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A geometric condition equivalent to commutativity in Banach algebras

by

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Abstract. Under certain geometric conditions, it is shown that the generalized spectrum and radical defined by T. Ransford may be put into the context of Banach algebra theory. As a consequence, it is proven that commutativity of $A/\mathrm{Rad}(A)$ is equivalent to the convexity of the set of invertible elements of A with respect to the family of functions $-\ln|\gamma|$, where γ is a continuous linear functional on A.

Introduction. Let E be a complex normed space, and Ω an open subset of E such that $0 \notin \Omega$ and $\lambda\Omega \subset \Omega$ for all $\lambda \in \boldsymbol{C}^*$ (nonzero complex numbers). Let 1 be any fixed element of Ω. Then T. J. Ransford has defined [5] the following notions of *Ω-spectrum* and *Ω-radical*:

$$\mathrm{sp}^{\Omega}(x) = \{\lambda\colon\ x-\lambda 1 \notin \Omega\}, \qquad \mathrm{Rad}^{\Omega}(E) = \{x\colon\ x+\Omega = \Omega\},$$

where Ω plays the part that the set of invertible elements would play if E were an algebra.

By imposing a geometric condition (pseudoconvexity) on Ω, Ransford goes surprisingly far into developing the usual theory of spectra for his generalized spectrum. Thus, it would seem, multiplication itself becomes redundant, provided one has enough knowledge of the geometry of Ω.

One is tempted to try to define a product on E compatible with the above definitions of Ω-spectrum and Ω-radical. Clearly, however, one cannot expect to obtain quite as much, for if E is any subspace of a unital algebra A, containing the unit, and Ω is the intersection of E and the set of invertible elements of A, then Ω is as above, but E need not be closed under multiplication. Instead, one should perhaps try to obtain the situation

$$h\colon\ E \to A$$

where A is a complex unital algebra, $h(1) = 1$, Ω is the preimage of the set of invertible elements of A, and the Ω-spectra and Ω-radical are similarly related to the classical notions of spectra and radical in A.

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In Theorem 1 we show that by imposing a "convexity" condition on Ω, this can be done, and with a commutative A; the important point being the relationship between the geometry of Ω and commutativity.

As it turns out, if A is any complex unital Banach algebra, and A^{-1} is its set of invertible elements, then the commutativity of $A/\mathrm{Rad}(A)$ is equivalent to the convexity of A^{-1} with respect to the family of functions

$$\mathscr{F} = \{-\ln|\gamma|: \gamma \text{ is a continuous linear functional on } A\}.$$

This is Theorem 2 of our paper. Note that $\mathscr{F}$ depends only on the Banach space structure of A. This condition for commutativity of $A/\mathrm{Rad}(A)$ is similar to that given by Fong and Sołtysiak [1], in that they both have to do with the existence of sufficiently many characters.

1. Throughout, E will denote a complex normed space, and E' its dual, with the weak* topology.

Take Ω as a nonempty open subset of E, with the following restriction: Ω is contained in the complement of some complex hyperplane of E. We will think of Ω as a set of invertible elements of some Banach algebra, and this condition is intended to prevent 0 from becoming "invertible". Denote by 1 any fixed arbitrary element of Ω, and define the following subset of E', which will obviously play the part of spectrum [2, 3, 6]:

$$M_\Omega = \{\gamma \in E': \gamma(1) = 1 \text{ and } \gamma(\Omega) \subset \mathbf{C}^*\}.$$

For example, if $\|1\| = 1$ and Ω is the open unit ball centered at 1, then M_Ω is the state space of E. It is not difficult to verify that M_Ω is a compact subset of E'. We will not prove this here, but let us mention the following related inequality.

PROPOSITION. *For all γ in M_Ω, $\|\gamma\| \leqslant d(1, \Omega^c)^{-1}$.*

Proof. Let x be an element of norm one, and set $d = d(1, \Omega^c)$. Then the ball centered at 1 with radius d is contained in Ω, so for all scalars λ with $|\lambda| < d$, $1+\lambda x$ is in Ω. Hence, $\gamma(1+\lambda x) \neq 0$ for every γ in M_Ω. Then $\lambda\gamma(x) \neq -1$. If we had $|\gamma(x)| > d^{-1}$, then we would have $|-(\gamma(x))^{-1}| < d$, so $-1 \neq -1$. Therefore, $|\gamma(x)| \leqslant d^{-1}$, so $\|\gamma\| \leqslant d^{-1}$ for every γ in M_Ω. ∎

