STUDIA MATHEMATICA, T. XCV (1989)

Factorization and extrapolation of pairs of weights

by

EUGENIO HERNÁNDEZ (Madrid)

In memory of J. L. Rubio de Francia

Abstract. General factorization results are given for pairs of weights μ and λ for which a positive sublinear operator T is bounded from $L^p(\mu)$ to $L^q(\lambda)$, $1 \leqslant p \leqslant q \leqslant \infty$. These results include the factorization theorem for A_p weights. When T is positive and linear we show the converse, which is a simple consequence of Hölder's inequality. Applications include sufficient conditions for some operators to be bounded from $L^p(\mu)$ to $L^q(\lambda)$. Extrapolation theorems as in [19] are also given in this general context.

1. Introduction. All nonnegative measurable functions μ and λ on $(0,\infty)$ for which the Hardy operator is bounded from $L^p(\mu)$ to $L^q(\lambda)$ are well understood. A characterization is given in [15], where it is also proved that each one of these functions can be factored as a product of two functions satisfying very simple inequalities (see Theorem 3 of [15]).

The characterization of the weights w for which the Hardy–Littlewood maximal operator and certain singular integral operators are bounded on $L^p(w)$ was obtained in [16], [10] and [4]. The problem of factoring these weights was solved by P. Jones in [12]. This result has several interesting applications which are beautifully collected in the monograph [8]. The long proof in [12] was shortened by J. L. Rubio de Francia using his knowledge of vector-valued inequalities. Shortly thereafter a simple proof of the factorization result was obtained in [5] by adapting an induction argument that appeared in [19]. A similar argument has already been used in [6] in order to obtain conditions for the boundedness of integral operators defined by a positive kernel.

All nonnegative measurable functions μ and λ on $(0,\infty)$ for which the Hardy operator is bounded from $L^p(\mu)$ to $L^q(\lambda)$, with $1 \leqslant p \leqslant q \leqslant \infty$, are characterized in [3]. Characterizations for other operators have been obtained in the last few years (see [22] and [23]).

In the second section of this paper we give a general factorization result for weights μ and λ for which an operator T is bounded from $L^p(\mu)$ to $L^q(\lambda)$ at the same time that an operator T' is bounded from $L^{q'}(\lambda^{-q'/q})$ to $L^{p'}(\mu^{-p'/p})$, with

The author was supported by a Fulbright/MEC fellowship and his research was done during a delightful stay at the Mathematical Sciences Research Institute in Berkeley.

$1 \leqslant p \leqslant q \leqslant \infty$; here T and T' are positive sublinear operators. The argument is an adaptation of the one in [5]. This argument has been used for the case $p = q$ in [2] and [20].

In the third section we prove the converse of the factorization result of Section 2 when T is a positive linear operator and T' is its dual. A counterexample is given to show that the converse is not true for positive sublinear operators. The case $p = q$ and $\mu = \lambda$ of our main result in this section appeared in [11], and it was obtained using the interpolation theorem for analytic families of operators proved in [25]. We show that this result can be proved using nothing more than Hölder's inequality even for the case $p \leqslant q$ and $\mu \neq \lambda$.

Some applications of the factorization result are mentioned briefly in the second section. An extensive account of this type of applications can be found in [20] for the case $p = q$. Instead we concentrate in the fourth section on applications of the converse of factorization. We give a simple proof of the results in [3] for the Hardy operator, the results in [1] and we also treat the case of the multidimensional Hardy operator.

The last section is dedicated to extrapolation. The first result of this kind was proved by J. L. Rubio de Francia in [19] using vector-valued inequalities. Constructive proofs have been given in [7] and [11]. Here we use the argument in [11] to prove that weighted inequalities can be extrapolated to include the case $p \leqslant q$. The case $p = q$ of some of these results has already appeared in [20].

2. Factorization. All operators considered here will be applied to (real or complex) functions defined on a measure space (X, dx), and their values will be functions defined on the same measure space. One of these operators, T, will be called *sublinear* if $|T(f+g)| \leqslant |T(f)| + |T(g)|$ and *positive* if $|f| \leqslant g \Rightarrow |T(f)| \leqslant T(g)$, where f and g are functions defined on X that belong to the domain of T.

Given a nonnegative function μ on X, $L^p(\mu)$ will denote the set of all measurable functions defined on X for which $\|f\|_{L^p(\mu)} = (\int_X |f(x)|^p \mu(x)\,dx)^{1/p} < \infty$ and $L^p_+(\mu)$ will denote the set of all nonnegative elements of $L^p(\mu)$.

THEOREM 2.1 (Factorization theorem). *Let T and T' be two positive sublinear operators, let μ and λ be two nonnegative functions on X and $1 < p \leqslant q < \infty$. Suppose that T is bounded from $L^p(\mu)$ to $L^q(\lambda)$ and that T' is bounded from $L^{q'}(\lambda^{-q'/q})$ to $L^{p'}(\mu^{-p'/p})$. Then, given g in $L_+^{(q/p)'}$ with $\|g\|_{L^{(q/p)'}} \leqslant 1$ and $K > \|T\|^{1/p'} + \|T\|^{1/p}$, there exist*

$$u_0 \in L^p_+(\mu), \quad v_0 \in L^p_+(\lambda^{p/q} g), \quad u_1 \in L^{p'}_+(\lambda^{-p'/q} g), \quad v_1 \in L^{p'}_+(\mu^{-p'/p})$$

such that $T(u_0) \leqslant K v_0$, $T'(u_1 g) \leqslant K v_1$ and $\mu = u_0^{-p/p'} v_1$, $\lambda = v_0^{-q/p'} u_1^{q/p}$.

Proof. Define the operator $S_1(f) = [\lambda^{1/q} g^{1/p} T(|f|^{p'} \mu^{-1/p})]^{1/p'}$. The boundedness of T and Hölder's inequality with exponent $q/p \geqslant 1$ imply the boundedness of S_1 on $L^{pp'}$:

$$\int_X |S_1(f)|^{pp'}\,dx = \int_X \lambda^{p/q} g\, |T(|f|^{p'} \mu^{-1/p})|^p\,dx$$

$$\leqslant \Big(\int_X \lambda\, |T(|f|^{p'} \mu^{-1/p})|^q\,dx\Big)^{p/q} \leqslant \|T\|^p \int_X |f|^{pp'}\,dx.$$

The operator $S_2(f) = [\mu^{-1/p} T'(|f|^p g^{1/p} \lambda^{1/q})]^{1/p}$ is bounded on $L^{pp'}$ also. This is a consequence of the boundedness of T' and Hölder's inequality with exponent $p'/q' \geqslant 1$:

$$\int_X |S_2(f)|^{pp'}\,dx = \int_X \mu^{-p'/p} |T'(|f|^p g^{1/p} \lambda^{1/q})|^{p'}\,dx$$

$$\leqslant \|T'\|^{p'} \Big(\int_X |f|^{pq'} g^{q'/p}\,dx\Big)^{p'/q'} \leqslant \|T'\|^{p'} \int_X |f|^{pp'}\,dx.$$

In the last inequality we have used the fact that $(q'/p)(p'/q')' = (q/p)'$.

