[8] N. Bourbaki, *Groupes et algèbres de Lie*, Chap. IV, *Groupes de Coxeter et systèmes de Tits*, Hermann, Paris 1968.
[9] M. Bożejko, *Positive definite functions on the free group and the noncommutative Riesz product*, Boll. Un. Mat. Ital. (6) 5-A (1986), 13–21.
[10] —, *Uniformly bounded representations of free groups*, J. Reine Angew. Math. 377 (1987), 170–186.
[11] —, *Positive definite bounded matrices and a characterization of amenable groups*, Proc. Amer. Math. Soc. 95 (1985), 357–360.
[12] M. Bożejko, T. Januszkiewicz and R. Spatzier, *An infinite Coxeter group is not Kazhdan*, J. Operator Theory 19 (1988), 63—68.
[13] I. M. Chiswell, *Abstract length functions in groups*, Math. Proc. Cambridge Philos. Soc. 80 (1976), 451–463.
[14] —, *Length functions and free products of groups*, Proc. London Math. Soc. 43 (1981), 42–58.
[15] M. Culler and J. W. Morgan, *Group actions on R-trees*, to appear.
[16] L. De Michele and A. Figà-Talamanca, *Positive definite functions on free groups*, Amer. J. Math. 102 (1980), 503–509.
[17] P. G. Dixon and S. W. Drury, *Unitary dilations, polynomial identities and the von Neumann inequality*, Math. Proc. Cambridge Philos. Soc. 99 (1986), 115–122.
[18] W. F. Donoghue, Jr., *Monotone Matrix Functions and Analytic Continuation*, Springer, Berlin 1974.
[19] A. Figà-Talamanca and M. A. Picardello, *Harmonic Analysis on Free Groups*, Lecture Notes in Pure and Appl. Math. 87, Dekker, New York 1983.
[20] U. Haagerup, *An example of a non nuclear C^*-algebra, which has the metric approximation property*, Invent. Math. 50 (1979), 279–293.
[21] N. Harrison, *Real length functions in groups*, Trans. Amer. Math. Soc. 174 (1972), 77–106.
[22] E. Hewitt and K. Ross, *Abstract Harmonic Analysis*, Vol. I, Springer, Berlin 1963.
[23] P. Julg and A. Valette, *K-theoretic amenability for $SL_2(Q_p)$, and the action on the associated tree*, J. Funct. Anal. 58 (1984), 194–215.
[24] R. A. Kunze, *Positive definite operator-valued kernels and unitary representations*, in: Functional Analysis, Proc. Conf., Irvine 1966, Thompson and Academic Press, 1967, 235–247.
[25] R. C. Lyndon, *Length functions in groups*, Math. Scand. 12 (1963), 209–234.
[26] J. W. Morgan and P. B. Shalen, *Valuations, trees, and degenerations of hyperbolic structures, I*, Ann. of Math. 120 (1984), 401–476.
[27] B. Sz.-Nagy and C. Foiaş, *Harmonic Analysis of Operators on Hilbert Space*, North-Holland, 1970.
[28] M. A. Picardello, *Positive definite functions and L^p convolution operators on amalgams*, Pacific J. Math., 123 (1) (1986), 209–221.
[29] J.-P. Serre, *Trees*, Springer, Berlin 1980.
[30] J. Tits, *A 'theorem of Lie–Kolchin' for trees*, in: Contributions to Algebra: A Collection of Papers Dedicated to Ellis Kolchin, Academic Press, New York 1977, 377–388.
[31] Y. Watatani, *Property (T) of Kazhdan implies property (FA) of Serre*, Math. Japon. 27 (1981), 97–103.

INSTYTUT MATEMATYCZNY UNIWERSYTETU WROCŁAWSKIEGO
INSTITUTE OF MATHEMATICS, WROCŁAW UNIVERSITY
Pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland

Received November 10, 1986
Revised version October 27, 1988 (2240)

STUDIA MATHEMATICA, T. XCV (1989)

Incomplete normed algebra norms on Banach algebras

by

BOHDAN J. TOMIUK (Ottawa, Ont.) and BERTRAM YOOD (University Park, Pa.)

Abstract. Let A be a semisimple Banach algebra. In various analytic situations one considers (incomplete) normed algebra norms on A and the completions of A in these norms. A study is made of all possible normed algebra norms and completions for classes of semisimple Banach algebras.

1. Introduction. An original impulse for this investigation came from the theory of generalized almost periodic functions. Let $A = \mathrm{AP}(G)$ be the set of all almost periodic functions on a topological group G considered as a Banach algebra under the sup norm, pointwise addition and convolution multiplication. For the classical case $G = \boldsymbol{R}$, the reals, and $1 \leqslant p < \infty$ the Stepanov S^p-almost periodic functions can be considered as the completion of $\mathrm{AP}(\boldsymbol{R})$ in an incomplete normed algebra norm $\|f\|_{S^p}$ on $\mathrm{AP}(\boldsymbol{R})$. Likewise the Besicovitch B^p-almost periodic functions arise in this way. Similarly we may consider completions in the noncommutative situation of $\mathrm{AP}(G)$. The question naturally arises whether there are any interesting properties shared by all possible completions of $\mathrm{AP}(G)$ in all possible normed algebra norms. Of course the same question occurs for the completions of other Banach algebras.

It is easy (see §4) to exhibit a commutative semisimple Banach algebra B with no nonzero idempotent and a normed algebra norm on B where the completion of B contains such an idempotent. For $\mathrm{AP}(G)$, or more generally any semisimple annihilator Banach algebra A, any idempotent in the completion of A in a normed algebra topology must already be in A.

Let $|x|_1$ and $|x|_2$ be two normed algebra norms on a semisimple Banach algebra A. We say that these norms are consistent if $|x_n - x|_1 \to 0$ and $|x_n - y|_2 \to 0$ imply that $x = y$ (where all the elements are in A). In view of the closed graph theorem the uniqueness of the norm theorem [4, Theorem 9, p. 130] can be expressed as saying that any two complete norms $|x|_1$ and $|x|_2$ are consistent.

On the other hand (see §3), it is easy to find two incomplete normed algebra norms on the disc algebra which are not consistent. Nevertheless, for many of the usual examples of semisimple Banach algebras, any two normed algebra norms are consistent. This is the case for C^*-algebras, annihilator algebras, regular commutative Banach algebras and other instances as shown in §3.

The set $N(A)$ of all normed algebra norms on a semisimple Banach algebra A is a partially ordered set where we say that $|x|_2$ follows $|x|_1$ if $|x|_1 \geq |x|_2$ for all x in A. Say a norm $|x|$ is continuous if the embedding of A into its completion A^c in the norm $|x|$ is continuous and that $|x|$ is semisimple if A^c is semisimple. In the "usual" examples mentioned above the set of continuous semisimple norms is cofinal in $N(A)$, but an example shows that such is not the case for all semisimple A.