Now if the elements of Ω are to be, in some sense, invertible, there will appear others that will necessarily be invertible as well. For example, if x is invertible, then so must be λx, for nonzero λ. We must replace Ω with something large enough to accomodate these other elements. Define

$$\tilde{\Omega} = \bigcap \{(\mathrm{Ker}\,\gamma)^c: \gamma \in M_\Omega\}.$$

Note that the condition we have imposed on Ω is equivalent to any of the following: $0 \notin \tilde{\Omega}$, $\tilde{\Omega} \neq E$, $M_\Omega \neq \emptyset$. An example with $M_\Omega = \emptyset$ can be obtained by putting $E = \mathbf{C}^{n\times n}$, $\Omega = \mathrm{GL}(n, \mathbf{C})$, $n > 1$.

Before we go on, we shall prove a few simple statements about $\tilde{\Omega}$.

PROPOSITION. *Let Ω be as above. Then:*

(i) *$\Omega \subset \tilde{\Omega}$, and $\Omega_1 \subset \Omega_2$ implies $\tilde{\Omega}_1 \subset \tilde{\Omega}_2$.*
(ii) *$\lambda\tilde{\Omega} \subset \tilde{\Omega}$, for all $\lambda \in \mathbf{C}^*$.*
(iii) *For each $y \notin \tilde{\Omega}$, there is a $\gamma \in M_\Omega$ with $\gamma(y) = 0$.*
(iv) *$\tilde{\Omega}$ does not depend on the element chosen as unit.*
(v) *$\tilde{\Omega}$ is open.*
(vi) *$\tilde{\tilde{\Omega}} = \tilde{\Omega}$.*

Proof. The first four are easy.

(v) Let y be in the closure of $\tilde{\Omega}^c$. There is a sequence $(y_n) \subset \tilde{\Omega}^c$ converging to y. For each n, there is by (iii) a $\gamma_n \in M_\Omega$ such that $\gamma_n(y_n) = 0$. Let (γ_{n_k}) be a subsequence of (γ_n) converging to an element $\gamma \in M_\Omega$. Then

$$\begin{aligned}|\gamma(y)| &= |\gamma_{n_k}(y_{n_k}) - \gamma(y)| \leqslant |\gamma_{n_k}(y_{n_k}) - \gamma_{n_k}(y)| + |\gamma_{n_k}(y) - \gamma(y)| \\ &= |\gamma_{n_k}(y_{n_k} - y)| + |\gamma_{n_k}(y) - \gamma(y)| \\ &\leqslant d(1, \Omega^c)^{-1}\|y_{n_k} - y\| + |\gamma_{n_k}(y) - \gamma(y)|\end{aligned}$$

which tends to 0 as k grows. So, $\gamma(y) = 0$, with $\gamma \in M_\Omega$, and therefore $y \in \tilde{\Omega}^c$.

(vi) Just check that $M_{\tilde{\Omega}} = M_\Omega$. ∎

We wish to describe $\tilde{\Omega}$ in terms of convexity with respect to a certain family of functions $f: E \to (-\infty, \infty]$.

Given any subset Q of E, define

$$Q^{\#} = \{x \in E: -\ln|\gamma(x)| \leqslant \sup_Q(-\ln|\gamma|), \text{ for all } \gamma \in E'\}.$$

Note that both sides of the inequality may well be infinite. Now if $Q \subset \Omega$, and $\gamma \in M_\Omega$, $\sup_Q(-\ln|\gamma|)$ may be infinite, although in this case the supremum is certainly never attained. We shall say Q is *well-contained* in Ω if this supremum is finite for each $\gamma \in M_\Omega$. Note that this does not depend on which element of Ω has been designated as 1. Any weakly compact subset of Ω is well-contained, but Ω is not well-contained in itself.

Now define

$$\hat{\Omega} = \bigcup \{Q^{\#}: Q \subset \Omega\}$$

where the union is taken over all well-contained subsets of Ω. We then obtain the following proposition.

PROPOSITION. *Let E be a normed space, and Ω an open subset of E as above, with $\lambda\Omega \subset \Omega$ for all $\lambda \in \mathbf{C}^*$. Then $\hat{\Omega} = \tilde{\Omega}$.*

Proof. $\subset$. Let $Q \subset \Omega$ be well-contained. It will be enough to see $Q^{\#} \subset \tilde{\Omega}$. Let $x \notin \tilde{\Omega}$, and take $\gamma \in M_\Omega$ such that $\gamma(x) = 0$. Now $\sup_Q(-\ln|\gamma|) < \infty = -\ln|\gamma(x)|$, so $x \notin Q^{\#}$.