The operator $S = S_1 + S_2$ is therefore bounded on $L^{pp'}$. Take $K > \|S\|$ and $f \in L^{pp'}_+$. The series $\sum_{n=0}^{\infty} (S^n f)/K^n$ converges to an element $\alpha \in L^{pp'}_+$. Moreover, the subadditivity of S, which is a consequence of the subadditivity of T and T', and the positivity of f imply $S\alpha \leqslant K\alpha$. Since S_1 and S_2 are positive operators we deduce the inequalities $S_1\alpha \leqslant K\alpha$ and $S_2\alpha \leqslant K\alpha$. These two conditions are equivalent to $T(\alpha^{p'} \mu^{-1/p}) \leqslant K\alpha^{p'} \lambda^{-1/q} g^{-1/p}$ and $T'(\alpha^p g^{1/p} \lambda^{1/q}) \leqslant K\alpha^p \mu^{1/p}$. The result follows by taking $u_0 = \alpha^{p'} \mu^{-1/p}$, $v_0 = \alpha^{p'} \lambda^{-1/q} g^{-1/p}$, $u_1 = \alpha^p \lambda^{1/q} g^{-1/p'}$ and $v_1 = \alpha^p \mu^{1/p}$. ∎

For the case $p = q$ in Theorem 2.1 we can take the largest g with norm less than or equal to 1 in L^∞, namely $g \equiv 1$. In this way we obtain Theorem A of [20], which is stated below for future reference:

COROLLARY 2.2. *Let T, T', λ, μ and p be as in Theorem 2.1. Suppose that T is bounded from $L^p(\mu)$ to $L^p(\lambda)$ and that T' is bounded from $L^{p'}(\lambda^{-p'/p})$ to $L^{p'}(\mu^{-p'/p})$. Then given $K > \|T\|^{1/p'} + \|T'\|^{1/p}$, there exist $u_0 \in L^p_+(\mu)$, $v_0 \in L^p_+(\lambda)$, $u_1 \in L^{p'}_+(\lambda^{-p'/p})$ and $v_1 \in L^{p'}_+(\mu^{-p'/p})$ such that $T(u_0) \leqslant K v_0$, $T'(u_1) \leqslant K v_1$ and $\mu = u_0^{1-p} v_1$, $\lambda = v_0^{1-p} u_1$.*

Remark. The proof of Theorem 2.1 shows that if $\mu = \lambda$ in Corollary 2.2 we have $u_0 = v_0$ and $u_1 = v_1$.

EXAMPLES. We give below some examples to which Theorem 2.1 and Corollary 2.2 can be applied. Our description is brief and we give it only to show the type of applications of these results. See [20] for a more detailed account.

1. If $T = T' = M$, where M denotes the Hardy–Littlewood maximal operator in $\mathbf{R}^n$, the remark that follows Corollary 2.2 gives the factorization theorem for A_p weights due to P. Jones ([12]).

2. Let $T = M^+$ where $M^+ f(x) = \sup_{h>0} (1/h) \int_x^{x+h} |f(t)|\,dt$ and $T' = M^-$ where $M^- f(x) = \sup_{h>0} (1/h) \int_{x-h}^{x} |f(t)|\,dt$. The remark that follows Corollary 2.2 gives the factorization of A_p^+ and A_p^- weights (see [23] and [14] for details).

3. Theorem 2.1 can be applied to integral operators with a positive kernel, that is, operators of the form $Tf(x) = \int_Y K(x, y) f(y)\,dy$ with $K(x, y) \geqslant 0$. In this case we take T' to be the dual of T, so that the boundedness of T^* from $L^{q'}(\lambda^{-q'/q})$ to $L^{p'}(\mu^{-p'/p})$ is equivalent to the boundedness of T from $L^p(\mu)$ to $L^q(\lambda)$ by a duality argument. Therefore the weights μ and λ for which T is bounded from $L^p(\mu)$ to $L^q(\lambda)$ have a factorization as in Theorem 2.1. Particular examples of this kind of operators are:

3.1. *Hardy operator*:

$$T_1 f(x) = \int_0^x f(y)\,dy.$$

3.2. *Multidimensional Hardy operator*:

$$T_n f(x_1, \ldots, x_n) = \int_0^{x_1} \ldots \int_0^{x_n} f(y_1, \ldots, y_n)\,dy_n \ldots dy_1$$

and its generalizations to cones in $\boldsymbol{R}^n$ (see [18]).

3.3. *Fractional integral operator*:

$$T_\alpha f(x) = \int_{\boldsymbol{R}^n} f(x-y) |y|^{\alpha - n}\,dy, \quad 0 \leqslant \alpha < n,$$

which is a selfadjoint operator.

There are substitutes for Theorem 2.1 for the cases $p = 1$ and $q = \infty$:

PROPOSITION 2.3. *Let T, T', μ and λ be as in Theorem 2.1.*

(1) *If $1 \leqslant q \leqslant \infty$ and T' is bounded from $L^{q'}(\lambda^{-q'/q})$ to $L^\infty(\mu^{-1})$ (here $f \in L^\infty(\mu^{-1}) \Leftrightarrow \|f\mu^{-1}\|_\infty < \infty$) we have $T'(\lambda^{1/q} g) \leqslant \mu$ for all $g \in L_+^{q'}$ with norm not exceeding* 1.

(2) *If $1 < p \leqslant \infty$ and T is bounded from $L^p(\mu)$ to $L^\infty(\lambda)$ (here $f \in L^\infty(\lambda) \Leftrightarrow \|f\lambda\|_\infty < \infty$) we have $T(g\mu^{-1/p}) \leqslant \lambda^{-1}$ for all $g \in L_+^p$ with norm not exceeding* 1.