2. Preliminaries. Consider a normed algebra norm $\|x\|_1$ defined on a semisimple Banach algebra A with given norm $\|x\|$. If $\|x\|_1$ is a complete norm then it is equivalent to the norm $\|x\|$ by the uniqueness of norm theorem [4, Theorem 9, p. 130]. Suppose that $\|x\|_1$ is an incomplete norm and that τ is the embedding of A into the completion A^c of A in the norm $\|x\|_1$. Examples abound in which τ is discontinuous or in which A^c is not semisimple. An unsolved problem is whether A^c semisimple forces τ to be continuous (see [13, p. 43]). We say that the norm $\|x\|_1$ is a *continuous norm* for A if τ is continuous and that $\|x\|_1$ is a *semisimple norm* if A^c is semisimple. For brevity we say that $\|x\|_1$ is a *smooth norm* if it is both continuous and semisimple.

PROPOSITION 2.1. *If $|x|$ is a semisimple norm on A and A is a left ideal in A^c, then $|x|$ is a smooth norm on A.*

Proof. This was pointed out in [10, p. 298]. In this case A is called an *abstract Segal algebra* in A^c.

Let $\mathrm{AP}(G)$ be the Banach algebra of all almost periodic functions on a locally compact group G where convolution $f*g$ is the multiplication and the norm is the sup norm. As in the classical case $G = \boldsymbol{R}$, for each p, $1 \leq p < \infty$, we can define the Stepanov S^p-norm $\|f\|_{S^p}$ (see [7] and [11]). As shown in [11, Theorem 7, p. 129]

$$\|f*g\|_{S^p} \leq \|f\|_{S^p} \|g\|_{S^p}$$

for all $f, g \in \mathrm{AP}(G)$. Thus the S^p-norm is a normed algebra norm. Moreover, by [11, Theorems 7 and 8, pp. 129 and 131], $\|f\|_{S^p} \leq \|f\|$, for all $f \in \mathrm{AP}(G)$, and the completion of $\mathrm{AP}(G)$ in the S^p-norm is semisimple. Hence $\|f\|_{S^p}$ is a smooth norm on $\mathrm{AP}(G)$. The same situation holds for the Besicovitch B^p-norm [11].

For the S^p- or B^p-norm $\|x\|_1$ on $\mathrm{AP}(G)$ we have the inequality $\|f*g\| \leq \|f\|_1 \|g\|$ for $f, g \in \mathrm{AP}(G)$ [11, Theorem 7, p. 129]. Therefore, by [10, Proposition 1.6, p. 299], $\mathrm{AP}(G)$ is an abstract Segal algebra in the completion.

In the Banach algebra $A = \mathrm{AP}(\boldsymbol{R})$, $A^2 \neq A$ inasmuch as every product $f*g$ has an absolutely and uniformly convergent Fourier expansion. Also A^2 is dense in A as A^2 contains every trigonometric polynomial. Thus there is a discontinuous linear functional $\phi(x)$ on A not identically zero where $\phi(A^2) = (0)$. Arguments used in [2, Example 1, p. 597] show that the normed algebra norm $\|f\|_1 = \|f\| + |\phi(f)|$ is not a continuous norm on $\mathrm{AP}(\boldsymbol{R})$. Also $\|f\|_1$ is not a semisimple norm.

Let A be a semisimple Banach algebra. A linear mapping $T: A \to A$ is called a *left multiplier* if $T(xy) = T(x)y$ for all $x, y \in A$. Let $M_l(A)$ be the algebra of all left multipliers on A. Since every left multiplier on A is continuous [9, p. 1072], $M_l(A)$ is a Banach algebra under the usual operator bound norm. For each $a \in A$, let L_a be the operator given by $L_a(x) = ax$, for all $x \in A$. Then $L_a \in M_l(A)$, for all $a \in A$, and the mapping $a \to L_a$ is a norm decreasing algebra isomorphism of A into $M_l(A)$ and embeds A as a left ideal of $M_l(A)$. In fact, let $T \in M_l(A)$ and $a \in A$. Then $T(a) \in A$ and $TL_a(x) = T(a)x = L_{T(a)}(x)$, for all $x \in A$. Hence, by the semisimplicity of A, $TL_a = L_{T(a)}$. Let L_A be the closure in $M_l(A)$ of the image of A by the mapping $a \to L_a$. We call L_A the *left regular representation* of A. In what follows we will identify A as a left ideal of $M_l(A)$ and as a dense left ideal of L_A.

PROPOSITION 2.2. *Let A be a semisimple Banach algebra. Then $\|x\|_1 = \|L_x\|$ is a smooth norm for A. More generally, any subalgebra B of $M_l(A)$ containing A is semisimple.*

Proof. Inasmuch as $\|L_x\| \leq \|x\|$ for x in A, only the last statement requires proof. As noted above we can consider A as a left ideal in $M_l(A)$. Let J be the radical of B. As A is a left ideal of B so is $J \cap A$. Every $x \in J \cap A$ is left quasi-regular in B so has a left quasi-inverse in A. Therefore $J \cap A = (0)$ as A is semisimple, so that also $JA = (0)$. Let $T \in J$. Then $0 = TL_x = L_{T(x)}$ for each x in A. From the semisimplicity of A we see that $T(x) = 0$ for all x in A and so $T = 0$.

Examples in which the norm $\|L_x\|$ is not equivalent to the given norm $\|x\|$ can be readily supplied.

Let A be a semisimple Banach algebra. By an ideal we shall always mean a two-sided ideal unless otherwise specified. For any subset S in A, $l_A(S)$ and $r_A(S)$ will denote, respectively, the left and right annihilators of S in A, and $\mathrm{cl}_A(S)$ will denote the closure of S in A. The socle of A will be denoted by S_A. We call A *modular annihilator* if every modular maximal left (right) ideal of A has a nonzero right (left) annihilator. A semisimple Banach algebra with dense socle is modular annihilator. We call A a *left annihilator algebra* if for each closed right ideal I, $I \neq A$, we have $l_A(I) \neq (0)$. An *annihilator algebra* is one which is both a left and right annihilator algebra.

If K is an ideal of A, then $l_A(K) = r_A(K)$ [16, p. 37]. We denote the common value of $l_A(K) = r_A(K)$ by K^a. A minimal left (right) ideal of A has the form Ae (eA), where e is an idempotent. Such an idempotent is called a *minimal idempotent* of A. If $S_A^a = (0)$ then every nonzero left (right) ideal of A contains a minimal idempotent [16, p. 37].

