STUDIA MATHEMATICA, T. XCIV (1989)

On the Forelli–Rudin construction and weighted Bergman projections

by

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Abstract. We apply the Forelli–Rudin construction to study the Sobolev spaces of holomorphic functions on some smooth Hartogs domains and the regularity of weighted Bergman projections on weakly regular pseudoconvex domains. We also study the kernels of weighted Bergman projections on strictly pseudoconvex domains.

1. Introduction. The main idea of the Forelli–Rudin construction [9] is to imbed the unit ball B_n of C^n into the unit ball B_{n+m} of C^{n+m} via $i(z) = (z, 0)$ and use the reproducing property of the Bergman kernel of B_{n+m} to obtain a new reproducing kernel on B_n, namely

$$c(n, m)\frac{(1-|t|^2)^m}{(1-\langle z, t\rangle)^{n+m+1}}.$$

In the case of the unit ball, this construction plus extension to an analytic family of operators leads to the family of reproducing kernels $c(n, s)\frac{(1-|t|^2)^s}{(1-\langle z, t\rangle)^{n+s+1}}$, $\operatorname{Re} s > -1$. Note that $\frac{c(n, s)}{(1+\langle z, t\rangle)^{n+s+1}}$ is the weighted Bergman kernel with respect to the weight $(1-|t|^2)^s$. (See Rudin's book [22] for further details.) In the present paper we apply the same method to the large class of Hartogs domains in order to study either the weighted Bergman kernels and projections or the Sobolev–Besov spaces of holomorphic functions on some Hartogs domains. We have already used this approach in [14]. The present work is thus a continuation of [14].

2. Bergman kernels of Hartogs domains. Let D be a bounded domain in C^n and φ a bounded positive continuous function on D. We define the Hartogs domain D_φ^m in C^{n+m} as $D_\varphi^m = \{(z, w) \in C^{n+m};\ z \in D,\ |w| < \varphi(z)\}$. Each holomorphic function f on D_φ^m is the sum of a locally uniformly convergent series $f(z, w) = \sum_{|\alpha|=0}^{\infty} f_\alpha(z) w^\alpha$, where the f_α are holomorphic on D. Consider the space $L^2\mathrm{Hol}(D_\varphi^m)$ of square-integrable holomorphic functions on D_φ^m. The following fact is basic for our study.

PROPOSITION 0. (a) *Let* $f \in L^2\mathrm{Hol}(D^m_\varphi)$. *Then*

$$f(z, w) = \sum_{|\alpha|=0}^{\infty} f_\alpha(z) w^\alpha,$$

where for every α, $f_\alpha \in L^2\mathrm{Hol}(D, g_\alpha)$, *the space of functions holomorphic on* D, *square-integrable with respect to the measure* $g_\alpha dV_{2n}$, *and*

$$g_\alpha(z) = \int_{B(0,\varphi(z))} |w_1|^{2\alpha_1} \dots |w_m|^{2\alpha_m} dV_{2m} = c_\alpha \varphi^{2|\alpha|+2m}(z).$$

($B(0, \varphi(z))$ *is the ball in* C^m, dV_{2n} *and* dV_{2m} *are the Lebesgue volume elements in* C^n *and* C^m *respectively.) Moreover,*

$$\|f(z, w)\|^2_{L^2(D^m_\varphi)} = \sum_{|\alpha|=0}^{\infty} \|f_\alpha\|^2_{L^2(D, g_\alpha)}.$$

(b) *Let* K^m_φ *be the Bergman kernel function of* D^m_φ *and let* K_α *be the reproducing kernel for* $L^2\mathrm{Hol}(D, g_\alpha)$, *the weighted Bergman kernel associated to the weight* g_α. *Then*

$$(*) \qquad K^m_\varphi[(z, w), (t, s)] = \sum_{|\alpha|=0}^{\infty} w^\alpha K_\alpha(z, t)\bar{s}^\alpha,$$

$$z, t \in D;\ w, s \in C^m,\ |w| < \varphi(z),\ |s| < \varphi(t).$$

Proposition 0 is of course well known and quite easy to prove. In order to check it one needs only to consider the sequence of domains $\Omega_k = D^m_{k,\varphi-\varepsilon_k}$, where $D_k \Subset D_{k+1} \Subset D$, $\bigcup_{k=1}^{\infty} D_k = D$, $\varepsilon_k \downarrow 0$, $\varphi - \varepsilon_k > 0$ on D_k. For each k we have

$$\|f(z, w)\|^2_{L^2(\Omega_k)} = \sum_{|\alpha|=0}^{\infty} \|f_\alpha(z) w^\alpha\|^2_{L^2(\Omega_k)}$$

since $f_\alpha(z)w^\alpha \perp f_\beta(z)w^\beta$ if $\alpha \neq \beta$, $\Omega_k \Subset D^m_\varphi$ and the series $\sum f_\alpha(z)w^\alpha$ converges uniformly on Ω_k. Thus $f_\alpha(z)w^\alpha \in L^2(D^m_\varphi)$ for every α and (a) is true. This implies that the right side of $(*)$ is a well-defined reproducing kernel for D^m_φ and hence must be equal to the Bergman kernel of D^m_φ.

If $m = 1$ then $(*)$ has the following form:

$$K^1_\varphi[(z, w), (t, s)] = \sum_{l=0}^{\infty} w^l K_l(z, t)\bar{s}^l \cdot (2l+2), \qquad z, t \in D;\ w, s \in C,$$

where $K_l(z, t)$ is the weighted Bergman kernel on D associated to the weight $\varphi^{2(l+1)}$.

Proposition 0 yields the following

COROLLARY 2.1. *Let* D *be a bounded domain in* C^n *and let* φ *be a positive continuous function on* D, $\psi = \sqrt{\varphi}$. *The restriction of the Bergman kernel function of* D^m_ψ *to the subspace* $\{w = 0\}$ *is equal to the weighted Bergman kernel on* D *associated to the weight* $\varphi^m/2m$.

Note that $D^m_\psi = \{(z, w)\};\ |w|^2 - \varphi < 0\}$. In the case of the unit ball D in C^n and $\varphi = 1 - |z|^2$ we get exactly the Forelli–Rudin construction.

3. The factorization of weighted Bergman kernels on strictly pseudoconvex domains. Let D be a bounded smooth strictly pseudoconvex domain and let ϱ be a defining function for D. (This means that $\varrho \in C^\infty(C^n)$, $D = \{z;\ \varrho(z) < 0\}$, $\mathrm{grad}\,\varrho \neq 0$ on ∂D.) We shall assume that ϱ is strictly plurisubharmonic in some neighborhood of $\bar{D}$. Put $\varphi(z) = \sqrt{-\varrho(z)}$. For every integer $k > 0$ we shall consider the spaces

$$A_k = L^2\mathrm{Hol}(D, |\varrho|^{k/2}), \qquad B_k = L^2\mathrm{Hol}(D, |\varrho|^{k+n+1}).$$

Let $L_k(z, t)$ denote the reproducing kernel for the space B_k.