Proof. To prove (1) observe that the boundedness of T' implies $|T'(\lambda^{1/q} g)\mu^{-1}| \leqslant \|g\|_{L^{q'}} \leqslant 1$, so that $T'(\lambda^{1/q} g) \leqslant \mu$. To prove (2) observe that the boundedness of T implies $|T(g\mu^{-1/p})\lambda| \leqslant \|g\|_{L^p} \leqslant 1$. From here the result follows. ∎

We finish this section by mentioning that Theorem 2.1 for $p < q$ can be deduced from Corollary 2.2 and the following proposition, which is a consequence of Hölder's inequality:

PROPOSITION 2.4. *Let $1 < p < q < \infty$ and let λ be a nonnegative measurable function on X. Then:*

(1) *$L^q(\lambda)$ is continuously embedded in $L^p(\lambda^{p/q} g)$ for all $g \in L^{(q/p)'}$ with inclusion norm not exceeding the norm of g, and*

(2) *$L^{p'}(\lambda^{-p'/q} g^{-p'/p})$ is continuously embedded in $L^{q'}(\lambda^{-q'/q})$ for all $g \in L^{(q/p)'}$ with inclusion norm not exceeding the norm of g.*

Proof. To prove (1) apply Hölder's inequality with $q/p > 1$. To prove (2) apply Hölder's inequality with $p'/q' > 1$ and use $(q'/p)(p'/q')' = (q/p)'$. ∎

We now show how Corollary 2.2 implies Theorem 2.1. Take $g \in L_+^{(q/p)'}$ with norm not exceeding 1. Since we are assuming that T is bounded from $L^p(\mu)$ to $L^q(\lambda)$, part (1) of Proposition 2.4 implies that T is bounded from $L^p(\mu)$ to $L^p(\lambda^{p/q} g)$. The assumption that T' is bounded from $L^{q'}(\lambda^{-q'/q})$ to $L^{p'}(\mu^{-p'/p})$ together with part (2) of Proposition 2.4 imply that T' is bounded from $L^{p'}(\lambda^{-p'/q} g^{-p'/p})$ to $L^{p'}(\mu^{-p'/p})$. We can now apply Corollary 2.2 to the operators T and T' and to the nonnegative functions μ and $\lambda^{p/q} g$ to obtain nonnegative measurable functions u_0, v_0, u_1 and v_1 with the integrability properties given in Corollary 2.2 such that $T(u_0) \leqslant K v_0$, $T'(u_1) \leqslant K v_1$ and $\mu = u_0^{-p/p'} v_1$, $\lambda^{p/q} g = v_0^{-p/p'} u_1$. Write $\lambda = v_0^{-q/p'} (u_1 g^{-1})^{q/p}$ and let $\tilde{u}_1 = u_1 g^{-1}$, so that u_0, v_0, $\tilde{u}_1$ and v_1 satisfy the conclusion of Theorem 2.1.

3. Converse of factorization.

When the operator T is *linear*, that is, $T(f+g) = T(f) + T(g)$, and *positive* and T' is the dual of T there is a converse of the factorization theorem 2.1:

THEOREM 3.1. *Let T be a positive linear operator, let μ and λ be two nonnegative measurable functions on X and $1 < p < q < \infty$. Suppose that there exist finite positive numbers K_1 and K_2 such that for all g in $L_+^{(q/p)'}$ with $\|g\|_{L^{(q/p)'}} \leqslant 1$ we can find nonnegative measurable functions u_0, v_0, u_1 and v_1 such that $T(u_0) \leqslant K_1 v_0$, $T^*(u_1 g) \leqslant K_2 v_1$ and $\mu = u_0^{-p/p'} v_1$, $\lambda = v_0^{-q/p'} u_1^{q/p}$. Then T is bounded from $L^p(\mu)$ to $L^q(\lambda)$ and T^* is bounded from $L^{q'}(\lambda^{-q'/q})$ to $L^{p'}(\mu^{-p'/p})$ with norms not exceeding $K_1^{1/p'} K_2^{1/p}$.*

Proof. The proof is based on *Hölder's inequality* for positive linear operators:

$$|T(fg)| \leqslant [T(|f|^p)]^{1/p} [T(|g|^{p'})]^{1/p'}, \quad 1 < p < \infty. \tag{3.1}$$

This inequality is proved as the usual Hölder inequality replacing the integral by the operator T, and details are therefore omitted.

It is enough to prove that T is bounded since the statement concerning T^* follows from the previous one by duality. Let $f \in L^p(\mu)$; by duality, there exist $g \in L^{(q/p)'}$ with $\|g\|_{L^{(q/p)'}} \leqslant 1$ such that

$$I = \Big(\int_X |T(f)|^q \lambda\,dx\Big)^{1/q} = \Big(\int_X |T(f)|^p \lambda^{p/q} g\,dx\Big)^{1/p}.$$

After writing $f = u_0^{1/p'} f u_0^{-1/p'}$ and using (3.1) we obtain

$$I \leqslant \Big(\int_X [T(u_0)]^{p/p'} [T(|f|^p u_0^{-p/p'})] \lambda^{p/q} g\,dx\Big)^{1/p}.$$

Using $T(u_0) \leqslant K_1 v_0$ and $\lambda = v_0^{-q/p'} u_1^{q/p}$ we deduce

$$I \leqslant K_1^{1/p'} \Big(\int_X [T(|f|^p u_0^{-p/p'})] u_1 g \, dx\Big)^{1/p}.$$

Using $T^*(u_1 g) \leqslant K_2 v_1$ we obtain

$$\begin{aligned} I &\leqslant K_1^{1/p'} \Big(\int_X |f|^p u_0^{-p/p'} [T^*(u_1 g)] \, dx\Big)^{1/p} \\ &\leqslant K_1^{1/p'} K_2^{1/p} \Big(\int_X |f|^p u_0^{-p/p'} v_1 \, dx\Big)^{1/p} \\ &\leqslant K_1^{1/p'} K_2^{1/p} \Big(\int_X |f|^p \mu \, dx\Big)^{1/p}, \end{aligned}$$

which is the desired result. ■

The case $p = q$ is simpler since it is enough to take $g \equiv 1$. We obtain in this way the converse of Corollary 2.2 when T is positive and linear and T' is the dual of T. The statement is given below, but the proof is omitted since it follows the same pattern as the proof of Theorem 3.1 with obvious modifications.

PROPOSITION 3.2. *Let T, μ, λ and p be as in Theorem 3.1. Suppose that there exist finite positive numbers K_1 and K_2 and nonnegative measurable functions u_0, v_0, u_1 and v_1 such that $T(u_0) \leqslant K_1 v_0$, $T^*(u_1) \leqslant K_2 v_1$ and $\mu = u_0^{1-p} v_1$, $\lambda = v_0^{1-p} u_1$. Then T is bounded from $L^p(\mu)$ to $L^p(\lambda)$ and T^* is bounded from $L^{p'}(\lambda^{-p'/p})$ to $L^{p'}(\mu^{-p'/p})$ with norms not exceeding $K_1^{1/p'} K_2^{1/p}$.*

Proposition 3.2 is proved in [11] for the case $\mu = \lambda$ and in [20] for the case $\mu \neq \lambda$. In both cases the proof uses the interpolation theorem for analytic families of operators due to E. M. Stein ([25]). In fact, the interpolation theorem with change of measures that appears in [25] can also be used. We have shown, however, that a simpler proof can be given using only Hölder's inequality.