A normed algebra A is called a *Q-algebra* if the set of quasi-regular elements of A is open in A. As in [15] we say that a normed algebra A is a *permanent Q-algebra* if it is a Q-algebra in all possible normed algebra norms. As noted in [15], every B^*-algebra is a permanent Q-algebra as is any semisimple modular annihilator algebra. A semisimple commutative regular Banach algebra is a permanent Q-algebra by [12, Corollary (3.7.6), p. 176] and [15, Lemma 2.5, p. 375].

3. Consistency and domination of norms. Let $|x|_1$ and $|x|_2$ be two normed algebra norms on an algebra A over the complex field (A is not necessarily complete in either norm). These norms are said to be *consistent* if $|x_n - x|_1 \to 0$ and $|x_n - y|_2 \to 0$ imply that $x = y$ (where all the elements are in A). The *consistency ideal* for these norms is the set Γ of all x in A for which there is a sequence $\{x_n\}$ in A where $|x_n|_1 \to 0$ and $|x_n - x|_2 \to 0$. The same set is obtained if the roles of the two norms are reversed and Γ is an ideal in A closed with respect to the two norms. The two norms are consistent if and only if $\Gamma = (0)$.

Let $\{p_n(x)\}$ be a sequence of polynomials with real coefficients converging uniformly to one on $[\frac{1}{2}, 1]$ and to zero on $[-1, -\frac{1}{2}]$. Consider the disc algebra A and let

$$|f|_1 = \sup\{|f(z)|: z \in [\tfrac{1}{2}, 1]\}, \qquad |f|_2 = \sup\{|f(z)|: z \in [-1, -\tfrac{1}{2}]\}.$$

Then $|1 - p_n|_1 \to 0$ and $|p_n|_2 \to 0$ so that the two norms are not consistent.

If $|x|_1$ and $|x|_2$ are two complete norms for a Banach algebra B with identity 1, then $1 \notin \Gamma$ by [12, Theorem (2.5.6), p. 72]. But in the example just cited, $1 \in \Gamma$. However, if one of the two norms is a Q-norm, this phenomenon cannot occur.

LEMMA 3.1. *Let $|x|_1$ and $|x|_2$ be two normed algebra norms on A one of which is a Q-norm. Then Γ contains no nonzero idempotent.*

Proof. Suppose $|x|_1$ is a Q-norm, e a nonzero idempotent in A, $|x_n|_1 \to 0$ and $|x_n - e|_2 \to 0$. Let $r(x) = \lim_{n\to\infty} |x^n|_2^{1/n}$. Then, as $e - ex_n e$ and $ex_n e$ permute,

$$1 = r(e) \leqslant r(e - ex_n e) + r(ex_n e) \leqslant |e - ex_n e|_2 + r(ex_n e)$$

for each $n = 1, 2, \ldots$ Now $r(ex_n e)$ is the spectral radius of $ex_n e$ in the completion A^c of A in the norm $|x|_2$. Therefore $r(ex_n e)$ is majorized by the spectral radius of $ex_n e$ in A which is, in turn, majorized by $|ex_n e|_1$ as $|x|_1$ is a Q-norm ([15]). Therefore

$$1 \leqslant |e - ex_n e|_2 + |ex_n e|_1 \to 0.$$

This contradiction shows that $e \notin \Gamma$.

LEMMA 3.2. *Let A be a Banach algebra in the norm $\|x\|$ and a normed algebra in the norm $|x|$. The following statements are equivalent.*

(i) *The norms $\|x\|$ and $|x|$ are consistent.*

(ii) *There is a continuous normed algebra norm $|x|_1$ on A where $|x| \geqslant |x|_1$ for all $x \in A$.*

(iii) *There is a continuous normed algebra norm $|x|_1$ on A and $c > 0$ where $|x| \geqslant c|x|_1$ for all $x \in A$.*

Proof. Let Σ be the separating ideal for the embedding mapping of A (in the norm $\|x\|$) into the completion A^c of A in the norm $|x|$. Specifically, Σ is the set of all $w \in A^c$ for which there is a sequence $\{x_n\}$ in A where $\|x_n\| \to 0$ and $|x_n - w| \to 0$. By [6, Theorem 4.6, p. 1101] the mapping $x + \Gamma \to x + \Sigma$ is a continuous isomorphism of A/Γ into A^c/Σ. Now assume (i). Then $\Gamma = (0)$ so that $\alpha(x) = x + \Sigma$ is a continuous mapping of A into A^c/Σ and $|x|_1 = |x + \Sigma|$ is a continuous norm satisfying (ii).

Assume (iii). Let $\|x_n - u\| \to 0$ and $|x_n - v| \to 0$ in A. Then, as $|x|_1$ is a continuous norm, $|x_n - u|_1 \to 0$. But also $|x_n - v|_1 \to 0$ so that $u = v$. Hence the norms $\|x\|$ and $|x|$ are consistent.

Given two norms $|x|_1$ and $|x|_2$ on A we say that $|x|_1$ *majorizes* $|x|_2$ if $|x|_1 \geqslant |x|_2$ for all x in A.

THEOREM 3.3. *Let A be a strongly semisimple Banach algebra with a discrete structure space. Then*

(i) *any two normed algebra norms on A one of which is a Q-norm are consistent,*

(ii) *any normed algebra norm on A majorizes a smooth norm.*

Proof. First of all any primitive ideal of A must be a modular maximal ideal by [17, Theorem 4.4, p. 187]. For each modular maximal ideal M of A, $M^a \neq (0)$ by [17, Theorem 3.14, p. 185]. As A is semisimple M cannot be dense in A in any normed algebra topology so that M is closed in all these topologies.

Let $|x|_1$ and $|x|_2$ be two normed algebra norms on A where $|x|_1$ is a Q-norm. Then A/M is a Q-algebra in the quotient norm $|x + M|_1$ and a normed algebra in $|x + M|_2$.

Let y be in the consistency ideal Γ for the norms $|x|_1$ and $|x|_2$. This puts $y + M$ in the consistency ideal Γ' for the norms $|x + M|_1$ and $|x + M|_2$ on A/M. As A/M is simple, either $\Gamma' = (0)$ or $\Gamma' = A/M$. In the latter case Γ' would contain an identity element contrary to Lemma 3.1. Theorefore $y + M = 0 + M$ and $\Gamma \subset M$. Inasmuch as A is strongly semisimple we see that $\Gamma = (0)$.