We now use Corollary 2.1 and a deep result of Boutet de Monvel and Sjöstrand to prove the following factorization theorem:

3.1. THEOREM. *For every* k,

$$L_k(z, t) = h_{0,k}(z, t) + h_{1,k}(z, t) \cdot h_{2,k}(z, t),$$

where the $h_{i,k}$ *are holomorphic in* z *and there exists a constant* $c(k)$ *such that*

$$\|h_i(\cdot, t)\|_{A_k} \leqslant \frac{c(k)}{|\varrho(t)|^{(2n+k)/4+1}}, \qquad i = 0, 1, 2.$$

Proof. Consider the domain D^m_φ for $m = k+n+1$. By Corollary 2.1 the kernel $L_k(z, t)$ is equal to $K^m_\varphi[(z, w), (t, s)]|_{w=0, s=0}$. (Recall that K^m_φ is the Bergman kernel function of D^m_φ.) The domain D^m_φ is a smooth bounded strictly pseudoconvex domain in C^{n+m} with defining function $\tilde{\varrho}(z, w) = \varrho(z) + |w|^2$, $z \in C^n$, $w \in C^m$. The function $\tilde{\varrho}$ is strictly plurisubharmonic on some neighborhood of D^m_φ.

The Boutet de Monvel–Sjöstrand theorem [5] implies that

$$K^m_\varphi[(z, w), (t, s)] = K^m_\varphi(u, v)$$

$$= \frac{F(u, v) + \psi(u, v) G(u, v) \ln \psi(u, v)}{\psi(u, v)^{n+m+1}}$$

$$= \frac{\Phi(u, v)}{[\psi(u, v)^{k/2+n+1}]^2}, \qquad u = (z, w),\ v = (t, s).$$

In the above formula the functions F, G, ψ are in $C^\infty(\bar{D}^m_\varphi \times \bar{D}^m_\varphi)$ and the *phase function* $\psi(u, v)$ has the following properties:

(a) $\psi(u, u) = -(1/i)\tilde{\varrho}(u)$.

(b) $\bar{\partial}_u \psi$ and $\partial_v \psi$ vanish to an infinite order along the diagonal in $\bar{D}^m_\varphi \times \bar{D}^m_\varphi$.

(c) $|\psi(u, v)| \geqslant c(|u-v|^2 - \tilde{\varrho}(u) - \tilde{\varrho}(v))$.

These properties of ψ yield that $\bar{\partial}_u(1/\psi^{k/2+n+1}) \in C^\infty(\bar{D}^m_\varphi \times \bar{D}^m_\varphi)$ and $\bar{\partial}_u(\Phi/\psi^{k/2+n+1}) \in C^\infty(\bar{D}^m_\varphi \times \bar{D}^m_\varphi)$. By Kohn's now classical estimates on the $\bar{\partial}$-problem (see [8]) there exist $g_1 \in C^\infty(\bar{D}^m_\varphi \times \bar{D}^m_\varphi)$ and $g_2 \in C^\infty(\bar{D}^m_\varphi \times \bar{D}^m_\varphi)$ such that

$$\tilde{h}_1 = \frac{\Phi}{\psi^{k/2+n+1}} - g_1 \quad \text{and} \quad \tilde{h}_2 = \frac{1}{\psi^{k/2+n+1}} - g_2$$

are holomorphic in u. We have

$$\begin{aligned} K^m_\varphi &= (\tilde{h}_1 + g_1)(\tilde{h}_2 + g_2) \\ &= \tilde{h}_1\tilde{h}_2 + \left(\frac{\Phi}{\psi^{k/2+n+1}} - g_1\right)g_2 + \left(\frac{1}{\psi^{k/2+n+1}} - g_2\right)g_1 + g_1g_2 \\ &= \tilde{h}_1\tilde{h}_2 + \tilde{h}_0. \end{aligned}$$

$\tilde{h}_0$ is holomorphic in u since K^m_φ, $\tilde{h}_1$ and $\tilde{h}_2$ are. We have

$$\tilde{h}_0 = \frac{\Phi g_2 + g_1}{\psi^{k/2+n+1}} - g_1 g_2.$$

It is now clear that there exists a constant c such that $|\tilde{h}_i| \leqslant c/|\psi|^{k/2+n+1}$ on $D^m_\varphi \times D^m_\varphi$, $i = 0, 1, 2$. Let

$$h_i(z, t) = \tilde{h}_i[(z, 0), (t, 0)].$$

Then

$$\begin{aligned} |h_i(z, t)| &\leqslant \frac{c}{|\psi[(z, 0), (t, 0)]|^{k/2+n+1}} \\ &\leqslant \frac{c'}{(|z-t|^2 - \varrho(t) - \varrho(z))^{k/2+n+1}}, \quad i = 0, 1, 2. \end{aligned}$$

Thus we get the following estimate:

$$\begin{aligned} \|h_i\|^2_{A_k} &= \int_D |h_i(z, t)|^2 |\varrho(z)|^{k/2} dV_z \leqslant c' \int_D \frac{|\varrho(z)|^{k/2} dV_z}{(|z-t|^2 - \varrho(z) - \varrho(t))^{k+2n+2}} \\ &\leqslant c'' \int_D \frac{dV_z}{(|z-t|^2 - \varrho(t))^{k/2+2n+2}} \leqslant \frac{c_2}{|\varrho(t)|^{k/2+n+2}}, \quad i = 0, 1, 2. \end{aligned}$$

Our theorem is thus proved.

The above theorem is an answer to a question posed to me by J. Peetre. It could be useful in the theory of weighted Hankel forms developed by Janson–Rochberg–Peetre [10], since the recent result of J. Peetre [21] shows that the factorization of weights is a sufficient condition for the weighted Hankel form theory to be valid.

4. The Szegö kernel in terms of weighted Bergman kernels. We now return to the more general Hartogs domains.

Let D be a bounded Lipschitz domain and let φ be a continuous function on $\bar{D}$, $\varphi > 0$ on D, $\varphi \equiv 0$ on ∂D. We also assume that φ is of class C^1 on $D \backslash E$, for some closed set E of measure 0, and that $\int_D |\operatorname{grad} \varphi| < \infty$.