Remark. There is also a converse of Proposition 2.3 when we assume that T is linear and T' is the dual of T. In fact the conclusion of (1) in Proposition 2.3 implies the T is bounded from $L^1(\mu)$ to $L^q(\lambda)$ and by duality T^* is bounded from $L^{q'}(\lambda^{-q'/q})$ to $L^\infty(\mu^{-1})$. Similarly, the conclusion of (2) in Proposition 2.3 implies that T^* is bounded from $L^1(\lambda^{-1})$ to $L^{p'}(\mu^{-p'/p})$ and by duality T is bounded from $L^p(\mu)$ to $L^\infty(\lambda)$.

Let M be the Hardy–Littlewood maximal operator on $\boldsymbol{R}^n$. In [2] it has been shown that if $w = u_0^{1-p} u_1$ with $Mu_j \leqslant K_j u_j$, $j = 0, 1$, then M is bounded on $L^p(w)$ for all $1 < p < \infty$. This is a converse of Corollary 2.2 for the sublinear operator M and equal weights $\mu = \lambda = w$. It is therefore natural to ask if Proposition 3.2 or Theorem 3.1 remains valid for sublinear operators and different weights. The answer is *negative*, for we shall show that even for the case $p = q = 2$ and different weights the analogue of Proposition 3.2 for M *cannot* be true.

The example we are going to give appears already in [16]. Let

$$u_0 = x^{-1} |\log(x)|^{-2} \chi_{(0,1/2)}(x), \quad v_1 \equiv 1, \quad \mu = u_0^{-1} v_1,$$
$$v_0 = (Mu_0)(x), \quad u_1 = \chi_{(0,1/2)}(x), \quad \lambda = v_0^{-1} u_1.$$

Thus μ and λ are factored. Observe that $(Mu_0)(x) = x^{-1} |\log(x)|^{-1}$ for $x \in (0, 1/2)$. Taking $f = u_0$ one sees immediately that $f \in L^2(\mu)$ but $Mf \notin L^2(\lambda)$. Thus M is not bounded from $L^2(\mu)$ to $L^2(\lambda)$.

There is a weak converse of Theorem 2.1 for sublinear operators that are "linearizable". A sublinear operator T is *linearizable by a family $\{U_f\}_{f \in C}$ of positive linear operators* if for each $f \in C$ there is a positive linear operator U_f such that

$$T(f) \leqslant C_1 U_f(|f|), \tag{3.2}$$
$$U_f(|g|) \leqslant C_2 T(g) \tag{3.3}$$

for all $g \in C$, with C_1 and C_2 constants independent of f, $g \in C$, where C is a dense subspace of $L^p(X)$ for every $p \in (1, \infty)$.

THEOREM 3.3. *Let T be a sublinear operator which is linearizable by the family $\{U_f\}_{f \in C}$ of positive linear operators, let μ and λ be two nonnegative measurable functions on X and $1 < p < q < \infty$. Suppose that there exist finite positive numbers K_1 and K_2 such that for all g in $L_+^{(q/p)'}$ with $\|g\|_{L^{(q/p)'}} < 1$ we can find $u_0 \in L_+^p(\mu)$, $v_0 \in L_+^p(\lambda^{p/q} g)$, $u_1 \in L_+^{p'}(\lambda^{-p'/q} g)$ and $v_1 \in L_+^{p'}(\mu^{-p'/p})$ such that $T(u_0) \leqslant K_1 v_0$, $U_f^*(u_1 g) \leqslant K_2 v_1$, with K_2 independent of $f \in C$, and $\mu = u_0^{-p/p'} v_1$, $\lambda = v_0^{-q/p'} u_1^{q/p}$. Then T is bounded from $L^p(\mu)$ to $L^q(\lambda)$.*

Proof. For $f \in C$, (3.3) implies $U_f(u_0) \leqslant C_2 T(u_0) \leqslant C_2 K_1 v_0$. Therefore we can apply Theorem 3.1 to U_f and U_f^* to deduce that U_f is bounded from $L^p(\mu)$ to $L^q(\lambda)$. From here, (3.2) and an obvious density argument imply the desired result. ■

Remark. There is a similar result for the case $p = q$. The necessary changes are as in Proposition 3.2 and are therefore omitted. Examples of operators covered by the above theorem are maximal operators and operators of the form $T^*f(x) = \sup_k T_k(|f|)(x)$, where T_k is a sequence of positive linear operators.

4. Applications. In this section we use the results of Section 3 to obtain sufficient conditions for boundedness of some positive linear operators between weighted Lebesgue spaces. In a particular case the sufficient condition is also necessary.

4.1. *Some integral operators with positive kernel.* Let $K(x, y) \geqslant 0$ be defined on $\Delta = \{(x, y) \in \boldsymbol{R}^2 \colon y < x\}$ and consider the operators

$$(T_K f)(x) = \int_{-\infty}^{x} K(x, y) f(y) \, dy, \quad (T_K^* f)(x) = \int_{x}^{\infty} K(y, x) f(y) \, dy. \tag{4.1}$$

We shall apply the results of Section 3 to prove the following theorem which appeared for the first time in [1]:

THEOREM 4.1. *Let T_K be the operator defined as in* (4.1) *with* $K(x,y) \geqslant 0$ *nonincreasing in x and nondecreasing in y. If* $1 < p \leqslant q < \infty$ *and* μ *and* λ *are nonnegative measurable functions on* $\boldsymbol{R}$ *which satisfy*

$$\text{(A)} \qquad \sup_{r>0} \left(\int_r^\infty [K(y,r)]^{q/p} \lambda(y)\,dy \right)^{1/q} \left(\int_{-\infty}^r K(r,y) [\mu(y)]^{-p'/p}\,dy \right)^{1/p'} = K < \infty,$$

then T_K is bounded from $L^p(\mu)$ to $L^q(\lambda)$ with norm not exceeding $K(p)^{1/q}(p')^{1/p'}$.

Proof. We give details for the case $p < q$, since the case $p = q$ requires only minor modifications. Let

$$u_0(x) = [\mu(x)]^{-p'/p} \left(\int_{-\infty}^x K(x,y)[\mu(y)]^{-p'/p}\,dy \right)^{-1/p},$$

$$v_1(x) = \left(\int_{-\infty}^x K(x,y)\,[\mu(y)]^{-p'/p}\,dy \right)^{-1/p'},$$

so that μ is factored as $u_0^{-p/p'} v_1$. Since $K(y,z)$ is nonincreasing in y we have

$$(T_K u_0)(x) = \int_{-\infty}^x K(x,y)\,[\mu(y)]^{-p'/p} \left(\int_{-\infty}^y K(y,z)\,[\mu(z)]^{-p'/p}\,dz \right)^{-1/p} dy$$

$$\leqslant \int_{-\infty}^x K(x,y)\,[\mu(y)]^{-p'/p} \left(\int_{-\infty}^y K(x,z)\,[\mu(z)]^{-p'/p}\,dz \right)^{-1/p} dy.$$