Next let $|x|$ be a normed algebra norm on A. We now know that it is consistent with the given norm $\|x\|$. Consider the continuous isomorphism $\alpha(x) = x + \Sigma$ of A into $B = A^c/\Sigma$ described in the proof of Lemma 3.2. Let J be the radical of B, and suppose $\alpha^{-1}(J) \neq (0)$. Then $\alpha^{-1}(J)$ is a nonzero ideal in the strongly semisimple algebra A so it is itself strongly semisimple. As A has a discrete structure space so does $\alpha^{-1}(J)$ by [12, Theorem (2.6.6), p. 79]. Let M_0 be a modular maximal ideal of $\alpha^{-1}(J)$. The annihilator of M_0 in $\alpha^{-1}(J)$ is nonzero. Now let us examine the isomorphic picture in B. Specifically, $\alpha(M_0)$ is a modular maximal ideal in the algebra $\alpha(A) \cap J$ and, by the above analysis, is closed in $\alpha(A)\cap J$. Thus there is a continuous homomorphism σ of $\alpha(A)\cap J$ onto a simple normed algebra W with identity e where, say, $\sigma(x_0)=e$. We can extend σ to be a continuous homomorphism τ of the closure Z of $\alpha(A)\cap J$ in B into the completion of W. Since $Z\subset J$ we see that Z is a radical Banach algebra. Therefore x_0 has a quasi-inverse y_0 in Z and

$$0=\tau(x_0+y_0-x_0y_0)=e+\tau(y_0)-\tau(y_0)=e.$$

This contradiction shows that $\alpha(A)\cap J=(0)$.

Now that we have $\alpha(A)\cap J=(0)$ we may use the quotient norm for B/J to see that $|x|_1=|\alpha(x)+J|$ is a normed algebra norm for A which is a continuous norm majorized by $|x|$. As $\alpha(A)+J$ is dense in B/J we see also that $|x|_1$ is a semisimple norm.

THEOREM 3.4. *In each of the following cases, any two normed algebra norms on A are consistent and any such norm majorizes a smooth norm.*

(a) *A is a semisimple modular annihilator Banach algebra.*

(b) *A is the Banach algebra of all bounded linear operators on a Banach space.*

(c) *A is a strongly semisimple Banach algebra which is a permanent Q-algebra.*

(d) *A is a semisimple commutative regular Banach algebra.*

(e) *A is a C^*-algebra.*

(f) *A is a semisimple Banach algebra with discrete structure space where A/P is finite-dimensional for each primitive ideal P of A.*

Proof. (a) As noted above in §2, every normed algebra norm on A is a Q-norm. Since A is a semisimple modular annihilator algebra, every nonzero ideal contains a nonzero idempotent. Hence, by Lemma 3.1, the consistency ideal for the two norms must be (0) and so the norms are consistent.

Let $|x|$ be a normed algebra norm for A and consider the continuous isomorphism $\alpha(x)=x+\Sigma$ of A into A^c/Σ described in the proof of Lemma 3.2. Let J be the radical of $B=A^c/\Sigma$. Now $\alpha^{-1}(J)$ cannot contain a nonzero idempotent so that $\alpha(A)\cap J=(0)$. Arguing as in the proof of Theorem 3.3 we see that $|x|_1=|\alpha(x)+J|$ is a smooth norm majorized by $|x|$.

(b) These arguments show that the same conclusions are valid for $B(X)$, the Banach algebra of all bounded linear operators on a Banach space X, inasmuch as $B(X)$ is a permanent Q-algebra by [15, Theorem 2.6, p. 375] and every nonzero ideal contains a nonzero idempotent.

(c) That two normed algebra norms on A are consistent follows from [15, Theorem 3.5, p. 379]. We use the notation and reasoning of Theorem 3.3. To see that a normed algebra norm $|x|$ must dominate a smooth norm it is sufficient to see that $\alpha^{-1}(J)=(0)$.

Let $x\in\alpha^{-1}(J)$. Then we have $\lim_{n\to\infty}|(x+\Sigma)^n|^{1/n}=0$. But $|x+\Sigma|$ is a normed algebra norm on A. As A is a permanent Q-algebra it follows that $\lim_{n\to\infty}\|x^n\|^{1/n}=0$ (for the given norm of A). Hence $\alpha^{-1}(J)$ is an ideal of A consisting entirely of quasi-regular elements. Since A is semisimple, $\alpha^{-1}(J)=(0)$.

(d) As noted in §2, a semisimple commutative regular Banach algebra is a permanent Q-algebra. Thus (d) is a special case of (c).

(e) Let $|x|_1$ and $|x|_2$ be two normed algebra norms for A and let $\|x\|$ be the given C^*-algebra norm. By [6, Theorem 5.4, p. 1104], there is a constant $M>0$ where $M\|x\|\leqslant|x|_k$, $k=1,2$, for all $x\in A$. If $|x_n-x|_1\to 0$ and $|x_n-y|_2\to 0$ in A we see that $\|x_n-x\|\to 0$, $\|x_n-y\|\to 0$ so that $x=y$. The norms are then consistent.

Let $|x|$ be any normed algebra norm. In the notation of Theorem 3.3 it follows from the arguments of [6, Theorem 5.4, p. 1104] that the mapping $x\to x+\Sigma$ is a bicontinuous mapping of A (in the norm $\|x\|$) onto A^c/Σ. Therefore $|x|_1=|x+\Sigma|$ is a smooth norm majorized by $|x|$.

(f) Let $|x|_1$ and $|x|_2$ be two normed algebra norms for A and let P be a primitive ideal of A. By [17, Theorem 3.14, p. 185], $P^a\neq(0)$ and then, by [17, Lemma 3.1, p. 182], $P=P^{aa}$. Therefore P is closed in both norm topologies. Suppose $|x_n-x|_1\to 0$ and $|x_n-y|_2\to 0$ in A. In terms of the quotient norms now available $|x_n+P-(x+P)|_1\to 0$ and $|x_n+P-(y+P)|_2\to 0$. But A/P has a unique normed linear space topology. Therefore $x+P=y+P$. As A is semisimple $x=y$. Hence the two norms are consistent. From our hypothesis that A/P is finite-dimensional it follows that P is a modular maximal ideal of A. That any normed algebra norm majorizes a smooth norm now follows from Theorem 3.3.

By algebraic theory, A/P is finite-dimensional for each primitive ideal P if A satisfies a polynomial identity.

For semisimple commutative Banach algebras it is not necessarily the case that a normed algebra norm must majorize a smooth norm as the following example shows.

Let N be the set of positive integers considered as an additive semigroup and let $l_1(N)$ be the discrete semigroup algebra of N in the terminology of [4, p. 9]. Here the norm of $f=f(n)$ in $l_1(N)$ is

$$\|f\|=\sum_{n=1}^{\infty}|f(n)|$$

and the multiplication is convolution. That $l_1(N)$ is semisimple can be seen directly and also follows from more general results [8, Theorem 5.8, p. 82].