Consider now as before the domain

$$D^m_\varphi = \{(z, w) \in C^n \otimes C^m;\ z \in D,\ |w| < \varphi(z)\}.$$

We define the Hardy space $H^2(D^m_\varphi)$ as the space of all holomorphic functions f on D^m_φ for which

$$\sup_{U \Subset D} \sup_{\varepsilon < \inf_{z \in U} \varphi} \int_{\substack{|w| = \varphi(z) - \varepsilon \\ z \in U}} |f|^2 < \infty.$$

If $f \in H^2(D^m_\varphi)$ then f has nontangential boundary values almost everywhere on ∂D^m_φ and

$$\|f\|^2_{H^2} = \int_{\partial D^m_\varphi} |f|^2 \, d\sigma.$$

Let $S[(z, w), (t, s)]$ be the Szegö kernel for $H^2(D^m_\varphi)$. We have the following

4.1. Proposition. *(a) Let* $f \in H^2(D^m_\varphi)$. *Then*

$$f(z, w) = \sum_{|\alpha|=0}^{\infty} f_\alpha(z) w^\alpha,$$

where

$$f_\alpha \in L^2 \operatorname{Hol}(D, \omega^\alpha), \quad \omega^\alpha = c_\alpha \varphi^{2\alpha+2m-2} \sqrt{\varphi^2 + |\nabla \varphi^2|^2/4},$$

and

$$\|f\|^2_{H^2(D^m_\varphi)} = \sum_{|\alpha|=0}^{\infty} \|f_\alpha\|^2_{L^2\operatorname{Hol}(D, w_\alpha)}.$$

(b) *We have*

$$S[(z, w), (t, s)] = \sum_{|\alpha|=0}^{\infty} w^\alpha K_\alpha(z, t) \bar{s}^\alpha,$$

where $K_\alpha(z, t)$ *is the weighted Bergman kernel associated to the weight* ω_α.

Proof. The proposition can be proved in the same manner as Proposition 0. It suffices to take the sequence of domains $\ldots D_k \Subset D_{k+1} \Subset D$, $\bigcup_{k=0}^{\infty} D_k = D$, and the sequence of surfaces $V_k = \{(z, w);\ z \in D_k, |w| = \varphi(z) - \varepsilon_k\}$, where $\varepsilon_k \searrow 0$ and $\varepsilon_k < \inf_{D_k} \varphi$. The rest of the proof is the same except that we must consider surface integrals over V_k instead of integrals over the domains Ω_k.

We now produce a very simple example, and show how the above proposition can be used to calculate the Szegö kernel of the l^1-ball in C^2, i.e., the set $\{|z_1| + |z_2| < 1\}$. We shall not need this in the sequel, but we think it is amusing.

We must first state Bell's transformation rule for proper holomorphic mappings.

Let Ω and D be two domains in C^n and let H be a proper holomorphic mapping from Ω onto D with finite multiplicity p. Let $\omega > 0$ be a weight function on D, and let K_ω denote the corresponding weighted Bergman kernel. Put $g = \omega \circ H$ and denote by K_g the corresponding weighted Bergman kernel on Ω. Let $z \in \Omega$, $\omega \in D$, $h^{-1}(w) = \{t_1, \ldots, t_p\} \subset \Omega$. We shall denote by JH the Jacobian of H. Then

$$p^{-1}\sum_i \overline{JH(t_i)^{-1}}\, K_g(z, t_i) = JH(z) K_\omega\big(H(z), t\big)$$

for all $z \in \Omega$ and $t \in D \setminus H(\{JH = 0\})$.

The above formula was proved by S. Bell [1] in the case $\omega \equiv 1$ (the Bergman kernel). The proof in our case is the same.

4.2. EXAMPLE. Let D be the unit disc Δ, $\varphi = 1-|z|$, $m = 1$. Then D^1_φ is the l^1-ball $|z|+|w| < 1$. Proposition 4.1 implies that the Szegö kernel S of D^1_φ can be expressed as

$$S[(z, w), (t, s)] = \frac{1}{2\pi\sqrt{2}} \sum_{p=0}^{\infty} w^p K_p(z, t)\bar{s}^p,$$

where $K_p(z, t)$ is the weighted Bergman kernel on the unit disc Δ, taken with respect to the weight $(1-|z|)^{2p+1}$. The Bell transformation formula yields that

$$K_p(z, t) = \frac{p+1}{2\pi}\cdot\frac{1}{\sqrt{z}\sqrt{\bar t}}\left(\frac{1}{(1-\sqrt{z}\sqrt{\bar t})^{2p+3}} - \frac{1}{(1+\sqrt{z}\sqrt{\bar t})^{2p+3}}\right).$$

This can easily be checked, by taking $\Omega = D = \Delta$, $H(z) = z^2$, $\omega = (1-|z|)^{2p+1}$. Then $g = (1-|z|^2)^{2p+1}$ and the reproducing kernel for g is equal to $\dfrac{2(p+1)}{\pi}\,\dfrac{1}{(1-\zeta\bar\tau)^{2p+3}}$. Thus the Szegö kernel $S[(z, w), (t, s)]$ is equal to

$$\frac{1}{\sqrt{2}\pi^2\sqrt{z}\sqrt{\bar t}}\cdot\frac{1}{(1-\sqrt{z}\sqrt{\bar t})^3}\left[\sum_{p=0}^{\infty}(p+1)\left(\frac{w\bar s}{(1-\sqrt{z}\sqrt{\bar t})^2}\right)^p - \sum_{p=0}^{\infty}(p+1)\left(\frac{w\bar s}{(1+\sqrt{z}\sqrt{\bar t})^2}\right)^p\right].$$

It is quite easy to calculate both sums.

As a final result we get

$$S[(z, w), (t, s)] = \frac{1}{2\sqrt{2}\pi^2}\,\frac{3-2z\bar t-2w\bar s-z^2\bar t^2-w^2\bar s^2+z\bar t w\bar s}{(1-2z\bar t-2w\bar s+z\bar t^2+w\bar s^2)^2}.$$

It is clear that the same method can be used to find the Szegö kernel for the l^1-ball $|z_1|+\ldots+|z_n| < 1$ in C^n, but the calculations will be more lengthy and tedious.

4.3. Remark. The method of expressing reproducing kernels as a sum of (weighted) reproducing kernels over a domain in lower dimension was already used by J. Peetre [20] in a study of reproducing kernels of Siegel domains of type II and by N. Jewell [11] in a study of Hardy spaces.

4.4. Remark. Results analogous to Propositions 0 and 4.1 can be proved for any weighted reproducing kernel on D^m_φ, taken with respect to a weight function $\omega(z, |w|)$.

5. The Sobolev spaces of holomorphic functions on some smooth Hartogs domains. Let D be a C^∞ smooth bounded domain in C^n. We shall denote by $\mathrm{Hol}^s_p(D)$ the *Sobolev–Besov spaces* of holomorphic functions on D.