The change of variables $f(y) = \int_{-\infty}^y K(x,z)\,[\mu(z)]^{-p'/p}\,dz$ allows us to compute the integral on the right-hand side of the above inequality so that we obtain

$$\text{(4.2)} \qquad (T_K u_0)(x) \leqslant p' \left(\int_{-\infty}^x K(x,z)\,[\mu(z)]^{-p'/p}\,dz \right)^{1/p'}.$$

Using (A) we obtain $T_K(u_0) \leqslant K p' v_0$ with $v_0(x) = (\int_x^\infty [K(y,x)]^{q/p} \lambda(y)\,dy)^{-1/q}$. Taking

$$u_1(x) = [\lambda(x)]^{p/q} \left(\int_x^\infty [K(y,x)]^{q/p} \lambda(y)\,dy \right)^{-p/(qp')}$$

we have factored λ as $v_0^{-q/p'} u_1^{q/p}$. Let g be any function in $L_{u_1}^{(q/p)'}$ with norm not exceeding 1. Hölder's inequality with exponent $q/p > 1$ and $K(z,y)$ being nondecreasing in y give

$$T_K^*(u_1 g)(x) \leqslant \left\{ \int_x^\infty [K(y,x)]^{q/p} \lambda(y) \left(\int_y^\infty [K(z,y)]^{q/p} \lambda(z)\,dz \right)^{-1/p'} dy \right\}^{p/q}$$

$$\leqslant \left\{ \int_x^\infty [K(y,x)]^{q/p} \lambda(y) \left(\int_y^\infty [K(z,x)]^{q/p} \lambda(z)\,dz \right)^{-1/p'} dy \right\}^{p/q}.$$

The change of variables $f(y) = \int_y^\infty [K(z,x)]^{q/p} \lambda(z)\,dz$ allows us to compute the last integral in the above inequality to obtain

$$\text{(4.3)} \qquad T_K^*(u_1 g)(x) \leqslant \left\{ p \left(\int_x^\infty [K(z,x)]^{q/p} \lambda(z)\,dz \right)^{1/p} \right\}^{p/q}.$$

Using (A) again we deduce $T_K^*(u_1 g) \leqslant K(p)^{p/q} v_1$. The result now follows as an application of Theorem 3.1. ∎

There is a corresponding result for T_K^* which can be proved by the same method as Theorem 4.1 or deduced from it by duality:

THEOREM 4.2. *Let T_K be the operator defined as in* (4.1) *with* $K(x,y) \geqslant 0$ *nonincreasing in x and nondecreasing in y. If* $1 < p \leqslant q < \infty$ *and* μ *and* λ *are nonnegative measurable functions on* $\boldsymbol{R}$ *which satisfy*

$$\text{(A*)} \qquad \sup_{r>0} \left(\int_{-\infty}^r K(r,y) \lambda(y)\,dy \right)^{1/q} \left(\int_r^\infty [K(y,r)]^{p'/q'} [\mu(y)]^{-p'/p}\,dy \right)^{1/p'} = K < \infty,$$

then T_K^* is bounded from $L^p(\mu)$ to $L^q(\lambda)$ with norm not exceeding $K(p)^{1/q}(p')^{1/p'}$.

The above results can be applied to the *Hardy operator* $T_1(f)(x) = \int_0^x f(y)\,dy$. In this case $K(x,y) = 1$ if $0 < y < x$ and $K(x,y) = 0$ otherwise. From Theorem 4.1 we deduce that the condition

$$\text{(B)} \qquad \sup_{r>0} \left(\int_r^\infty \lambda(y)\,dy \right)^{1/q} \left(\int_0^r [\mu(y)]^{-p'/p}\,dy \right)^{1/p'} = K < \infty$$

is sufficient for T_1 to be bounded from $L^p(\mu)$ to $L^q(\lambda)$. This result was proved in [15] for the case $p = q$ and in [3] for the case $p < q$. We emphasize that here we have proved the result as a simple consequence of the converse of factorization. It is easy to see that (B) is also necessary for the boundedness of T_1 from $L^p(\mu)$ to $L^q(\lambda)$ (see [3]).

Besides the Hardy operator, the above results can be applied to *convolution operators* of the form $T(f)(y) = \int_{-\infty}^x K(x-y) f(y)\,dy$, where K is a nonnegative and nonincreasing function defined on $\boldsymbol{R}$. A particular case is the *fractional integral of Riemann–Liouville* defined by $I_\alpha(f)(x) = \int_0^x f(y)(x-y)^{\alpha-1}\,dy$, $0 \leqslant \alpha < 1$. With some previous work they can be also be applied to the *Laplace transform* $L(f)(x) = \int_0^\infty e^{-xy} f(y)\,dy$ (see [1] for details).

4.2. *Multidimensional Hardy operator.* The simplest way to define an analogue of Hardy's operator on $\boldsymbol{R}^n$ is

$$\text{(4.4)} \qquad T_n f(x_1, \ldots, x_n) = \int_0^{x_1} \ldots \int_0^{x_n} f(y_1, \ldots, y_n)\,dy_n \ldots dy_1.$$

To simplify notation we shall write $x = (x_1, \ldots, x_n)$ for a point in $\boldsymbol{R}^n$ and

$\int_{\langle 0,x\rangle} f(y)\,dy$ will denote the integral that appears in (4.4). An examination of the proof of Theorem 4.1 shows that the conditions

$$\text{(C)}\qquad \sup_{r>0}\Big(\int_{\langle r,\infty\rangle} \lambda(y)\,dy\Big)^{1/q}\Big(\int_{\langle 0,r\rangle} [\mu(y)]^{-p'/p}\,dy\Big)^{1/p'} = K < \infty,$$

$$\text{(D)}\qquad \int_{\langle 0,x\rangle} [\mu(z)]^{-p'/p}\Big(\int_{\langle 0,z\rangle} [\mu(y)]^{-p'/p}\,dy\Big)^{-1/p}dz \leqslant C_1(p)\Big(\int_{\langle 0,x\rangle} [\mu(y)]^{-p'/p}\,dy\Big)^{1/p'},$$

$$\text{(E)}\qquad \int_{\langle x,\infty\rangle} \lambda(z)\Big(\int_{\langle z,\infty\rangle} \lambda(y)\,dy\Big)^{-1/p'}dz \leqslant C_2(p)\Big(\int_{\langle x,\infty\rangle} \lambda(y)\,dy\Big)^{1/p},$$

where $r = (r_1, \ldots, r_n) > 0$ means $r_j > 0$ for all $j = 1, \ldots, n$, are sufficient for the operator T_n to be bounded from $L^p(\mu)$ to $L^q(\lambda)$, $1 < p \leqslant q < \infty$. In fact, (D) and (E) is all that is needed to obtain the analogues of (4.2) and (4.3) for the operator T_n. Condition (C) is necessary for the boundedness of T_n, but it is not sufficient if $n > 1$. An example can be found in [24].