For each $r\in N$ set $\alpha(r)=2^{-r!}$. Note that $\alpha(r+s)\leqslant\alpha(r)\alpha(s)$ for every $r,s\in N$. We introduce the normed algebra norm

$$\|f\|_1=\sum_{n=1}^{\infty}|f(n)|\,\alpha(n)$$

on $l_1(N)$. Take the specific element $g \in l_1(N)$ where $g(1) = 1$ and $g(n) = 0$, $n > 1$. Then (see [4, p. 96]) $\lim_{n\to\infty} \|g^n\|_1^{1/n} = 0$. Inasmuch as the completion A^c of $l_1(N)$ in the norm $\|f\|_1$ is commutative, we see that g lies in the radical of A^c. Let $\|f\|_2$ be any normed algebra norm on $l_1(N)$, with $\|f\|_2 \leqslant \|f\|_1$ for all $f \in l_1(N)$. Then $\lim_{n\to\infty} \|g^n\|_2^{1/n} = 0$ so that g is in the radical of the completion of $l_1(N)$ in the norm $\|f\|_2$. Thus $\|f\|_2$ is not a smooth normed algebra norm on $l_1(N)$.

4. On idempotents and socles. Let A_0 be the closed ideal of the disc algebra A consisting of all f in A with $f(0) = 0$. Consider the normed algebra norm $|f| = \sup\{|f(z)|: z \in [\frac{1}{2}, 1]\}$ on A_0. There exists a sequence $p_n(z)$ of polynomials in A_0 converging to one uniformly on $[\frac{1}{2}, 1]$. The semisimple Banach algebra A_0 has no nonzero idempotent but the completion A_0^c of A_0 in the norm $|f|$ has a nonzero idempotent, the limit in A_0^c of the sequence $\{p_n\}$.

In spite of this example we show that, for A in a class of semisimple Banach algebras properly containing all semisimple annihilator Banach algebras, any idempotent in the completion of A in a normed algebra norm must already be in A. A feature here is that for these algebras (as well as for C^*-algebras) the socle S_A of A is an ideal in every completion A^c.

Throughout Section 4, A is a semisimple Banach algebra. Let e be a minimal idempotent of A so that $eAe = \{\lambda e: \lambda \text{ complex}\}$. For each $ex \in eA$ define the linear functional f_{ex} on the Banach space Ae by the rule

$$f_{ex}(ye)\,e = exye \qquad (y \in A).$$

Then f_{ex} is a bounded linear functional on Ae and, by [5, Theorem 13, p. 161], the mapping W_e defined by $W_e(ex) = f_{ex}$ is a continuous one-to-one linear mapping of eA into $(Ae)^*$.

Likewise, for $xe \in Ae$, consider the linear functional g_{xe} defined on eA by

$$g_{xe}(ey)\,e = eyxe \qquad (y \in A).$$

The mapping V_e given by $V_e(xe) = g_{xe}$ is a continuous one-to-one linear mapping of Ae into $(eA)^*$.

Note that the norm of f_{ex} is given by

$$\|f_{ex}\| = \sup\{|f_{ex}(ye)|: \|ye\| \leqslant 1,\ y \in A\}$$
$$= \|e\|^{-1} \sup\{\|exye\|: \|ye\| \leqslant 1,\ y \in A\}$$

and that $\|f_{ex}\|$ defines a normed linear space norm on eA which, for later purposes, we also denote by $\|ex\|'$.

Likewise,

$$\|g_{xe}\| = \|e\|^{-1} \sup\{\|eyxe\|: \|ey\| \leqslant 1,\ y \in A\}$$

and we set $\|g_{xe}\| = \|xe\|''$. Inasmuch as $\|exye\| \leqslant \|e\|\,\|ex\|\,\|ye\|$, we see that

$$(1) \qquad \|ex\|' \leqslant \|ex\|, \qquad \|xe\|'' \leqslant \|xe\|.$$

Also W_e (V_e) is an isometric linear mapping of eA (Ae) in the norm $\|ex\|'$ ($\|xe\|''$) into $(Ae)^*$ ($(eA)^*$).

Definition. We say that the minimal idempotent e is *right full* if there is a constant $K > 0$ so that $\|xe\|'' \geqslant K\|xe\|$, for all $x \in A$, and that e is *left full* if there is a constant $K > 0$ so that $\|ex\|' \geqslant K\|ex\|$, for all $x \in A$. (Right full is the notion of full as given by Barnes [3, p. 174].)

In view of the completeness of eA (Ae) one readily sees the following.

Proposition 4.1. *The following statements about the minimal idempotent e are equivalent:*

(a) *e is left full.*
(b) *$\|ex\|'$ is a complete norm on eA.*
(c) *The range of W_e is closed in $(Ae)^*$.*

Proposition 4.2. *Suppose that e is a minimal idempotent in a left (right) annihilator Banach algebra A. Then the range of W_e (V_e) is all of $(Ae)^*$ ($(eA)^*$) and e is left (right) full.*

Proof. We use the argument of [5, Theorem 10, p. 160]. Suppose that A is a semisimple left annihilator algebra. For each $ex \in eA$ consider the operator L_{ex} on Ae defined by

$$L_{ex}(ye) = exye = f_{ex}(ye)\,e.$$

As shown in [5, p. 160] every finite-dimensional bounded linear operator on Ae is of the form L_{ex} for a suitable $ex \in eA$. Let $g \in (Ae)^*$. The one-dimensional linear operator defined by $ye \to g(ye)\,e$ has the form L_{ex} for a suitable x. Then $g(ye) = f_{ex}(ye)$ for all $ye \in Ae$ so that g is in the range of W_e. That e is left full is a consequence of Proposition 4.1.

Consider the completion A^c of A in some normed algebra norm $\|x\|_1$ for A. Let e be a minimal idempotent of A. If $w \in A$ then $ewe = \lambda e$ for some λ complex so that

$$(2) \qquad \|e\|^{-1}\|ewe\| = \|e\|_1^{-1}\|ewe\|_1.$$

For $z \in A^c$ set, analogously to the notion of $\|ex\|'$ and $\|xe\|''$,

$$\|ez\|_1' = \|e\|_1^{-1} \sup\{\|ezye\|_1: \|ye\|_1 \leqslant 1,\ y \in A^c\},$$
$$\|ze\|_1'' = \|e\|_1^{-1} \sup\{\|eyze\|_1: \|ey\|_1 \leqslant 1,\ y \in A^c\}.$$