It follows from [15]–[19] that

$$\mathrm{Hol}^s_p(D) = B^s_{pp}(D)\cap \mathrm{Hol}(D) = W^s_p(D)\cap\mathrm{Hol}(D) \quad \text{if } 1 \leqslant p < \infty,\ s \in R,$$

$$\mathrm{Hol}^s_\infty(D) = B^s_{\infty\infty}(D)\cap\mathrm{Hol}(D) = \begin{cases} \Lambda_s(D)\cap\mathrm{Hol}(D), & s > 0,\\ L^\infty(D, |\varrho|^{-s})\cap\mathrm{Hol}(D), & s < 0,\\ \{f\in\mathrm{Hol}(D):\ \sup_{z\in D}|\varrho(z)\operatorname{grad} f(z)| < \infty\}, & s = 0.\end{cases}$$

(Here ϱ denotes the defining function for D as in (3).) It was proved in [16]–[19] that if $1 < p < \infty$, $s < 1/p$, or $p = 1$, $0 \leqslant s \leqslant 1$, then $\mathrm{Hol}^s_p(D) = L^p(D, |\varrho|^{-sp})\cap\mathrm{Hol}(D)$ and that $\mathrm{Hol}^s_1(D)$ is equal to the closure $\mathrm{Harm}^s_1(D) = \overline{L^2\mathrm{Harm}(D)}$ in $L^1(D, |\varrho|^{-s})$ if $s < 0$.

We now consider the smooth Hartogs domains of the following type: Let D be as above and let ϱ be a defining function for D. Let $\varphi_k = |\varrho|^{1/2k}$. We shall study the Sobolev–Besov spaces of holomorphic functions on the domains $D^m_k = D^m_{\varphi_k}$. Note that each D^m_k is a smooth domain, since its defining function is equal to $\varrho(z)+|w|^{2k}$, $z\in C^n$, $w \in C^m$.

5.1. THEOREM. *Let f belong to $\mathrm{Hol}^s_p(D^m_k)$. The function f can be expressed as*

$$f(z, w) = \sum_{|\alpha|=0}^{\infty} f_\alpha(z) w^\alpha,$$

where $f\in\mathrm{Hol}^{s-|\alpha|/2k-m/kp}_p(D)$. The mapping R_α: $f\mapsto f_\alpha$ is continuous from $\mathrm{Hol}^s_p(D^m_k)$ into $\mathrm{Hol}^{s-|\alpha|/2k-m/kp}_p(D)$.

Each mapping R_α is onto because of the following

5.2. PROPOSITION. *The function $f_\alpha(z)w^\alpha$ belongs to $\mathrm{Hol}^s_p(D^m_k)$ iff $f_\alpha(z)\in\mathrm{Hol}^{s-|\alpha|/2k-m/kp}_p(D)$. The mapping E_α: $f_\alpha\mapsto f_\alpha w^\alpha$ is continuous from $\mathrm{Hol}^{s-|\alpha|/2k-m/kp}_p(D)$ into $\mathrm{Hol}^s_p(D)$.*

Proof of Theorem 5.1 and Proposition 5.2. Let $s < 1/p$. In this case $\mathrm{Hol}_p^s(D_k^m) = L^p(D_k^m, |\varrho+|w|^{2k}|^{-ps}) \cap \mathrm{Hol}(D_k^m)$. Thus

$$\|f\|_p^s \simeq \sup|\langle f, \varphi\rangle_0|, \quad q = \frac{p}{p-1},$$

where the supremum is over $\varphi \in L^q(D_k^m, |\varrho(z)+|w|^{2k}|^{qs})$ with $\|\varphi\| \leqslant 1$.

Now observe that the mapping $u(z) \mapsto u(z)w^\alpha(\varrho(z)-|w|^{2k})^{-s}$ is an isomorphic imbedding of $L^q(D, |\varrho|^{|\alpha|q/2k+m/k})$ into $L^q(D_k^m, |\varrho(z)+|w|^{2k}|^{qs})$. This implies that

$$\|f\|_p^s \gtrsim \sup\Big|\int_{D_k^m} f\bar{u}\bar{w}^\alpha(-\varrho(z)-|w|^{2k})^{-s}\Big|$$
$$= \sup\Big|\int_{D_k^m} f_\alpha \bar{u}|w^\alpha|^2(-\varrho(z)-|w|^{2k})^{-s}\Big|$$
$$= c_{\alpha,s}\sup\Big|\int_D f_\alpha \bar{u}|\varrho|^{(|\alpha|+m)/k-s}\Big|$$
$$= c_{\alpha,s}\sup\Big|\int_D (\bar{u}|\varrho|^{|\alpha|/2k+m/kq})(f_\alpha|\varrho|^{\alpha/2k+m/kp-s})\Big|,$$

where the suprema are over u in the unit ball of $L^q(D, |\varrho|^{|\alpha|q/2k+m/k})$.

The mapping $u \mapsto u|\varrho|^{|\alpha|/2k+m/kq}$ is an isomorphism between $L^q(D, |\varrho|^{|\alpha|q/2k+m/k})$ and $L^q(D)$. Hence

$$\|f\|_p^s \gtrsim \|f_\alpha|\varrho|^{\alpha/2k+m/kp-s}\|_{L^p(D)}$$
$$= \|f_\alpha\|_{L^p(D,|\varrho|^{\alpha p/2k+m/k-sp})} = \|f_\alpha\|_{\mathrm{Hol}_p^{s-|\alpha|/2k-m/kp}(D)}.$$

(Above we have used the simple fact that $L^p(D_k^m, |\varrho+|w|^{2k}|^{\beta p})$ is dual to $L^p(D_k^m, |\varrho+|w|^{2k}|^{-\beta q})$ via the usual L^2-scalar product $\langle\cdot,\cdot\rangle_0$, $1 < p \leqslant \infty$, $\beta \in \mathbf{R}$.)

In order to prove our assertions for $s \geqslant 1/p$ we must consider the derivatives in the z direction. Let $f(z, w) \in \mathrm{Hol}_p^s(D_k^m)$. Then

$$\frac{\partial^\beta}{\partial z^\beta} f = \sum_{|\alpha|=0}^{\infty} \frac{\partial^\beta}{\partial z^\beta} f_\alpha(z) w^\alpha.$$

Thus

$$\frac{\partial^\beta}{\partial z^\beta} f_\alpha(z) \in \mathrm{Hol}_p^{s-|\beta|-|\alpha|/2k-m/kp}(D)$$

provided $|\beta| > s-1/p$. Hence $f_\alpha \in \mathrm{Hol}_p^{s-|\alpha|/2k-m/kp}(D)$ and, by the closed graph theorem, the mapping $f \mapsto f_\alpha$ is continuous.

Proposition 5.2 is already proved for $s < 1/p$. We can now consider the derivatives $D^\beta(f_\alpha w^\alpha)$ for $|\beta| > |\alpha|+s-1/p$ and prove the rest of it.

Let us now consider the Hardy–Sobolev spaces $H_p^l(\partial D)$ of holomorphic functions whose lth derivatives belong to $H^p(\partial D)$ ($l \geqslant 0$ is an integer).