The simplest example of weights that satisfy (C), (D) and (E) is $\mu = \prod_{j=1}^n \mu_j$ and $\lambda = \prod_{j=1}^n \lambda_j$, where each pair (μ_j, λ_j) satisfies the one-dimensional version of (C), i.e. condition (B).

The importance of obtaining weighted inequalities for T_n is due to the fact that they play the same role in n-dimensional proofs as T_1 plays in the one-dimensional case: some examples can be found in [9], [13], [17] and [21]. The problem of characterizing the weights μ and λ for which T_n is bounded from $L^p(\mu)$ to $L^q(\lambda)$, $p \leqslant q$, is still open for $n \geqslant 2$. It was solved in [24] for $n = 2$.

A more intricate way to generalize the Hardy operator to $\boldsymbol{R}^n$ is through the use of *cones*, as in [18]. In what follows we use the same notation as in [18], to which we refer the reader for the precise definitions. An *open, convex, homogeneous and selfadjoint cone* V defines a partial ordering in $\boldsymbol{R}^n$ which is denoted by $<_V$. The Hardy operator associated to V is

$$T_V(f)(x) = \int_{\langle 0,x\rangle} f(y)\,dy, \qquad x \in V,$$

where $\langle 0,x\rangle = \{y \in V\colon 0 <_V y <_V x\}$. The function $\Delta(x) = \int_{\langle 0,x\rangle} dy$ plays in this context the same role as the weight x plays in the classical (one-dimensional) Hardy inequality. In particular, the following result appears in Theorem 2 of [18]:

PROPOSITION 4.3. *Let $1 < p < \infty$. Let V be an open, convex, homogeneous and selfadjoint cone in $\boldsymbol{R}^n$ and $\sigma = \max(-1, -2/(n-1))$. Then for $\gamma < -\sigma p - 1$ we have*

$$\int_V \Big(\int_{\langle 0,x\rangle} f(y)\,dy\Big)^p [\Delta(x)]^{\gamma-p}\,dx \leqslant C\int_V [f(x)]^p [\Delta(x)]^\gamma\,dx.$$

An easy proof of this proposition can be given using our Proposition 3.2. We need the equalities

$$\text{(4.5)}\qquad \int_{\langle 0,x\rangle} [\Delta(t)]^\alpha\,dt = C[\Delta(x)]^{\alpha+1}$$

valid for $\alpha > \sigma$ (see Lemma 4 of [18]) and

$$\text{(4.6)}\qquad \int_{\langle x,\infty\rangle} [\Delta(t)]^\alpha\,dt = C[\Delta(x)]^{\alpha+1}$$

valid for $\alpha < -\sigma-2$. This last equality can be deduced from (4.5) using Lemma 3 of [18]. The proof is as follows: take $u_0 = \Delta^{-(\gamma+1)/p}$, $v_1 = \Delta^{\gamma/p - 1/p'}$, $v_0 = \Delta^{-\gamma/p + 1/p'}$, and $u_1 = \Delta^{\gamma/p - 1 - 1/p'}$, so that $\Delta^\gamma = u_0^{-p/p'} v_1$ and $\Delta^{\gamma-p} = v_0^{-p/p'} u_1$. From (4.5) we deduce $T_V(u_0) \leqslant Cv_0$ and from (4.6) we obtain $T_V^*(u_1) \leqslant Cv_1$. The result now follows by applying Proposition 3.2.

We mention that other results that appear in [18] can also be proved using the results of Section 3. We leave the details to the interested reader.

5. Extrapolation. For $1 \leqslant p \leqslant q \leqslant \infty$ and T any mapping defined on an appropriate class of measurable functions on a measure space X, we denote by $V_{p,q}(T)$ the set of all pairs (μ, λ) of nonnegative measurable functions on X for which T is bounded from $L^p(\mu)$ to $L^q(\lambda)$. Here $L^\infty(w) = \{f\colon X \to \boldsymbol{C}\colon \|fw\|_\infty < \infty\}$ for any nonnegative measurable function w defined on X. When $p = q$ we write $V_p(T)$ instead of $V_{p,p}(T)$. We start by showing how to extrapolate from $p = 1 = q$.

THEOREM 5.1. *Let U be a positive linear operator and let T be any mapping which is bounded from $L^1(\tilde\mu)$ to $L^1(\tilde\lambda)$ for all pairs of nonnegative measurable functions such that $U^*(\tilde\lambda) \leqslant C\tilde\mu$. Then for $1 < p \leqslant q < \infty$, T is bounded from $L^p(\mu)$ to $L^q(\lambda)$ for all $(\mu, \lambda) \in V_{p,q}(U)$.*

Proof. *Case* I: $p = q$. Let $(\mu, \lambda) \in V_p(U)$ and $f \in L^p(\mu)$. By duality there exists $g \in L^{p'}(\lambda)$ with norm not exceeding 1 such that

$$\Big(\int_X |Tf|^p \lambda\,dx\Big)^{1/p} = \int_X |Tf|\,\lambda g\,dx.$$

The pair $(\tilde\mu, \tilde\lambda)$ with $\tilde\mu = U^*(\lambda g)$ and $\tilde\lambda = \lambda g$ satisfies $U^*(\tilde\lambda) \leqslant C\tilde\mu$ with $C = 1$. Hence the right-hand side of the above formula is dominated by $C\int_X |f|\,U^*(\lambda g)\,dx$. Using Hölder's inequality we obtain

$$\Big(\int_X |Tf|^p\lambda\,dx\Big)^{1/p} \leqslant C\Big(\int_X |f|^p\mu\,dx\Big)^{1/p}\Big(\int_X |U^*(\lambda g)|^{p'}\mu^{-p'/p}\,dx\Big)^{1/p'}.$$

By duality $(\mu, \lambda) \in V_p(U)$ is equivalent to $(\lambda^{-p'/p}, \mu^{-p'/p}) \in V_{p'}(U^*)$. Hence the last integral in the above expression is dominated by a constant. This proves the result for $p = q$.

Case II: $p \leqslant q$. Take $(\mu, \lambda) \in V_{p,q}(U)$. By part (1) of Proposition 2.4, $(\mu, \lambda^{p/q} g) \in V_p(U)$ for all $g \in L_+^{(q/p)'}$. By Case I, T is bounded from $L^p(\mu)$ to $L^p(\lambda^{p/q} g)$ for all g as above. Therefore, for some $g \in L_+^{(q/p)'}$ with $\|g\|_{L^{(q/p)'}} \leqslant 1$ we have

$$\Big(\int_X |Tf|^q\lambda\,dx\Big)^{1/q} \leqslant \Big(\int_X |Tf|^p \lambda^{p/q} g\,dx\Big)^{1/p} \leqslant \Big(\int_X |f|^p\mu\,dx\right)^{1/p}.$$

This ends the proof of Theorem 5.1. ∎

We can also extrapolate from weak type (1,1). In what follows the Lorentz space $L^{q,\infty}$ will be denoted by wL^q. The proof of the next theorem is similar to the proof of Theorem 5.1. The case $p = q$ can be found in [20].