For $x \in A$, $\|ex\|'$ and $\|ex\|_1'$ make sense and we wish to compare them as well as $\|xe\|''$ and $\|xe\|_1''$. By a result of Bachelis [1, Theorem 2.1, p. 308] the embedding of A into A^c is automatically continuous when restricted to Ae or eA. Therefore there is a constant $D>0$ so that

$$(3)\qquad \|ex\|_1 \leqslant D\|ex\|, \quad \|xe\|_1 \leqslant D\|xe\|$$

for all x in A. Note that, by (2),

$$\|ex\|_1' = \|e\|^{-1}\sup\{\|exye\|: \|ye\|_1 \leqslant 1,\ y\in A\}.$$

By (3), $\|ye\| \leqslant 1$ implies that $\|ye\|_1 \leqslant D$ so that

$$D\|ex\|_1' \geqslant \|e\|^{-1}\sup\{\|exye\|: \|ye\| \leqslant 1,\ y\in A\}$$

or $D\|ex\|_1' \geqslant \|ex\|'$. Then, using (1) and (3), we get

$$(4)\qquad \|ex\|' \leqslant D\|ex\|_1' \leqslant D\|ex\|_1 \leqslant D^2\|ex\|.$$

THEOREM 4.3. *Suppose that e is a left full minimal idempotent of A. Then $eA = eA^c$ and all four norms $\|ex\|$, $\|ex\|_1$, $\|ex\|'$ and $\|ex\|_1'$ are equivalent on eA.*

Proof. By definition there is a constant $K>0$ so that $\|ex\|' \geqslant K\|ex\|$, $x\in A$. Formula (4) then shows all the norms to be equivalent on eA. Note also that if A^c is semisimple then e is also a left full minimal idempotent of the Banach algebra A^c.

COROLLARY 4.4. *If every minimal idempotent of A is left (right) full, then S_A is a right (left) ideal of A^c.*

Proof. This is immediate from Theorem 4.3. Therefore, by Proposition 4.2, S_A is a right (left) ideal of A^c if A is a left (right) annihilator Banach algebra.

THEOREM 4.5. *Suppose that A has dense socle and every minimal idempotent is both left and right full. Let A^c be the completion of A in the normed algebra norm $|x|$. Then every idempotent in A^c is already in A.*

Proof. First we verify the conclusion in the case where $|x|$ is a smooth norm. By Corollary 4.4, the socle S_A of A is a dense ideal in A^c and $S_A \subset S_{A^c}$. For a minimal idempotent p of A^c, $pS_Ap = \{\lambda p: \lambda \text{ complex}\}$ so that $p\in S_A$. Hence $S_A = S_{A^c}$. As A^c has dense socle and is semisimple, A^c is a modular annihilator algebra and A^c/S_{A^c} is a radical algebra by [16, Theorem 3.4, p. 38]. Hence every idempotent of A^c lies in S_{A^c} and thus in A.

Suppose $|x|$ is not a smooth norm. Let Σ be the separating ideal for the embedding mapping of A, in the norm $\|x\|$, into A^c and set $x' = x+\Sigma$ for $x\in A^c$. As shown in §3, the mapping $x\to x'$ is a continuous isomorphism of A into $B = A^c/\Sigma$. Let J be the radical of B. As in the proof of Theorem 3.4, $|x|_1 = |x'+J|$ is a smooth norm for A.

Let p be an idempotent in A^c. Then $p'+J$ is an idempotent in B/J so that, by the smooth norm case, there is an idempotent q in A such that $q'+J = p'+J$. Since A is a modular annihilator algebra, $q\in S_A$. By Corollary 4.4, pq and qp lie in S_A. Now $(pq)'+J = q'+J = (qp)'+J$. As the mapping $x\to x'+J$ is an isomorphism on A, we see that $pq = qp = q$. Therefore $p-q$ is an idempotent in A^c. Then $p'-q'$ is an idempotent in the radical J so that $p-q\in\Sigma$. However, Σ contains no nonzero idempotent by [13, Theorem 6.16, p. 42]. Therefore $p = q$.

By Proposition 4.2 and [12, Cor. (2.8.16), p. 100], every semisimple annihilator algebra satisfies the hypotheses of Theorem 4.5. There are other Banach algebras which also do so. Let $B(X)$ be the Banach algebra of all bounded linear operators on a Banach space X. Let $F(X)$ denote the closure in $B(X)$ of all $T\in B(X)$ with finite-dimensional range. It follows from [12, Theorem (2.4.18), p. 69] that $F(X)$ is a left annihilator Banach algebra. However, by [12, Theorem (2.8.23), p. 104], $F(X)$ is an annihilator Banach algebra if and only if X is reflexive.

PROPOSITION 4.6. *$F(X)$ satisfies the hypotheses of Theorem 4.5.*

Proof. As noted above $F(X)$ is a left annihilator Banach algebra. Hence, by Proposition 4.2, every minimal idempotent p must be left full. That p is right full was noted by Barnes [3, Example 2.4, p. 176].

There are semisimple Banach algebras with dense socle where no minimal idempotent is left or right full, as can be seen from [3, Example 2.9, p. 177]. However, the next result shows that if A is semisimple with dense socle, then its norm dominates a normed algebra norm $\|x\|_1$ where the completion A^c is semisimple, has dense socle and all its minimal idempotents are left and right full. In what follows the norm of a Banach algebra A will be denoted by $\|\cdot\|_A$.

THEOREM 4.7. *Let A be a semisimple Banach algebra with dense socle. Then there exists a normed algebra norm $\|\cdot\|_1$ on A such that (1) $\|x\|_1 \leqslant \|x\|$ for all $x\in A$ and (2) the completion of A in the norm $\|\cdot\|_1$ is a semisimple Banach algebra in which every minimal idempotent is left and right full.*

Proof. Suppose first that A is topologically simple. Let $I = Ae$, $e=e^2$, be a minimal left ideal in A, and let $B(I)$ be the Banach algebra of all bounded linear operators on I in the uniform topology. For each $a\in A$, let T_a be the operator on I defined by $T_a(xe) = axe$, $x\in A$, and let B be the closure in $B(I)$ of the image of the mapping $a\to T_a$. Since the mapping $a\to T_a$ is a norm reducing algebra isomorphism of A into $B(I)$, we may consider B as the completion of A in the norm $\|\cdot\|_B$. Since (see notation in Section 4) for all $a\in A$,

$$\begin{aligned}\|ea\|_B &= \sup\{\|eabe\|_A: \|be\|_A \leqslant 1,\ b\in A\}\\ &= \|e\|\sup\{|f_{ea}(be)|: \|be\|_A \leqslant 1,\ b\in A\}\\ &\leqslant \|e\|\sup\{|f_{ea}(be)|: \|be\|_B \leqslant 1,\ b\in A\} = \|e\|\,\|ea\|_B',\end{aligned}$$

it follows that e is left full in A in the norm $\|\cdot\|_B$.