5.3. Theorem. *Let* $f \in H_p^l(\partial D_k^m)$. *Then*

$$f(z, w) = \sum_{|\alpha|=0}^{\infty} f_\alpha(z) w^\alpha,$$

where $f_\alpha \in \mathrm{Hol}_p^{l+1/p-|\alpha|/2k-m/kp}(D)$. *The mapping* $f \mapsto f_\alpha$ *is continuous.*

5.4. Proposition. *The function* $f_\alpha \in \mathrm{Hol}_p^{l+1/p-|\alpha|/2k-m/kp}(D)$ *iff* $f_\alpha w^\alpha \in H_p^l(\partial D_k^m)$. *The mapping* $f_\alpha \mapsto f_\alpha w^\alpha$ *is continuous.*

Proof of Theorem 5.3 and Proposition 5.4. Let us prove our assertions for $l = 0$. Let $\varphi(z, w) = \psi(z)w^\alpha$. We have

$$\int_{\partial D_k^m} |\varphi|^q d\sigma = \int_D |\psi|^q |\varrho|^{(|\alpha|q+2m-2)/2k}\sqrt{|\varrho|^{1/k}+|\nabla\varrho|^2/\varrho^{2-2/k}}$$
$$= \int_D |\psi|^q |\varrho|^{|\alpha|q/2k+m/k-1}\sqrt{|\varrho|^{2-1/k}+|\nabla\varrho|^2}.$$

This implies that $\varphi \in L^q(D_k^m)$ iff $\psi \in L^q(D, |\varrho|^{|\alpha|q/2k+m/k-1})$. Let $f(z, w) \in H_p^0(\partial D_k^m)$ $(= H^p(\partial D_k^m))$. Then, if $1/p+1/q = 1$,

$$\|f(z, w)\|_{H_p^0(\partial D_k^m)} = \sup_\varphi\Big|\int_{\partial D_k^m} f\bar{\varphi}\Big| \gtrsim \sup_\psi\Big|\int_{\partial D_k^m} \bar{\psi} f_\alpha |w^\alpha|^2\Big|$$
$$\gtrsim \sup_\psi\Big|\int_D \bar{\psi} f_\alpha |\varrho|^{|\alpha|/k+m/k-1}\Big|$$
$$= \sup_\psi\Big|\int_D (\bar{\psi}|\varrho|^{|\alpha|/2k+m/kq-1/q})(f_\alpha|\varrho|^{|\alpha|/2k+m/kp-1/p})\Big|$$
$$\simeq \|f_\alpha|\varrho|^{|\alpha|/2k+m/kp-1/p}\|_{L^p(D)} \simeq \|f_\alpha\|_{\mathrm{Hol}_p^{-|\alpha|/2k-m/kp+1/p}(D)},$$

where the suprema are over φ and ψ in the unit balls of $L^q(D_k^m)$ and $L^q(D, |\varrho|^{|\alpha|q/2k+m/k-1})$ respectively.

In order to prove our assertions for $l > 0$, it suffices to consider the derivatives as in the proof of Theorem 5.1 and Proposition 5.2. This follows from the well-known fact that a *holomorphic* function belongs to $H_p^l(\partial D)$ iff its trace on the boundary belongs to $W_p^l(\partial D)$.

5.5. Remark. Theorem 5.1 yields in particular that if $f \in \mathrm{Hol}_p^s(D_k^m)$, then its trace on $D = \{(z, w);\ w = 0\}$ belongs to $\mathrm{Hol}_p^{s-m/kp}(D)$. This shows that the smoothness of the trace of a holomorphic function on a lower-dimensional complex subspace depends on the geometrical shape of the domain (if $p < \infty$).

5.6. Problem. What are the necessary and sufficient conditions on a sequence of functions $f_\alpha \in \mathrm{Hol}_p^{s-|\alpha|/2k-m/kp}(D)$ to ensure that

$$f(z, w) = \sum_{|\alpha|=0}^{\infty} f_\alpha(z) w^\alpha \in \mathrm{Hol}_p^s(D_k^m)\ ?$$

The same question can be asked in the case

$$f_\alpha \in \mathrm{Hol}_p^{l-|\alpha|/2k-m/kp+1/p}(D), \quad f(z, w) = \sum_{|\alpha|=0}^{\infty} f_\alpha w^\alpha \in H_p^l(\partial D_k^m).$$

The above question seems to be a difficult one.

Observe that if $f(z, w) \in H_p^0(\partial D_k^m)$, then $f_\alpha \in \mathrm{Hol}_p^{-|\alpha|/2k-m/kp+1/p}(D)$, and if $f(z, w) \in \mathrm{Hol}_p^{1/p}(D_k^m)$, then f_α also belongs to $\mathrm{Hol}_p^{-|\alpha|/2k-m/kp+1/p}(D)$. However, $H_p^0(\partial D_k^m) \neq \mathrm{Hol}_p^{1/p}(D_k^m)$ if $p \neq 2$. Hence it is exactly the type of convergence of the series $\sum f_\alpha w^\alpha$ which makes these two spaces different.

6. Weighted Bergman projections on weakly regular domains. The class of weakly regular domains was introduced by D. Catlin in [6]. Recently N. Sibony [23] proved that weakly regular domains are in fact pseudoconvex B-regular domains and found various conditions on domains which are equivalent to the weak regularity. We shall use the simplest one as a definition.

6.1. Definition ([22], Th. 2.1). The bounded smooth domain D in C^n is *weakly regular* if there exists an exhaustion function $\sigma \in C(\bar{D}) \cap C^\infty(D)$ such that

$$L_\sigma(z, t) = \sum_{i,j} \frac{\partial^2 \sigma}{\partial z_i \partial \bar{z}_j} t_i \bar{t}_j \geqslant |t|^2, \quad z \in D,\ t \in C^n.$$

Recall that σ is an *exhaustion function* for D if $\sigma \equiv 0$ on ∂D, $\sigma < 0$ on D, $D = \bigcup_{\varepsilon > 0} \{\sigma < -\varepsilon\}$ and $\{\sigma < -\varepsilon_1\} \Subset \{\sigma < -\varepsilon_2\}$ iff $\varepsilon_1 < \varepsilon_2$.

Catlin proved in [6] that if D is weakly regular, then the compact estimate for the $\bar\partial$ Neumann problem holds. This implies in particular that the Bergman projection is bounded from $W_2^s(D)$ into itself, $0 \leqslant s < \infty$. We shall prove in this part of the present paper that weighted Bergman procjections with respect to weights $(-\varrho)^s$, $s > 0$, s rational, where ϱ is some defining function, are also bounded in $W_2^s(D)$ norms.

Let D be weakly regular domain in C^n and let ϱ be a defining function for D such that $-(-\varrho)^{1/k}$ is strictly plurisubharmonic inside D if k is sufficiently large. Diederich and Fornæss proved in [7] that such a defining function exists for every smooth bounded pseudoconvex domain.