$\supset$. We shall see $\hat{\Omega}^c \subset \tilde{\Omega}^c$; i.e., if $x \in E$ is such that for each well-contained $Q \subset \Omega$, $x \notin Q^{\#}$, then $x \notin \tilde{\Omega}$.

We define, for each $n \in \mathbf{N}$,

$$U_n = \{y \in \Omega\colon |\gamma(y)| > 1/n, \text{ for all } \gamma \in M_\Omega\},$$

$$Q_n = \{y \in \Omega\colon |\gamma(y)| \geqslant 1/n, \text{ for all } \gamma \in M_\Omega\}.$$

Note that all U_n are open in the norm topology of E. Also, the Q_n are well-contained in Ω. We have $\ldots \subset U_n \subset Q_n \subset U_{n+1} \subset \ldots \subset \Omega$, and $\bigcup_n U_n = \Omega$, so

$$\ldots \supset M_{U_n} \supset M_{Q_n} \supset M_{U_{n+1}} \supset \ldots \supset M_\Omega, \quad \text{and} \quad \bigcap_n M_{U_n} = M_\Omega.$$

Also, all the M_{U_n} are compact, just as M_Ω is. Since, for each $n \in \mathbf{N}$, $x \notin Q_n^{\#}$, there are γ_n such that

$$-\ln|\gamma_n(x)| > \sup_{Q_n}(-\ln|\gamma_n|), \quad \text{i.e.,} \quad |\gamma_n(x)| < \inf_{Q_n}|\gamma_n|.$$

Note that 1 belongs to Q_n, so $\gamma_n(1) \neq 0$. Multiplying by convenient scalars, we may suppose $\gamma_n(1) = 1$ for all n. Also, since γ_n is never null on Q_n, it is never null on U_n, and we have $\gamma_n \in M_{U_n}$. There is, therefore, a subsequence (γ_{n_k}) converging to, say, γ. We shall prove that $\gamma \in M_\Omega$: if this were not the case, there would exist disjoint open neighborhoods U, V of M_Ω and γ respectively. (M_{U_n}) is a decreasing sequence of compact sets whose intersection M_Ω is contained in U, so $M_{U_n} \subset U$ for sufficiently large n. This implies $M_{U_n} \cap V = \emptyset$, which contradicts the convergence of γ_{n_k} to γ. We then have a $\gamma \in M_\Omega$ for which

$$|\gamma(x)| \leqslant \inf\{|\gamma(\omega)|\colon \omega \in \Omega\} = 0.$$

Hence x is an element of $\operatorname{Ker}\gamma$. But $\gamma \in M_\Omega$, so $x \notin \tilde{\Omega}$. ■

DEFINITION. Let $\mathcal{F}$ be the family of functions

$$\mathcal{F} = \{-\ln|\gamma|\colon \gamma \in E'\}$$

and Ω an open subset of E such that $\lambda\Omega \subset \Omega$ for all $\lambda \in \mathbf{C}^*$. We say Ω is *$\mathcal{F}$-convex* if one, and therefore all, of the following hold:

(a) $Q^{\#} \subset \Omega$, for all well-contained $Q \subset \Omega$.
(b) $\hat{\Omega} = \Omega$.
(c) $\tilde{\Omega} = \Omega$.

We have the following theorem.

THEOREM 1. *Let E be a normed space, and Ω an open subset contained in the complement of some hyperplane. Then there are a commutative unital Banach algebra A and a continuous linear function $h\colon E \to A$ such that $h(1) = 1$ and $\tilde{\Omega} = h^{-1}(A^{-1})$. Also:*

(i) *$\tilde{\Omega}$ is the smallest open set containing Ω that has this property.*
(ii) $\operatorname{Rad}^{\tilde{\Omega}}(E) = \bigcap\{\operatorname{Ker}\gamma\colon \gamma \in M_\Omega\} = \operatorname{Ker} h$.
(iii) $\operatorname{sp}^{\tilde{\Omega}}(x) = \operatorname{sp}_A(h(x))$.