THEOREM 5.2. *Let S be a positive sublinear operator and let T be any mapping which is bounded from $L^1(\tilde{\mu})$ to $wL^1(\tilde{\lambda})$ for all pairs of nonnegative measurable functions such that $S(\tilde{\lambda}) \leqslant C\tilde{\mu}$. Then for $1 < p \leqslant q < \infty$, T is bounded from $L^p(\mu)$ to $wL^q(\lambda)$ for all (μ, λ) such that $(\lambda^{-q'/q}, \mu^{-p'/p}) \in V_{q',p'}(S)$.*

Extrapolation from $p_0 > 1$ is more complicated. We need to use the induction argument of J. L. Rubio de Francia. The case $p = q$ and equal weights of the next two results can be found in [11], from which our proof borrows heavily.

THEOREM 5.3. *Let $1 \leqslant p_0 \leqslant \infty$ and suppose that S is a positive sublinear operator and T is a mapping with $V_{p_0}(T) \supset V_{p_0}(S)$. Then if $1 \leqslant p \leqslant p_0$ and $p \leqslant q < \infty$, $V_{p,q}(T) \supset V_{p,q}(S)$, and if $p_0 < p < \infty$ and $p \leqslant q < \infty$, $V_{p,q}(T) \supset V_{p,q}(S)$ if S satisfies "$(\mu, \lambda) \in V_p(S) \Leftrightarrow (\lambda^{-p'/p}, \mu^{-p'/p}) \in V_{p'}(S)$ for all $1 < p < \infty$".*

Proof. *Case* I: $p = q$. Suppose $1 \leqslant p < p_0$ and take $(\mu, \lambda) \in V_p(S)$. Thus, the operator $S_1(f) = \lambda^{1/p} S(f\mu^{-1/p})$ is bounded on L^p with norm not exceeding $\|S\|$. Hence, given $h \in L_+^p$ with norm not exceeding 1, the function

$$H = \sum_{n=0}^{\infty} S_1^n(h)/(2\|S\|)^n$$

belongs to L_+^p and satisfies

$$S_1(H) \leqslant 2\|S\| H, \qquad \|H\|_{L^p} \leqslant 2, \qquad h \leqslant H. \tag{5.1}$$

Observe that $S_1(H) \leqslant 2\|S\| H$ is exactly $S(H\mu^{-1/p}) \leqslant CH\lambda^{-1/p}$ so that the positivity of S implies

$$\|S(f) H^{-1} \lambda^{1/p}\|_{L^\infty} \leqslant \|f H^{-1} \mu^{1/p}\|_{L^\infty}. \tag{5.2}$$

From $(\mu, \lambda) \in V_p(S)$ we deduce

$$\|S(f) \lambda^{1/p}\|_{L^p} \leqslant \|f\mu^{1/p}\|_{L^p}. \tag{5.3}$$

Interpolating between (5.2) and (5.3) we see that S is bounded from $L^{p_0}(\mu^{p_0/p} H^{p-p_0})$ to $L^{p_0}(\lambda^{p_0/p} H^{p-p_0})$. This together with the assumption $V_{p_0}(T) \supset V_{p_0}(S)$ allows us to obtain

$$(\mu^{p_0/p} H^{p-p_0}, \lambda^{p_0/p} H^{p-p_0}) \in V_{p_0}(T). \tag{5.4}$$

Take now $f \in L^p(\mu)$. By the converse of Hölder's inequality there is a $g \in L_+^{p/(p_0-p)}(\mu)$ with norm 1 such that

$$\Big(\int_X |f|^p \mu\, dx\Big)^{1/p} = \Big(\int_X |f|^{p_0} g^{-1} \mu\, dx\Big)^{1/p_0}. \tag{5.5}$$

With $h = g^{1/(p_0-p)} \mu^{1/p} \in L^p$ find $H \in L_+^p$ satisfying (5.1). In particular, the condition $h \leqslant H$ implies

$$g^{-1}\mu \geqslant \mu^{p_0/p} H^{p-p_0}. \tag{5.6}$$

Hölder's inequality with exponent $p_0/p \geqslant 1$ together with (5.4)–(5.6) imply the desired result since

$$\begin{aligned}
\Big(\int_X |Tf|^p \lambda\, dx\Big)^{1/p} &\leqslant \Big(\int_X |Tf|^{p_0} \lambda^{p_0/p} H^{p-p_0}\, dx\Big)^{1/p_0} \|H\|_{L^p}^{(p_0-p)/(p_0 p)} \\
&\leqslant C\Big(\int_X |Tf|^{p_0} \lambda^{p_0/p} H^{p-p_0}\, dx\Big)^{1/p_0} \\
&\leqslant C\Big(\int_X |f|^{p_0} \mu^{p_0/p} H^{p-p_0}\, dx\Big)^{1/p_0} \\
&\leqslant C\Big(\int_X |f|^{p_0} g^{-1}\mu\, dx\Big)^{1/p_0} = C\Big(\int_X |f|^p \mu\, dx\Big)^{1/p}.
\end{aligned}$$

The case $p_0 = \infty$ is similar, except that the interpolation argument leading to (5.4) is not needed. Details are left to the reader.

Consider now the case $p_0 < p < \infty$. Let $(\mu, \lambda) \in V_p(S)$. By duality there exists $g \in L^{(p/p_0)'}(\lambda)$ with norm not exceeding 1 such that

$$\Big(\int_X |Tf|^p \lambda\, dx\Big)^{1/p} = \Big(\int_X |Tf|^{p_0} \lambda g\, dx\Big)^{1/p_0}. \tag{5.7}$$

Since $(\mu, \lambda) \in V_p(S)$ if and only if $(\lambda^{-p'/p}, \mu^{-p'/p}) \in V_{p'}(S)$, and $p' < p_0'$, we can proceed as above; so that for $h = g^{(p/p_0)'/p'} \lambda^{1/p'} \in L^{p'}$ we can find $H \in L^{p'}$ such that

$$g\lambda \leqslant \lambda^{p_0/p} H^{p'/(p/p_0)'} \tag{5.8}$$

and $(\lambda^{-p_0'/p} H^{p'-p_0'}, \mu^{-p_0'/p} H^{p'-p_0'}) \in V_{p_0'}(S)$. Our hypotheses imply

$$(\mu^{p_0/p} H^{p'/(p/p_0)'}, \lambda^{p_0/p} H^{p'/(p/p_0)'}) \in V_{p_0}(S) \subset V_{p_0}(T). \tag{5.9}$$