Let D, ϱ and k be as above. We have

6.2. Proposition. *For every integer $N > 0$, the domain D_{Nk}^m is weakly regular.*

Proof. In [23], Th. 2.1, it was proved that weak regularity is equivalent to the fact that for each $z_0 \in \partial D$ there exists $u \in C(\bar{D})$, u plurisubharmonic on D, such that $u(z_0) = 1$ and $u < 1$ on $\bar{D} \backslash z_0$. Since the boundary of D_{Nk}^m is defined by $|w|^2 = (-\varrho)^{1/Nk}$, all points of $\partial D_{Nk}^m \backslash \{w = 0\}$ are points of strict pseudoconvexity and have the above property. If $w = 0$ then $z_0 \in \partial D$ and we can take $u(z)$ with a peak point at z_0 since D is weakly regular.

6.3. Theorem. *Let D be a weakly regular domain and let ϱ be as above. The weighted Bergman projection B_s with respect to the weight $|\varrho|^s$, $s > 0$, maps $L^2(D, |\varrho|^{-2r}) = \{\varrho^r L^2(D)\}$ continuously into $W_2^r(D)$ provided s is rational and $-s/2 \leqslant r$.*

6.4. Theorem. *Let D and ϱ be as above and let $s > 0$ be rational. The mapping $h \mapsto B(\varrho^s h)$ is an isomorphism between $\mathrm{Hol}_2^r(D)$ and $\mathrm{Hol}_2^{r+s}(D)$ for every $r \in \boldsymbol{R}$.*

6.5. Theorem. *Let D and ϱ be as above and let $s > 0$ be rational. The mapping B_s maps $W_2^r(D)$ continuously into itself for $r \geqslant -s$.*

Proof of Theorem 6.3. If s is rational we can always find m and N such that $s = m/Nk$. Consider the domain D_{Nk}^m. Let $\tilde{B}$ denote the Bergman projection for D_{Nk}^m and let $\tilde{\varrho} = -\varrho(z) - |w|^{2Nk}$. Then $\tilde{\varrho}$ is a defining function for D_{Nk}^m. It was proved in [16] that if $r' \geqslant 0$, then $f \mapsto \tilde{B}(\tilde{\varrho}^{r'} f)$ maps $L^2(D_{Nk}^m)$ continuously into $W_2^{r'}(D_k^m)$. Let $f(z) \in L^2(D, |\varrho|^{m/Nk})$. Hence $f(z) \in L^2(D_{Nk}^m)$ and the results of Sections 2 and 5 of the present work imply that

$$\begin{aligned}\tilde{B}(\tilde{\varrho}^{r'} f) &= \int_{D_{Nk}^m} K[(z, w), (t, s)]\, f(t)(-\varrho(t) - |s|^{2Nk})^{r'} \\ &= c_{r'} \int_D K_0(z, t)\, f(t)(-\varrho(t))^{r'} |\varrho|^{m/kN} \\ &= B_s(|\varrho|^{r'-m/2kN}(|\varrho|^{m/2kN} f)) \in \mathrm{Hol}_2^{r'-m/2kN}(D).\end{aligned}$$

Put $r = r' - m/2Nk$. Since $f \mapsto |\varrho|^{m/2kN} f$ is an isomorphism between $L^2(D, |\varrho|^{m/kN})$ and $L^2(D)$, we have proved our theorem.

Proof of Theorem 6.4. Bell's construction [2] implies that for every $r \geqslant -s$ and $h \in \mathrm{Hol}_2^{r+s}(D)$ there is $u \in L^2(D)$ such that $h = B(|\varrho|^{r+s} u) = B(|\varrho|^s B_s(|\varrho|^r u))$ (see Boas [4], Bell and Boas [3], Komatsu [13] and also [2], [17]). We have $B_s(|\varrho|^r u) \in \mathrm{Hol}_2^r(D)$ by Theorem 6.3. The mapping $h \mapsto B(|\varrho|^s h)$ thus maps $\mathrm{Hol}_2^r(D)$ *onto* $\mathrm{Hol}_2^{r+s}(D)$ and is continuous (see [17]–[19]). It is also one-to-one and thus must be an isomorphism.

In order to prove our theorem for $r < -s$ it suffices to observe that $h \mapsto B(|\varrho|^s h)$ is a selfadjoint operator on $\mathrm{Hol}_2^0(D)$ and apply the duality between $\mathrm{Hol}_2^s(D)$ and $\mathrm{Hol}_2^{-s}(D)$ (see Bell and Boas [3], Komatsu [13], [16]).

Proof of Theorem 6.5. The negative Sobolev space $W_2^{-s}(D)$, $s > 0$, is defined here, following [25], as the space of restrictions to D of distributions in the negative Sobolev space $W_2^{-s}(C^n)$. Let $\mathring{W}_2^s(D)$ be the space dual to $W_2^{-s}(D)$. The results of [25] yield that $\mathring{W}_2^s$ is equal to the closure of $C_0^\infty(D)$ in $W_2^s(D)$ for all $s \neq l + \frac{1}{2}$, l an integer. We are going to prove that $\varphi \mapsto B(\varrho^s \varphi)$ maps $W_2^{-s}(D)$ continuously into $L^2(D)$. This is equivalent, via duality, to the statement that $h \mapsto \varrho^s h$ maps $\mathrm{Hol}_2^0(D)$ continuously into $\mathring{W}_2^s(D)$. The latter was proved by Boas in [4], and follows easily from considering the analytic family of operators

$h \mapsto \varrho^z h$ and the fact that if $l \geq 0$ is an integer, then the mappings $h \mapsto \varrho^{l+it} h$ form a uniformly bounded family of continuous operators from $\mathrm{Hol}_2^0(D)$ into $\mathring{W}_2^l(D)$. It was proved in [17] and [18] that $\varphi \mapsto B(\varrho^s \varphi)$ maps $W_2^r(D)$ continuously into $W_2^{r+s}(D)$, $r \geq 0$, provided B is regular in the $W_2^{r+s}(D)$ norm. Thus in our case

$$B_s(\varphi) = [B(\varrho^s \cdot)]^{-1}(B(\varrho^s \varphi))$$

is continuous in all $W_2^r(D)$ norms, $r \geq 0$, and in the $W_2^{-s}(D)$ norm. Hence, by interpolation, it must be regular in all $W_2^r(D)$ norms for $r \geq -s$.

6.6. PROBLEM. The assumption that s is rational seems to be quite artificial. The main obstacle to extend our results to all real s is that we do not any longer know how to extend the family of operators B_s, $s \in \mathbf{R}_+$, to an analytic family B_z defined for z from the halfplane $\operatorname{Re} z > -1$.

