APPROXIMATING ENTROPY OF MAPS OF THE INTERVAL

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This paper extends some results of M. Misiurewicz [M], Y. Takahashi [T], and L.-S. Young [Y], showing that the topological entropy of a continuous map of the interval is determined by its periodic orbits and its symbolic dynamics.

Let f denote a continuous map of a compact interval I to itself, and let $h(\cdot)$ denote topological entropy [AKM]. If $P = \{x_1 < x_2 < \dots < x_N\}$ is a periodic orbit of f , let L_P denote the "connect-the-dots" map determined by P , i.e., $L_P: [x_1, x_N] \rightarrow [x_1, x_N]$ agrees with f on P , and is linear on each interval $[x_i, x_{i+1}]$.

THEOREM A. $h(f) = \sup h(L_P)$, where the supremum is taken over all periodic orbits P of f .

Theorem A is due originally to Takahashi [T, Theorem 3], although his proof appears to be valid only for piecewise monotone maps (see MR 821:58057).

THEOREM B. Suppose $h(f) > 0$, and let $\varepsilon > 0$. There is a collection $K_1, \dots, K_n$ of pairwise disjoint closed subintervals of I such that $\log \lambda \geq h(f) - \varepsilon$, where λ is the largest eigenvalue of the $n \times n$ matrix B defined by $B_{ij} = 1$ if $f(K_i) \supseteq K_j$, and $B_{ij} = 0$ otherwise.

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In this case, there is a closed invariant subset X (contained in the set of points whose orbits lie in $K_1 \cup \dots \cup K_n$) such that $f|X$ is semiconjugate to S , the shift of finite type associated to the matrix B . Since $h(f|X) \geq h(S) = \log \lambda$, we may approximate the topological entropy of f arbitrarily well from its symbolic dynamics. For piecewise monotone f , Young [Y, Lemma 2.7] showed that given $\varepsilon > 0$, there is a closed invariant set X such that $h(f|X) \geq h(f) - \varepsilon$ and $f|X$ is actually conjugate to a shift of finite type.

Our proofs of both theorems are based on the following theorem of Misiurewicz.

THEOREM 1 [M, Theorem 1]. *Let $\varepsilon > 0$. There are positive integers n and d , a subinterval J of I , and subintervals $J_1, \dots, J_d$ of J with pairwise disjoint interiors, such that $f^n(J_i) \supseteq J$ for $i = 1, \dots, d$, and $(1/n) \log d \geq h(f) - \varepsilon$.*

(By shrinking J a bit, we can choose $J_1, \dots, J_d$ to be pairwise disjoint, thus obtaining symbolic dynamics for f^n .)

We also use the following corollary of Theorem 1.

THEOREM 2 [M, Theorem 2]. *Topological entropy, regarded as a function on the space of all continuous maps from I to itself, is lower semi-continuous.*

We will use the following two lemmas. The first follows from [BGM, Lemma 1.5].

LEMMA 3. *If P is a periodic orbit of f , then $h(f) \geq h(L_P)$.*

LEMMA 4. *If P is a periodic orbit of f and $Q \subseteq P$ is a periodic orbit of f^n , then $h(L_P) \geq (1/n)h(L_Q)$.*

Proof. Since Q is a periodic orbit of L_P^n , it follows from Lemma 3 that $h(L_P^n) \geq h(L_Q)$. But $h(L_P^n) = nh(L_P)$ [AKM, Theorem 2].

We will associate to a periodic orbit a *Markov graph* and a *type*. Let $P = \{x_1 < x_2 < \dots < x_N\}$ be a periodic orbit of f . Let $I_1 = [x_1, x_2], \dots, I_{N-1} = [x_{N-1}, x_N]$. The Markov graph associated to P is the directed graph with vertices $I_1, \dots, I_{N-1}$, and an edge (arrow) from I_i to I_j if and only if $L_P(I_i) \supseteq I_j$. Given a loop in the graph, there is a periodic orbit of f which goes through the vertices in the prescribed order. The *type* of P is the permutation π of $\{1, \dots, N\}$ defined by $\pi(i) = j$ if $f(x_i) = x_j$. Then $h(L_P)$ is the logarithm of the largest eigenvalue of the adjacency matrix of the Markov graph associated to P , and in particular, if P and Q have the same type, then $h(L_P) = h(L_Q)$.

We may also form a Markov graph using any finite set of points in place of P . In this case too, loops in the graph yield periodic orbits of f .

Proof of Theorem A. It follows from Lemma 3 that $h(f) \geq \sup h(L_P)$. To obtain the reverse inequality, it suffices to prove the following.

- (*) For any $\varepsilon > 0$ and any positive integer M , there is a periodic orbit P of f , containing at least M points, such that $h(L_P) \geq h(f) - \varepsilon$.

(The condition that P have at least M points will be used in the proof of Theorem B, but not in the proof of Theorem A.)

