad = E[(Z_\infty/Z_T)\exp\{\alpha(A_\infty - A_{T-})\}\mid \mathfrak{F}_T] \\ &\qquad \leqslant E[(Z_\infty/Z_T)^r\mid \mathfrak{F}_T]^{1/r}E[\exp\{\alpha s(A_\infty - A_{T-})\}\mid \mathfrak{F}_T]^{1/s}. \end{aligned}$$

If $M \in \mathrm{BMO}$, then Z satisfies the reverse Hölder inequality (5), i.e. the first term in the last expression is dominated by $C_r^{1/r}$. On the other hand, Lemma 1 implies that if $0 \leqslant \alpha < 1/(cs)$, then $\alpha s < 1/c$ and so the second term is dominated by $(1-\alpha cs)^{-1/s}$. Thus the proof is complete.

The following lemma is of fundamental importance in our investigation. For the proof, see [6].

LEMMA 3. *Let X and Y be positive random variables. If there are two constants $a > 0$ and $c > 0$ such that for $\lambda > 0$ and $\gamma > 1$*

$$P(X > \gamma\lambda,\ Y \leqslant \lambda) \leqslant ce^{-a\gamma}P(X > \lambda),$$

then for $0 \leqslant b < a$ and $0 < p < \infty$

$$E[X^p\exp(bX/Y)] \leqslant C_{b,p}E[X^p].$$

Proof of Theorem 1. For each $\lambda > 0$, we first define the stopping times τ and σ as follows:

$$\tau = \inf\{t;\ U_t > \lambda\}, \qquad \sigma = \inf\{t;\ V_t > \lambda\}.$$

Obviously $V_{\sigma-} \leqslant \lambda$ and so $E[U^\sigma_\infty - U^\sigma_{T-} \mid \mathfrak{F}_T] \leqslant \varkappa\lambda$ by the condition (2), where T is an arbitrary stopping time. Let now $0 \leqslant \alpha < 1/(\varkappa s)$ and $\alpha\varkappa < \delta < 1/s$. Then from Lemma 2 the inequality

$$E_Q\left[\exp\left\{\frac{\delta}{\varkappa\lambda}(U^\sigma_\infty - U^\sigma_{T-})\right\}\middle|\ \mathfrak{F}_T\right] \leqslant C_r^{1/r}(1-\delta s)^{-1/s}$$

follows at once. Combining this with the fact that $U_{\tau-} \leqslant \lambda$, we have

$$\begin{aligned} Q(U_\infty > \gamma\lambda, V_\infty \leqslant \lambda) &\leqslant Q\{U_\infty - U_{\tau-} > (\gamma-1)\lambda,\ \sigma = \infty,\ \tau < \infty\} \\ &\leqslant Q\left\{\frac{\delta}{\varkappa\lambda}(U^\sigma_\infty - U^\sigma_{\tau-}) > \frac{\delta(\gamma-1)}{\varkappa},\ \tau < \infty\right\} \\ &\leqslant \exp\left\{-\frac{\delta(\gamma-1)}{\varkappa}\right\} E_Q\left[E_Q\left[\exp\left\{\frac{\delta}{\varkappa\lambda}(U^\sigma_\infty - U^\sigma_{\tau-})\right\}\middle|\ \mathfrak{F}_\tau\right]:\ \tau < \infty\right] \\ &\leqslant C\exp\left(-\frac{\delta}{\varkappa}\gamma\right) Q(U_\infty > \lambda), \end{aligned}$$

where $C = C_r^{1/r} e^{\delta/\varkappa}(1-\delta s)^{-1/s}$. As $\alpha < \delta/\varkappa$, we obtain (3) by Lemma 3.

Furthermore, observing that if $\gamma > 0$ and $0 \leqslant \beta < 1$, then

$$\exp(\alpha x^\beta) \leqslant C(\alpha, \beta, \gamma)\exp(\gamma x) \qquad (0 \leqslant x < \infty)$$

for every $\alpha \geqslant 0$, (4) follows immediately from (3). This completes the proof.

With the help of Theorem 1 we shall give some improvements of the ratio inequalities obtained in [4], [5] and [8]. Recently, Barlow and Yor proved in [1] that

$$c_p E[\langle X\rangle_t^{p/2}] \leqslant E[(L_t^*)^p] \leqslant C_p E[\langle X\rangle_t^{p/2}] \qquad (0 < p < \infty)$$

for any continuous martingale X with the family $(L_t^a)_{t \geqslant 0, a \in \mathbf{R}}$ of local times. On the other hand, the Burkholder–Davis–Gundy inequality

$$c_p E[\langle X\rangle_t^{p/2}] \leqslant E[(X_t^*)^p] \leqslant C_p E[\langle X\rangle_t^{p/2}] \qquad (0 < p < \infty)$$

is now well known. Combining the conditional forms of these inequalities for $p = 1$ shows that any two of the three increasing processes X^*, $\langle X\rangle^{1/2}$ and L^* satisfy the condition (2). Therefore, the following is an immediate consequence of Theorem 1.