6.7. Remark. One can ask if it is possible to obtain the estimates in $W_p^s(D)$, $p \neq 2$, or in Hölder norms for Bergman and weighted Bergman projections on weakly regular domains. The recent example of a weakly regular domain with only one point of weak pseudoconvexity for which there are no Hölder or uniform estimates for solutions of the $\bar\partial$ problem suggests that the answer could be negative. This example was constructed by N. Sibony in [24].

7. The kernels of weighted Bergman projections for strictly pseudoconvex domains with boundary of class C^4. Let D be a bounded strictly pseudoconvex domain with C^4-smooth boundary and let ϱ be a defining function of class C^4, strictly plurisubharmonic in a neighborhood of $\bar D$. In this case we are able to write down an explicit formula for the most singular term of $K_s(z, t)$, the kernel of the weighted Bergman projection B_s with weight $|\varrho|^s$, for every real $s > -1$. We shall do it by combining the methods from [15] and the Forelli–Rudin method for the unit ball [22].

Consider the domain D_1^m in $\mathbf{C}^n \times \mathbf{C}^m$ defined as $D_1^m = \{(z, w);\ z \in D, \varrho(z) + |w|^2 < 0\}$; it is strictly pseudoconvex with C^4 boundary. Let

$$F_1(z, t) = \sum_i \frac{\partial \varrho(t)}{\partial t_i}(t_i - z_i) - \frac{1}{2}\sum_{i,j} \frac{\partial^2 \varrho}{\partial t_i \partial t_j}(t)(t_i - z_i)(t_j - z_j).$$

We have

$$\operatorname{Re} F_1(z, t) \geq \frac{\varrho(t)}{2} - \frac{\varrho(z)}{2} + \frac{c}{2}|z-t|^2 \quad \text{if } \varrho(t) < \delta_0 \text{ and } |z-t| < \varepsilon_0.$$

Let $\psi(|z-t|)$ be a cut-off function such that $\psi = 1$ if $|z-t| < \varepsilon_0/4$ and $\psi = 0$ if $|z-t| > \varepsilon_0/2$. Let

$$F(z, t) = \psi(|z-t|)F_1(z, t) + (1 - \psi(|z-t|))M|z-t|^2,$$

where $M > 0$ is a constant such that $(M-1)\varepsilon_0^2/16 - |\varrho(z)| > 1$ on $D_{\delta_0} = \{z;\ \varrho(z) < \delta_0\}$. Let

$$g_i(z, t) = \psi(|z-t|)\left(\frac{\partial \varrho}{\partial t_i}(t) + \frac{1}{2}\sum_j \frac{\partial^2 \varrho}{\partial t_j \partial t_i}(z_i - t_i)\right) + (1 - \psi(|z-t|))M(\bar t_i - \bar z_i).$$

We shall now consider the function Φ on $D_1^m \times D_1^m$ defined by

$$\Phi[(z, w), (t, s)] = F(z, t) + \sum_{i=1}^m \bar s_i(s_i - w_i).$$

Also, set

$$\tilde g_i = \begin{cases} g_i(z, t), & i \leq n, \\ \bar s_i, & n+1 \leq i \leq n+m, \end{cases}$$

and $\tilde\varrho(z, w) = \varrho(z) + |w|^2$. We have

$$\begin{aligned} \operatorname{Re}\Phi[(z, w), (t, s)] - \tilde\varrho(t, s) &= \psi(|z-t|)\big(\operatorname{Re}(F_1(z, t) + \sum_i \bar s_i(s_i - w_i)) - \varrho(t) - |s|^2\big) \\ &\quad + (1 - \psi(|z-t|))\big(\operatorname{Re}(M|z-t|^2 + \sum \bar s_i(s_i - w_i)) - \varrho(t) - |s|^2\big) \\ &\geq \psi(|z-t|)\left(-\frac{\tilde\varrho(z, w) + \tilde\varrho(t, s)}{2} + \frac{c}{2}|z-t|^2 + \frac{1}{2}|w-s|^2\right) \\ &\quad + (1 - \psi(|z-t|))\left(|z-t|^2 + \frac{1}{2}|w-s|^2 - \frac{\varrho(z) + \varrho(t)}{2}\right). \end{aligned}$$

This implies that the form

$$\begin{aligned} N[(z, w), (t, s)] &= c(n, m)\sum_i (-1)^{i-1}\frac{\tilde g_i}{(\Phi - \tilde\varrho(t, s))^{n+m}}\bar\partial_t \tilde g_1 \ldots \widehat{\bar\partial \tilde g_i} \ldots \bar\partial \tilde g_{n+m}\, dt\, ds \end{aligned}$$

is a Cauchy–Fantappiè form for $(t, s) \in \partial D_1^m$ and thus for every holomorphic $f(z, w)$ in $C^1(\bar D_1^m)$,

$$f(z, w) = \int_{D_1^m} f(t, s)\bar\partial_{(t,s)} N((z, w), (t, s)).$$

We have now

$$\bar\partial_{(t,s)} N = \frac{c(n, m)\det\begin{vmatrix} -\varrho(t) - |s|^2 & \tilde g_i \\ \partial\tilde\varrho/\partial\bar t_i & \partial\tilde g_i/\partial\bar t_i \end{vmatrix}}{(\Phi - \tilde\varrho(t, s))^{n+m+1}} = c(n, m)\frac{L(z, t)}{(\Phi - \tilde\varrho(t, s))^{n+m+1}},$$

where

$$L(z,t) = \det\begin{vmatrix} -\varrho(t) & g_i(z,t) \\ \partial\varrho/\partial\bar{t}_i & (\partial g_i/\partial\bar{t}_i)(z,t)\end{vmatrix}.$$

We also have $\Phi - \tilde{\varrho}(t,s) = F(z,t) - \varrho(t) - \langle w, s\rangle$ and hence

$$\bar{\partial}_{(t,s)} N = c(n,m)\frac{L(z,t)}{(F(z,t)-\varrho(t)-\langle w,s\rangle)^{n+m+1}}.$$

If $f(z)$ is a holomorphic function in $C^1(\bar{D})$, then

$$f(z,0) = c(n,m)\int_{D_T^m}\frac{L(z,t)f(t)}{(F(z,t)-\varrho(t))^{n+m+1}}$$
$$= c_1(n,m)\int_D \frac{L(z,t)f(t)(-\varrho(t))^m}{(F(z,t)-\varrho(t))^{n+m+1}}.$$

Hence the kernel

$$F(m)(z,t) = c_1(n,m)\frac{L(z,t)}{(F(z,t)-\varrho(t))^{n+1}}\left(\frac{-\varrho(t)}{F(z,t)-\varrho(t)}\right)^m$$

is the reproducing kernel for functions in $\mathrm{Hol}(D)\cap C^1(\bar{D})$. We can now simply apply the original Forelli–Rudin idea: consider the analytic family of operators F_τ with kernels

$$F(\tau)(z,t) - c(n,\tau)\frac{L(z,t)}{(F(z,t)-\varrho(t))^{n+1}}\left(\frac{-\varrho(t)}{F(z,t)-\varrho(t)}\right)^\tau,$$

$\mathrm{Re}(1+\tau) > 0$. $c(n,\tau)$ is a holomorphic function equal to $c_1(n,m)$ for $\tau = m$, $m = 0, 1, \ldots$ ($c(n,\tau)$ must be in fact equal to the function $c(n,s)$ considered in [22] for the unit ball, since these coefficients do not depend on the domain in question).