Let R be the radical of B. Then either (1) $A\cap R=(0)$ or (2) $A\cap R\neq(0)$. If $A\cap R\neq(0)$ then $A\cap R$ is a nonzero ideal of A and therefore contains a minimal idempotent of A since $S_A^a=(0)$ (see [16, Lemma 3.1, p. 37]). This is impossible as the radical does not contain any nonzero idempotents. Therefore $A\cap R=(0)$. For convenience let $E=B/R$. Then the mapping $x\to x+R$ is a (norm reducing) algebra isomorphism of A onto a dense subset of E. Hence we may identify E with the completion of A in the norm $\|x\|_E=\|x+R\|_B$, $x\in A$. Since $\mathrm{cl}_E(EeE)\supset\mathrm{cl}_A(AeA)=A$ and A is dense in E, we get $\mathrm{cl}_E(EeE)=E$ so that E is a topologically simple, semisimple Banach algebra.

We show next that e is left full in E. We know that e is left full in A in the norm $\|\cdot\|_B$. For $w\in A$, we have (see Section 4)

$$\|e\|_B^{-1}\,\|ewe\|_B=\|e\|_E^{-1}\,\|ewe\|_E. \tag{1}$$

Also

$$\|ex\|_B'\leqslant\|ex\|_B,\qquad \|ex\|_E'\leqslant\|ex\|_E \tag{2}$$

for all $x\in A$. Clearly

$$\|x\|_E\leqslant\|x\|_B \tag{3}$$

for all $x\in A$. Using (1) and (3) and the fact that Ae is dense in Ee and Be, we have, for all $x\in A$,

$$\begin{aligned}\|ex\|_E'&=\|e\|_E^{-1}\sup\{\|exye\|_E:\ \|ye\|_E\leqslant 1,\ y\in A\}\\&=\|e\|_B^{-1}\sup\{\|exye\|_B:\ \|ye\|_E\leqslant 1,\ y\in A\}\\&\geqslant\|e\|_B^{-1}\sup\{\|exye\|_B:\ \|ye\|_B\leqslant 1,\ y\in A\}\end{aligned} \tag{**}$$

or $\|ex\|_E'\geqslant\|ex\|_B'$. Then, using (2) and (3), we get

$$\|ex\|_B'\leqslant\|ex\|_E'\leqslant\|ex\|_E\leqslant\|ex\|_B \tag{4}$$

for all $x\in A$. As e is left full in A in the norm $\|\cdot\|_B$, there is a constant $D>0$ so that $D\|ex\|_B\leqslant\|ex\|_B'$ for all $x\in A$. Hence, by (3) and (4), we obtain

$$D\|ex\|_E\leqslant D\|ex\|_B\leqslant\|ex\|_B'\leqslant\|ex\|_E'\leqslant\|ex\|_E$$

for all $x\in A$. Thus e is left full in A in the norm $\|\cdot\|_E$ and therefore left full in E.

Now let J be the minimal right ideal, $J=eE$. To each $b\in E$, associate the operator U_b on J defined by $U_b(ey)=eyb$, $y\in E$. Then the mapping $b\to U_b$ is a norm reducing algebra isomorphism of E into $B(J)$. (Here we take the composition of mappings in reverse order.) Let E' be the closure in $B(J)$ of the image of the mapping $b\to U_b$. Then we may consider E' as the completion of E in the norm $\|\cdot\|_{E'}$. Since

$$\begin{aligned}\|be\|_{E'}&=\sup\{\|eybe\|_E:\ \|ey\|_E\leqslant 1,\ y\in E\}\\&=\|e\|_E\sup\{|g_{be}(ey)|:\ \|ey\|_E\leqslant 1,\ y\in E\}\\&\leqslant\|e\|_E\sup\{|g_{be}(ey)|:\ \|ey\|_{E'}\leqslant 1,\ y\in E\}=\|e\|_E\|be\|_{E'}'',\end{aligned}$$

for all $b\in E$, it follows that e is right full in E in the norm $\|\cdot\|_{E'}$. Let R' be the radical of E'. As above we can show that $E\cap R'=(0)$ so that the mapping $x\to x+R'$ is a norm reducing algebra isomorphism of E onto a dense subset of E'/R'. Let $\boldsymbol{B}=E'/R'$. Thus we may identify $\boldsymbol{B}$ with the completion of E (and hence of A) in the norm $\|x\|_{\boldsymbol{B}}=\|x+R'\|_{E'}$, $x\in E$. Clearly $\boldsymbol{B}$ is a topologically simple, semisimple Banach algebra and, by Theorem 4.3, e is left full in $\boldsymbol{B}$ since e is left full in E. Now since e is right full in E in the norm $\|\cdot\|_{E'}$, by an argument similar to the one used above to prove that e is left full in E, we can show that e is also right full in $\boldsymbol{B}$. In fact from (**), with an obvious change in notation, we obtain $\|xe\|_{\boldsymbol{B}}''\geqslant\|xe\|_{E'}''$, for all $x\in A$ (eA is dense in $e\boldsymbol{B}$ and eE'). Since e is right full in E in the norm $\|\cdot\|_{E'}$, there exists a constant $D'>0$ such that $D'\|xe\|_{E'}\leqslant\|xe\|_{E'}''$, for all $x\in A$. Hence

$$D'\|xe\|_{\boldsymbol{B}}\leqslant D'\|xe\|_{E'}\leqslant\|xe\|_{E'}''\leqslant\|xe\|_{\boldsymbol{B}}''\leqslant\|xe\|_{\boldsymbol{B}}$$

for all $x\in A$. Thus e is right full in A in the norm $\|\cdot\|_{\boldsymbol{B}}$ and therefore right full in $\boldsymbol{B}$. Therefore, by [3, Remark 2.10, p. 178], every minimal idempotent of $\boldsymbol{B}$ is left and right full in $\boldsymbol{B}$. This completes the proof for the case where A is topologically simple, with $\|x\|_1=\|x\|_{\boldsymbol{B}}$. We observe that $\|a\|_{\boldsymbol{B}}\leqslant\|T_a\|$, for all $a\in A$. We shall use this fact below.