Starting with (5.7) the proof can be finished by using (5.8), (5.9) and Hölder's inequality with exponent $p/p_0 > 1$; in fact,

$$\begin{aligned}
\Big(\int_X |Tf|^p \lambda\, dx\Big)^{1/p} &= \Big(\int_X |Tf|^{p_0} \lambda g\, dx\Big)^{1/p_0} \\
&\leqslant \Big(\int_X |Tf|^{p_0} \lambda^{p_0/p} H^{p'/(p/p_0)'}\, dx\Big)^{1/p_0} \leqslant \Big(\int_X |f|^{p_0} \mu^{p_0/p} H^{p'/(p/p_0)'}\, dx\Big)^{1/p_0} \\
&\leqslant C\Big(\int_X |f|^p \mu\, dx\Big)^{1/p} \|H\|_{L^{p'}}^{p'/p_0(p/p_0)'} \leqslant C\Big(\int_X |f|^p \mu\, dx\Big)^{1/p}.
\end{aligned}$$

Case II: $p < q < \infty$. If $(\mu, \lambda) \in V_{p,q}(S)$, part (1) of Proposition 2.4 implies $(\mu, \lambda^{p/q} g) \in V_p(S)$ for all $g \in L^{(q/p)'}$. By Case I, $(\mu, \lambda^{p/q} g) \in V_p(T)$ for all $g \in L^{(q/p)'}$. Given $f \in L^p(\mu)$, choose $g \in L_+^{(q/p)'}$ with norm not exceeding 1 such that

$$\Big(\int_X |Tf|^q \lambda\, dx\Big)^{1/q} = \Big(\int_X |Tf|^p \lambda^{p/q} g\, dx\Big)^{1/p}.$$

For this particular g, $(\mu, \lambda^{p/q} g) \in V_p(T)$ so that the right-hand side of the above equality is dominated by $C\big(\int_X |f| \mu\, dx\big)^{1/p}$. This proves $(\mu, \lambda) \in V_{p,q}(T)$ and finishes the proof of the theorem. ∎

For the case of positive linear operators the statement of the extrapolation theorem is simpler:

THEOREM 5.4. *Let $1 \leqslant p_0 \leqslant \infty$ and suppose that U is a positive linear operator and T is a mapping with $V_{p_0}(T) \supset V_{p_0}(U)$. Then $V_{p,q}(T) \supset V_{p,q}(U)$ for all $1 \leqslant p \leqslant q < \infty$.*

The case $p = q$ needs again the J. L. Rubio de Francia argument ; details can be found in [20]. The case $p < q$ follows from this one using the same argument as in the corresponding case of Theorem 5.3.

After the first version of this paper was typed I have learned that E. Harboure, R. A. Macias and C. Segovia (*Extrapolation results for classes of weights*, to appear in Amer. J. Math.) and C. Segovia and J. L. Torrea (*Extrapolation for pairs of related weights*, preprint) have obtained particular cases of Theorem 5.3. They treat the case $p = q$ and equal weights, when S is the Hardy–Littlewood maximal operator or a modification of it. I am indebted to my colleagues J. García-Cuerva and J. L. Torrea for pointing out these results to me.

References

[1] K. F. Andersen and H. P. Heinig, *Weighted norm inequalities for certain integral operators*, SIAM J. Math. Anal. 14 (1983), 834–844.

[2] S. Bloom, *Solving weighted norm inequalities using the Rubio de Francia algorithm*, Proc. Amer. Math. Soc. 101 (1987), 306–312.

[3] S. Bradley, *Hardy inequalities with mixed norms*, Canad. Math. Bull. 21 (1978), 405–408.

[4] R. R. Coifman and C. Fefferman, *Weighted norm inequalities for maximal functions and singular integrals*, Studia Math. 51 (1974), 241–250.

[5] R. R. Coifman, P. Jones and J. L. Rubio de Francia, *Constructive decomposition of BMO functions and factorization of A_p weights*, Proc. Amer. Math. Soc. 87 (1983), 665–666.

[6] E. Gagliardo, *On integral transformations with positive kernel*, Proc. Amer. Math. Soc. 16 (1965), 429–434.

[7] J. García-Cuerva, *An extrapolation theorem in the theory of A_p weights*, ibid. 87 (1983), 422–426.

[8] J. García-Cuerva and J. L. Rubio de Francia, *Weighted Norm Inequalities and Related Topics*, North-Holland, 1985.

[9] H. P. Heinig, *Weighted norm inequalities for classes of operators*, Indiana Univ. Math. J. 33 (1984), 573–582.

[10] R. Hunt, B. Muckenhoupt and R. Wheeden, *Weighted norm inequalities for the conjugate function and Hilbert transform*, Trans. Amer. Math. Soc. 176 (1973), 227–251.

[11] B. Jawerth, *Weighted inequalities for maximal operators: linearization, localization and factorization*, Amer. J. Math. 108 (1986), 361–414.

[12] P. Jones, *Factorization of A_p weights*, Ann. of Math. 111 (1980), 511–530.

[13] W. B. Jurkat and G. Sampson, *On rearrangement and weight inequalities for the Fourier transform*, Indiana Univ. Math. J. 33 (1984), 257–270.

[14] F. J. Martín-Reyes, P. Ortega and A. de la Torre, *Weighted inequalities for one-sided maximal functions*, preprint.

[15] B. Muckenhoupt, *Hardy's inequality with weights*, Studia Math. 44 (1972), 31–38.

[16] —, *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc. 165 (1972), 207–226.

[17] B. Muckenhoupt and R. L. Wheeden, *Two weight function norm inequalities for the Hardy–Littlewood maximal function and the Hilbert transform*, Studia Math. 55 (1976), 279–294.

[18] T. Ostrogorski, *Analogues of Hardy's inequality in* $\boldsymbol{R}^n$, ibid. 88 (1988), 209–219.

[19] J. L. Rubio de Francia, *Factorization theory and A_p weights*, Amer. J. Math. 106 (1984), 533–547.

[20] F. J. Ruiz and J. L. Torrea, *The use of the Rubio de Francia algorithm in a general context*, preprint.

[21] C. Sadosky and R. L. Wheeden, *Some weighted norm inequalities for the Fourier transform of functions with vanishing moments*, Trans. Amer. Math. Soc. 300 (1987), 521–533.

[22] E. T. Sawyer, *A characterization of a two-weight norm inequality for maximal operators*, Studia Math. 75 (1982), 1–11.

[23] —, *Weighted inequalities for the one-sided Hardy–Littlewood maximal functions*, Trans. Amer. Math. Soc. 297 (1986), 53–61.

[24] —, *Weighted inequalities for the two-dimensional Hardy operator*, Studia Math. 82 (1985), 1–16.

[25] E. M. Stein, *Interpolation of linear operators*, Trans. Amer. Math. Soc. 83 (1956), 482–492.

UNIVERSIDAD AUTÓNOMA DE MADRID
MATEMÁTICAS
28049 Madrid, Spain

Received August 5, 1988 (2469)
Revised version January 9, 1989