It now suffices to repeat word by word the proof of Proposition 7.1.2 from [22] to check that the kernels $F(\tau)(z,t)$ have the reproducing property for functions from $\mathrm{Hol}(D)\cap C^1(\bar{D})$. Thus for $s > 0$, the operator F_s can approximate the projection B_s well. We are now going to check this.

We shall follow the idea of Kerzman and Stein [12]. F_s is an operator acting on functions from $L^2(D, |\varrho|^s)$, which is a Hilbert space with scalar product $\langle f, g\rangle_s = \int f\bar{g}|\varrho|^s$. The kernel of F_s with respect to this product is

$$\psi_s(z,t) = c(n,s)\frac{L(z,t)}{(F(z,t)-\varrho(t))^{n+1+s}}.$$

Let us introduce a correction term $Q_s(z,t)$. The form $\bar{\partial}_z\psi_s(z,t)$ is of class $C^\infty\times C^1$ on $\bar{D}\times\bar{D}$. We define $Q_s(z,t) = -T_z\bar{\partial}_z\psi_s(z,t)$, where T_z is Henkin's operator solving the $\bar{\partial}$ problem on $\bar{D}$. Let $G_s(z,t) = \psi_s(z,t) + Q_s(z,t)$. The kernel $G_s(z,t)$ is holomorphic in z. Moreover, if $f\in\mathrm{Hol}(D)\cap C^1(\bar{D})$ we have

$$\int Q_s(z,t)f(t)|\varrho(t)|^s = -\int T_z\bar{\partial}_z\psi_s(z,t)f(t)|\varrho(t)|^s$$
$$= -T_z\bar{\partial}_z\int\psi_s(z,t)f(t)|\varrho(t)|^s = -T_z\bar{\partial}_z f = 0.$$

The kernel $Q_s(z,t)\in C^1(\bar{D}\times\bar{D})$ and therefore is nonsingular. Let G_s be an integral operator with kernel $G_s(z,t)$ and let $A_s = G^* - G$ be an integral operator with kernel $\overline{G_s(z,t)} - G_s(z,t) = A_s(z,t)$. (The formal adjoint above is considered with respect to $\langle\cdot,\cdot\rangle_s$.)

By using the properties of $F(z,t)$, namely that $|F(z,t) - \varrho(t) - \overline{F(t,z)} + \varrho(z)| \leqslant C|z-t|^3$, we can prove that

$$|A_s(z,t)| \leqslant \frac{|z-t|^\alpha}{|(F(z,t)-\varrho(t))|^{n+s+1}} \leqslant \frac{1}{(-\varrho(z)/2-\varrho(t)/2+C|z-t|^2)^{n+s-\alpha/2+1}},$$

where $\alpha = s - [s]$ and $[s]$ is the integer part of s. This implies that $A_s(z,t)|\varrho(t)|^s$ is a smoothing kernel of order $\alpha/2$. In particular, the operator $A_s f = \int A_s(z,t)f(t)|\varrho(t)|^s$ is compact from $L^2(D, |\varrho|^s)$ into itself. Now it suffices to repeat word by word the considerations from [15] (or [12]) to find that G_s is a continuous projection from $L^2(D, |\varrho|^s)$ onto $L^2\mathrm{Hol}(D, |\varrho|^s)$ and that $B_s = G_s(I-A_s)^{-1} = (I+A_s)^{-1}G_s^*$. Hence

$$B_s = G_s - G_sA(1-A_s)^* = G_s + K,$$

where K is an $\alpha/2$-smoothing operator, $\alpha = s - [s]$.

The description given above of weighted Bergman projections could be used to obtain estimates for B_s in Sobolev and Hölder norms. It can also be used to study weighted Bergman–Toeplitz operators on strictly pseudoconvex domains. We shall deal with these matters in a subsequent paper.

Acknowledgements. The present paper was written during the Special Year on Complex Analysis (1987/1988) in the Mittag-Leffler Institute. The author is very grateful to the Institute and especially to Professor C. O. Kiselman for help and hospitality.

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Received March 15, 1988
Revised version July 4, 1988 (2421)

STUDIA MATHEMATICA, T. XCIV (1989)

A multidimensional Wolff theorem

by

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Abstract. We prove a Wolff theorem for n-tuples of Banach spaces. First we establish a general Wolff theorem for Aronszajn–Gagliardo orbit and coorbit functors and later we specialize this general result to the case of Sparr's spaces. Furthermore, we show an alternative proof for this last situation which works also in the quasi-Banach case. Some remarks on multiparameter scales are given as well.

0. Introduction. The question of extending Wolff's theorem [10] to a multidimensional context appears in [7]. Accordingly, we prove in this paper such a result. In fact, once the "geometry" of the new situation is understood it is easy to adapt directly the proof of the general Wolff theorem in [7] for Aronszajn–Gagliardo orbit and coorbit functors. This is done in Sec. 1, where we also review the basic terminology connected with interpolation of several Banach spaces. Similarly, in Sec. 2 we review the essentials of Sparr's theory of real interpolation of several Banach spaces [8]. In Sec. 3 we then specialize the general result in Sec. 1 to this case. This first calls for realizing the K- and J-functors as suitable coorbits respectively orbits, similarly to the case of Banach couples [6]. However, this is possible, strictly speaking, only in the Banach case. Accordingly, we sketch in Sec. 4 an alternative approach, closer to Wolff's original proof [10] (the argument there really goes back to Stafney [9]), which works also in the quasi-Banach case. At an early stage of our investigation we had thought that our Wolff theorem could be applied in the context of multiparameter scales of interpolation spaces, in a similar way to that in the one parameter case (see [7], Sec. 4), but we soon met unexpected difficulties of geometric nature, which we have not been able to overcome. So perhaps a more refined result might be needed. In Sec. 5 we have included a brief sketch of what kind of applications we had in mind.

1. A general Wolff theorem. We begin by fixing the terminology (cf. [8]).

By a *Banach n-tuple* we mean a family $\bar{A} = \{A_1, \ldots, A_n\}$ consisting of n Banach spaces A_i ($i = 1, \ldots, n$) all continuously embedded in some Hausdorff topological vector space $\mathscr{A}$, in symbols: $A_i \subset \mathscr{A}$.

* Supported in part by Ministerio de Educación y Ciencia (SAB-87-0172).