Let ε and M be given. By Theorem 1, there are positive integers n and d , a subinterval J of I , and subintervals $J_1, \dots, J_d$ of J with pairwise disjoint interiors, such that $f^n(J_i) \supseteq J$ for $i = 1, \dots, d$, and $(1/n) \log d \geq h(f) - \varepsilon/2$. We may assume that J and $J_1, \dots, J_d$ are closed (by taking closures if necessary).

We introduce a piecewise linear map L , which will serve as a model for the relevant dynamics of f^n on J . Let $J = [a, b]$. There are points $w_0 < w_1 < \dots < w_d$ in J such that either

- (1) $f^n(w_i) = a$ for i even, and $f^n(w_i) = b$ for i odd, or
- (2) $f^n(w_i) = a$ for i odd, and $f^n(w_i) = b$ for i even.

Without loss of generality, we assume the former.

Let L be the piecewise linear map of $[0, 1]$ to itself satisfying

- (3) $L(0) = 0$, $L(1/d) = 1$, $L(2/d) = 0$, $L(3/d) = 1, \dots$,

and

- (4) The restriction of L to the intervals $[0, 1/d], \dots, [(d-1)/d, 1]$ is linear.

Note that $h(L) = \log d$.

We first claim that given $\delta > 0$, there is a periodic orbit of L which meets every subinterval of $[0, 1]$ of length δ . To see this, take a finite open cover of $[0, 1]$ by intervals of length less than $\delta/2$. There is a positive integer m such that $L^m(C) = [0, 1]$ for every C in this cover. Therefore, in the Markov graph associated to the set of endpoints of the members of the cover, there is a loop which passes through every vertex, and hence there is a periodic orbit of L which meets every member of the cover. But any interval of length δ contains a member of the cover.

Choose δ small enough so that the periodic orbit R described above contains at least M points. Extend L_R to a map $\bar{L}_R$ of $[0, 1]$ to itself by defining $\bar{L}_R$ to be constant on $[0, \min R]$ and $[\max R, 1]$. Then $h(\bar{L}_R) = h(L_R)$ and $\bar{L}_R$ is within $d\delta$ of L in the C^0 -metric. By Theorem 2, we may choose δ small enough so that $h(L_R) \geq \log d - \varepsilon/2$.

We next claim that f^n has a periodic orbit Q of the same type as R . To prove this, we define a nested sequence $\{A_k\}$ of finite sets, by

- (5) $A_0 = \{0, 1/d, 2/d, \dots, (d-1)/d, 1\}$,
- (6) $A_k = A_{k-1} \cup L^{-1}(A_{k-1})$, $k \geq 1$.

Since $A_k = \{j/d^{k+1} | 0 \leq j \leq d^{k+1}\}$, it follows that $\bigcup_{0 \leq k < \infty} A_k$ is dense, and so there is a positive integer k such that any interval joining adjacent points of A_k contains at most one point of R .

Let $B_0 = \{w_0, \dots, w_d\}$. Note that B_0 and A_0 have the same number of points. Choose a subset B_1 of $B_0 \cup f^{-n}(B_0)$ such that

$$(7) \quad B_1 \supseteq B_0,$$

$$(8) \quad B_1 \text{ and } A_1 \text{ have the same number of points,}$$

$$(9) \quad \text{If } A_1 = \{y_0 < \dots < y_m\} \text{ and } B_1 = \{z_0 < \dots < z_m\}, \text{ then } y_i \in A_0 \text{ if and only if } z_i \in B_0, \text{ and for } y_i \in A_1 - A_0, L(y_i) = y_j \text{ if and only if } f^n(z_i) = z_j.$$

We continue inductively, obtaining $B_2, \dots, B_k$ satisfying the appropriate analogous conditions. Then, the Markov graph determined by the points of A_k and L can be embedded in the Markov graph determined by the points of B_k and f^n .

There is a loop in the Markov graph determined by the points of A_k and L which corresponds to the periodic orbit R . Hence the same loop appears in the Markov graph determined by the points of B_k and f^n . This loop yields a periodic orbit Q of f^n which has the same type as R . The claim is established.

Let P be the unique periodic orbit of f which contains Q . By Lemma 4, $h(L_P) \geq (1/n)h(L_Q)$. Thus

$$h(L_P) \geq (1/n)(\log d - \varepsilon/2) \geq (1/n)\log d - \varepsilon/2 \geq h(f) - \varepsilon.$$

This proves (*) and hence Theorem A.

Proof of Theorem B. By Theorem A, there is a periodic orbit R of f such that $h(L_R) \geq h(f) - \varepsilon/2$. It follows from (*) above, applied to L_R , that L_R has a periodic orbit $Q = \{q_1 < \dots < q_N\}$, with $N > 2$, $Q \neq R$ (and hence $Q \cap R = \emptyset$), and $h(L_Q) \geq h(L_R) - \varepsilon/2$.