Proof. Take $A = C(M_\Omega)$, and $h\colon E \to A$ the Gelfand transform, $h(x)(\gamma) = \gamma(x)$. Then h is a continuous linear map, its norm being $\leqslant d(1, \Omega^c)^{-1}$. Also, $h(1) = 1$, and

$$h^{-1}(A^{-1}) = \{x \in E\colon \gamma(x) \neq 0 \text{ for all } \gamma \in M_\Omega\} = \bigcap\{(\operatorname{Ker}\gamma)^c\colon \gamma \in M_\Omega\} = \tilde{\Omega}.$$

(i) Suppose B is a commutative unital Banach algebra, and $g\colon E \to B$ a continuous linear function such that $g(1) = 1$ and $\Omega \subset g^{-1}(B^{-1})$. Let us see that we also have $\tilde{\Omega} \subset g^{-1}(B^{-1})$:

Let ψ be a character of B; then $\psi g \in M_\Omega$, for $\psi(g(1)) = \psi(1) = 1$, and $\psi(g(\Omega)) \subset \psi(B^{-1}) \subset \mathbf{C}^*$. Then for each $x \in \tilde{\Omega}$, $\psi(g(x)) \neq 0$. Since this is so for any character, $g(x) \in B^{-1}$.

(ii) The second equality is trivial. Let us see the first.

$\subset$. Let $x \in \operatorname{Rad}^{\tilde{\Omega}}(E)$, and suppose γ is an element of M_Ω with $\gamma(x) \neq 0$. Then $-x/\gamma(x) \in \operatorname{Rad}^{\tilde{\Omega}}(E)$, and $(-x/\gamma(x)) + 1 \in \tilde{\Omega}$, but

$$\gamma\big((-x/\gamma(x)) + 1\big) = (-\gamma(x)/\gamma(x)) + 1 = 0;$$

absurd, for $\gamma \in M_\Omega = M_{\tilde{\Omega}}$, so $\gamma(\tilde{\Omega}) \subset \mathbf{C}^*$. Then $\gamma(x) = 0$.

$\supset$. Let $x \notin \operatorname{Rad}^{\tilde{\Omega}}(E)$. Set $a \in \tilde{\Omega}$ such that $x + a \notin \tilde{\Omega}$. Now let $\gamma \in M_\Omega$ such that $\gamma(x + a) = 0$. We have $\gamma(x) = -\gamma(a) \neq 0$, so $x \notin \operatorname{Ker}\gamma$.

(iii) $\operatorname{sp}^{\tilde{\Omega}}(x) = \{\lambda \in \mathbf{C}\colon x - \lambda 1 \notin \tilde{\Omega}\} = \{\lambda \in \mathbf{C}\colon x - \lambda 1 \notin h^{-1}(A^{-1})\}$
$= \{\lambda \in \mathbf{C}\colon h(x) - \lambda 1 \notin A^{-1}\} = \operatorname{sp}_A(h(x))$. ■

Note that if Ω is $\mathcal{F}$-convex, then the problem we posed in the introduction is solved with a commutative Banach algebra.

We end with the following characterization of commutativity.

THEOREM 2. *Let A be a complex Banach algebra with unit, and denote by Γ its set of invertible elements. Then the following are equivalent:*

(i) *Γ is $\mathcal{F}$-convex.*
(ii) *$A/\operatorname{Rad}(A)$ is commutative.*

Proof. (i)$\Rightarrow$(ii). By [2, 3, 6], the elements of M_Γ are multiplicative linear functionals of A, so $h\colon A \to C(M_\Gamma)$ is an algebra morphism. But $C(M_\Gamma)$ is commutative, so

$$ab - ba \in \operatorname{Ker} h = \operatorname{Rad}^{\tilde{\Gamma}}(A) = \operatorname{Rad}^{\Gamma}(A) = \operatorname{Rad}(A),$$

the last equality by [4].

(ii)$\Rightarrow$(i). Let $\pi\colon A \to A/\operatorname{Rad}(A)$ be the projection map. We must see that $\tilde{\Gamma} \subset \Gamma$. Take $x \notin \Gamma$. Then $0 \in \operatorname{sp}_A(x) = \operatorname{sp}_{A/\operatorname{Rad}(A)}(\pi(x))$, so $\pi(x)$ is singular. By the commutativity of $A/\operatorname{Rad}(A)$, there is a maximal ideal J, a hyperplane of

$A/\mathrm{Rad}(A)$, to which $\pi(x)$ belongs. Let $H = \pi^{-1}(J)$. H is a hyperplane of A to which x belongs, and $H \cap \Gamma = \emptyset$, for if $y \in H \cap \Gamma$, $\pi(y)$ would be an invertible element of J; absurd. Take $\gamma \in A'$ such that $\gamma(1) = 1$ and $\operatorname{Ker}\gamma = H$. Then $\gamma \in M_\Gamma$ and $\gamma(x) = 0$, i.e., $x \notin \tilde{\Gamma}$. ∎

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