Now suppose that A is any semisimple Banach algebra with dense socle. We lean on the proof of [14, Theorem 6.5, p. 270]. Let $K_1,\ldots,K_n$ be a finite set of different minimal closed ideals in A. For each $j=1,\ldots,n$, let e_j be a minimal idempotent in K_j and set $I_j=Ae_j=K_je_j$. Each K_j is a topologically simple, semisimple Banach algebra with dense socle. Let $y\to T_y^{(j)}$ be the isomorphism of K_j into $B(I_j)$ defined by $T_y^{(j)}(xe_j)=yxe_j$, for all $x\in K_j$. By the proof above each K_j has a completion $\boldsymbol{B}_j$ which is a topologically simple, semisimple Banach algebra in which every minimal idempotent is left and right full; moreover,

$$\|y\|_{\boldsymbol{B}_j}\leqslant\|T_y^{(j)}\|\leqslant\|y\| \tag{5}$$

for all $y\in K_j$. Let $y_j\in K_j$, $j=1,\ldots,n$. By [14, p. 272]

$$\max\{\|T_y^{(j)}\|:\ j=1,\ldots,n\}\leqslant\|y_1+\ldots+y_n\|. \tag{6}$$

Hence, by (5) and (6),

$$\max\{\|y_j\|_{\boldsymbol{B}_j}:\ j=1,\ldots,n\}\leqslant\|y_1+\ldots+y_n\|. \tag{7}$$

Let $\{K_\alpha:\ \alpha\in\Lambda\}$ be the family of all minimal closed ideals in A. By the first part of the proof, for each $\alpha\in\Lambda$, there is a norm reducing algebra isomorphism T_α of K_α onto a dense subalgebra $\boldsymbol{D}_\alpha$ of a topologically simple, semisimple Banach algebra $\boldsymbol{B}_\alpha$ in which every minimal idempotent is left and right full. Let $\boldsymbol{B}$ be the $B(\infty)$-sum of the algebras $\boldsymbol{B}_\alpha$. Then $\boldsymbol{B}$ is a semisimple Banach algebra with dense socle in which every minimal idempotent is left and right full.

Denote the norm in $\boldsymbol{B}$ by $\|\cdot\|_1$. Using inequality (7), we can show by an argument similar to that in [14, p. 273] that there is a norm reducing algebra isomorphism T' of A onto a dense subalgebra of $\boldsymbol{B}$. This completes the proof.

References

[1] G. F. Bachelis, *Homomorphisms of Banach algebras with minimal ideals*, Pacific J. Math. 41 (1972), 307–311.

[2] W. G. Bade and P. C. Curtis, Jr., *Homomorphisms of commutative Banach algebras*, Amer. J. Math. 82 (1960), 589–608.

[3] B. A. Barnes, *Representations of normed algebras with minimal left ideals*, Proc. Edinburgh Math. Soc. (2) 19 (1974), 173–190.

[4] F. F. Bonsall and J. Duncan, *Complete Normed Algebras*, Springer, Berlin 1973.

[5] F. F. Bonsall and A. W. Goldie, *Annihilator algebras*, Proc. London Math. Soc. (3) 4 (1954), 154–167.

[6] S. B. Cleveland, *Homomorphisms of non-commutative $*$-algebras*, Pacific J. Math. 13 (1963), 1097–1109.

[7] H. W. Davis, *Stepanoff and Weyl AP functions on locally compact groups*, Duke Math. J. 34 (1967), 535–548.

[8] E. Hewitt and H. S. Zuckerman, *The l_1-algebra of a commutative semigroup*, Trans. Amer. Math. Soc. 83 (1956), 70–97.

[9] B. E. Johnson and A. M. Sinclair, *Continuity of derivations and a problem of Kaplansky*, Amer. J. Math. 90 (1965), 1067–1073.

[10] M. Leinert, *A contribution to Segal algebras*, Manuscripta Math. 10 (1973), 297–306.

[11] J. W. McCoy, *Generalized almost periodic functions*, Colloq. Math. 22 (1970), 121–133.

[12] C. E. Rickart, *General Theory of Banach Algebras*, Van Nostrand, New York, 1960.

[13] A. M. Sinclair, *Automatic continuity of linear operators*, Cambridge Univ. Press, 1976.

[14] B. J. Tomiuk and B. Yood, *Topological algebras with dense socle*, J. Funct. Anal. 28 (1978), 254–277.

[15] B. Yood, *Homomorphisms on normed algebras*, Pacific J. Math. 8 (1958), 373–381.

[16] —, *Ideals in topological rings*, Canad. J. Math. 16 (1964), 28–45.

[17] —, *Structure spaces of rings and Banach algebras*, Michigan J. Math. 31 (1984), 181–189.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF OTTAWA
Ottawa, Ontario K1N6N5, Canada

DEPARTMENT OF MATHEMATICS
THE PENNSYLVANIA STATE UNIVERSITY
University Park, Pennsylvania 16802, U.S.A.

Received April 28, 1987
Revised version November 29, 1988
(2308)

STUDIA MATHEMATICA, T. XCV (1989)

Factoring the identity operator on a subspace of l_n^∞

by

PIOTR MANKIEWICZ (Warszawa)

Abstract. It is proved that a "random n-dimensional subspace E of l_{2n}^∞" has the property that for every factorization of the identity on E through a Banach space Y, $S_1\colon E\to Y$ and $S_2\colon Y\to E$, one has $\mathrm{bc}(Y)\,\|S_1\|\,\|S_2\| \geqslant cn^{3/2}/\dim Y$, where $c>0$ is a numerical constant.

A. Pełczyński ([4], Prop. 11.1) proved that for every n-dimensional Banach space X_n there is a Banach space Y with basis constant $\mathrm{bc}(Y)=1$, $\dim Y \leqslant n^{3/2}$, and there are operators $S_1\colon X_n\to Y$, $S_2\colon Y\to X_n$, such that $S_2S_1 = \mathrm{Id}_{X_n}$ and $\|S_1\|\,\|S_2\| \leqslant 3$. In this context he asked whether the estimate on the dimension of Y is optimal. Essentially the same proof yields the following more general result:

For every n-dimensional Banach space X_n and every $m \leqslant n\,\mathrm{bc}(X_n)$ there is an m-dimensional Banach space Y_m and operators $S_1\colon X_n\to Y_m$, $S_2\colon Y_m\to Y_n$ such that $S_2S_1=\mathrm{Id}_{X_n}$ and

$$\mathrm{bc}(Y_m)\,\|S_1\|\,\|S_2\| \leqslant 3n\,\mathrm{bc}(X_n)/m. \tag{1}$$

One can ask whether the estimate (1) is optimal. S. J. Szarek ([6], Prop. 5.1), using the technique introduced in [1] and developed in [5], proved that there are real n-dimensional Banach spaces X_n such that for every factorization of identity on X_n through an m-dimensional Banach space one has (in the notation above)

$$\mathrm{bc}(Y_m)\,\|S_1\|\,\|S_2\| \geqslant \frac{cn}{m}\log^{3/2} n, \tag{2}$$

where $c>0$ is a numerical constant. The complex variant of (2) was done in a similar way in [2] by the author. The aim of this note is to show that (both in the real and complex case) the estimate (1) is optimal "up to a multiplicative numerical constant" even if we restrict our interest to the case when X_n is an n-dimensional subspace of l_{2n}^∞ with basis constant of order $\sqrt{n}$. The same argument yields that (1) is optimal "up to a multiplicative numerical constant" for n-dimensional subspaces of l_{2n}^p with basis constant of order $n^{1/2-1/p}$ for $p \geqslant 2$.