We will first find a collection of pairwise disjoint closed intervals $J_i \subseteq [q_i, q_{i+1}]$, $i = 1, \dots, N-1$, such that if $L_Q[q_i, q_{i+1}] \supseteq [q_j, q_{j+1}]$, then $L_R(J_i) \supseteq J_j$.

Since $N > 2$, there is an integer n , $1 < n < M$, such that $L_R(q_n) = q_1$ or q_N . Relabel Q as $\{\alpha_1, \dots, \alpha_N\}$, where $\alpha_1 = q_n$, and $L_R(\alpha_i) = \alpha_{i-1}$ for $i = 2, \dots, N$. Note that L_R is strictly monotone at each α_i . Let $\sigma_0 = +1$ or -1 according as $L_R(q_n) = q_N$ or q_1 , and for $i = 1, \dots, N$, let $\sigma_i = +1$ or -1 according as L_R is increasing or decreasing at α_i . Finally, let $\tau_i = \sigma_0 \sigma_1 \dots \sigma_i$. Then L_R maps the τ_1 side of α_1 outside of $[q_1, q_N]$, and for $i = 2, \dots, N$, it maps the τ_i side of α_i to the τ_{i-1} side of α_{i-1} .

We define a collection of points $\beta_1, \dots, \beta_N$, inductively, as follows. If $\tau_1 = -1$, let β_1 be the largest point in $L_R^{-1}(R)$ which is less than α_1 , and if $\tau_1 = +1$, let β_1 be the smallest point in $L_R^{-1}(R)$ which is greater than α_1 .

Assuming that $\beta_1, \dots, \beta_{i-1}$ have been defined, if $\tau_i = -1$, let β_i be the largest point in $L_R^{-1}(\beta_{i-1})$ which is less than α_i , and if $\tau_i = +1$, let β_i be the smallest point in $L_R^{-1}(\beta_{i-1})$ which is greater than α_i .

Note that $L_R(\beta_1) \notin [q_1, q_N]$, and that $\langle \alpha_i, \beta_i \rangle \cap Q = \emptyset$ for $i = 1, \dots, N$, where $\langle \alpha_i, \beta_i \rangle$ denotes the open interval with endpoints α_i and β_i . Also, $L_R \langle \alpha_1, \beta_1 \rangle \cap [q_1, q_N] = \emptyset$ and $L_R \langle \alpha_i, \beta_i \rangle = \langle \alpha_{i-1}, \beta_{i-1} \rangle$ for $i = 2, \dots, N$. Finally, if $L_Q[q_i, q_{i+1}] \supseteq [q_j, q_{j+1}]$, then

$$L_R([q_i, q_{i+1}] - \langle \alpha_1, \beta_1 \rangle) \supseteq [q_j, q_{j+1}]$$

and

$$L_R([q_i, q_{i+1}] - \langle \alpha_k, \beta_k \rangle) \supseteq [q_j, q_{j+1}] - \langle \alpha_{k-1}, \beta_{k-1} \rangle, \quad k = 2, \dots, N.$$

For $i = 1, \dots, N-1$, let $J_i = [q_i, q_{i+1}] - \bigcup_{1 \leq j \leq N} \langle \alpha_j, \alpha_j \rangle$. Then

$J_1, \dots, J_{N-1}$ is the desired collection of pairwise disjoint closed intervals such that $L_R(J_i) \supseteq J_j$ whenever $L_Q[q_i, q_{i+1}] \supseteq [q_j, q_{j+1}]$.

As in [B, Theorem 3.3], there is a periodic orbit P of f such that the action of f on $P \cup R$ is the same as the action of L_R on $Q \cup R$. By this we mean that P and Q have the same number of points, and if $P \cup R = \{x_1 < \dots < x_M\}$ and $Q \cup R = \{y_1 < \dots < y_M\}$, then $f(x_i) = x_j$ if and only if $L_R(y_i) = y_j$. As in [B, Theorem 3.3] again, there are points $\gamma_1, \dots, \gamma_N$, corresponding to $\beta_1, \dots, \beta_N$, such that the action of f on $P \cup R \cup \{\gamma_1, \dots, \gamma_N\}$ is the same as the action of L_R on $Q \cup R \cup \{\beta_1, \dots, \beta_N\}$. Thus we obtain a collection of pairwise disjoint closed intervals $K_1, \dots, K_{N-1}$ (defined analogously to $J_1, \dots, J_{N-1}$), such that $f(K_i) \supseteq K_j$ whenever $L_Q[q_i, q_{i+1}] \supseteq [q_j, q_{j+1}]$.

Finally, let A be the adjacency matrix of the Markov graph associated to Q , and let μ denote the largest eigenvalue of A . Let B and λ be as in the statement of the theorem. Since $B_{ij} = 1$ whenever $A_{ij} = 1$, we have $\log \lambda \geq \log \mu = h(L_Q) \geq h(f) - \varepsilon$. This completes the proof of the theorem.

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