THEOREM 2. *Assume that $M \in \mathrm{BMO}$. If U and V are any two of the three increasing processes X^*, $\langle X\rangle^{1/2}$ and L^*, then for some $\alpha > 0$ the ratio inequality*

$$E_Q[U^p_\infty \exp(\alpha U_\infty/V_\infty)] \leqslant C_{\alpha,p} E_Q[U^p_\infty] \qquad (0 < p < \infty)$$

holds for all continuous martingales X.

Moreover, if $0 \leqslant \beta < 1$, then for every $\alpha \geqslant 0$

$$E_Q[U^p_\infty \exp\{\alpha(U_\infty/V_\infty)^\beta\}] \leqslant C(\alpha, \beta, p) E_Q[U^p] \qquad (0 < p < \infty).$$

We especially remark the following.

COROLLARY 1. *Assume that $M \in \mathrm{BMO}$. Then for every $0 \leqslant \alpha < \infty$ and every $0 < p < \infty$ we have*

$$E_Q[\langle X\rangle^p_\infty \exp(\alpha\langle X\rangle^{1/2}_\infty / X^*_\infty)] \leqslant C_{\alpha,p} E_Q[\langle X\rangle^p_\infty], \tag{6}$$

$$E_Q[\langle X\rangle^p_\infty \exp(\alpha\langle X\rangle^{1/2}_\infty / L^*_\infty)] \leqslant C_{\alpha,p} E_Q[\langle X\rangle^p_\infty]. \tag{7}$$

Proof. The usual stopping argument enables us to assume that X is an L^2-bounded martingale. Observe first that $E[\langle X\rangle_\infty - \langle X\rangle_T \mid \mathfrak{F}_T] \leqslant E[(X^*_\infty)^2 \mid \mathfrak{F}_T]$ for any stopping time T. Then, applying the latter part of Theorem 1 to the case where $U = \langle X\rangle$, $V = (X^*)^2$ and $\beta = 1/2$ we can obtain (6). The same argument proves (7), because $E[\langle X\rangle_\infty - \langle X\rangle_T \mid \mathfrak{F}_T] \leqslant CE[(L^*_\infty)^2 \mid \mathfrak{F}_T]$. Thus the proof is complete.

It is natural to ask if the inequality for another pair holds for any $\alpha \geqslant 0$ as is stated in Corollary 1. But we cannot settle this question so far.

Finally, we improve a result given by Sekiguchi. He proved in [7] that if $M \in \mathrm{BMO}$, then the inequality

$$E_Q[(X^*_\infty)^p] \leqslant C_p E_Q[\langle X\rangle^{p/2}_\infty] \qquad (0 < p < \infty) \tag{8}$$

holds for all continuous martingales X. Consequently, combining his result with Theorem 2 gives the following.

COROLLARY 2. *If $M \in \mathrm{BMO}$, then there is a constant $\alpha > 0$ independent of p such that the ratio inequality*

$$E_Q[(X^*_\infty)^p \exp(\alpha X^*_\infty / \langle X\rangle^{1/2}_\infty)] \leqslant C_{\alpha,p} E_Q[\langle X\rangle^{p/2}_\infty] \qquad (0 < p < \infty)$$

holds for all continuous martingales X.

Sekiguchi proved there that the converse is also true. Precisely speaking, his claim is that if the inequality (8) is valid for $p = 1$, then $M \in \mathrm{BMO}$.

References

[1] M. T. Barlow and M. Yor, *(Semi-) Martingale inequalities and local times*, Z. Wahrsch. Verw. Gebiete 55 (1981), 237–254.

[2] C. Dellacherie et P. A. Meyer, *Probabilités et Potentiel, Chap. V à VIII*, Hermann, Paris 1980.
[3] C. Doléans-Dade et P. A. Meyer, *Inégalités de normes avec poids*, in: Sém. de Probabilités VIII, Lecture Notes in Math. 721, Springer, 1979, 313–331.
[4] R. Gundy, *The density of the area integral*, in: Conference on Harmonic Analysis in Honor of Antoni Zygmund, Vol. 1, Wadsworth, Belmont, Calif., 1982, 138–149.
[5] N. Kazamaki and M. Kikuchi, *Quelques inégalités des rapports pour martingales continues*, C. R. Acad. Sci. Paris Sér. I 305 (1987), 37–38.
[6] T. Murai and A. Uchiyama, *Good λ inequalities for the area integral and the nontangential maximal function*, Studia Math. 83 (1986), 251–262.
[7] T. Sekiguchi, BMO-*martingales and inequalities*, Tôhoku Math. J. 31 (1979), 355–358.
[8] M. Yor, *Application de la relation de domination à certains renforcements des inégalités de martingales*, in: Sém. de Probabilités XVI, Lecture Notes in Math. 920, Springer, 1982, 221–233.

DEPARTMENT OF MATHEMATICS
TOYAMA UNIVERSITY
Gofuku, Toyama 930, Japan

Received November 30, 1987 (2387)

STUDIA MATHEMATICA

Contents of forthcoming issues

Volume XCIV, number 2

Volume XCIV, number 3

Volume XCV, number 1