ON THE RATE OF POINTWISE CONVERGENCE OF SZÁSZ-MIRAKYAN OPERATORS

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1. Preliminaries

Let φ be a complex-valued function bounded on some interval $[a, b]$. Introduce the modulus of variation of φ on the interval $[a, b]$, defined for positive integers n by

$$v(n, \varphi; a, b) = \sup_{\pi_n} \sum_{j=0}^{n-1} |\varphi(x_{2j+1}) - \varphi(x_{2j})|,$$

where the supremum is taken over all systems π_n of non-overlapping intervals $(x_{2j}, x_{2j+1}) \subset (a, b)$, $j = 0, 1, \dots, n-1$. If $n = 0$ we shall take $v(0, \varphi; a, b) = 0$. Some basic properties of this modulus can be found in [1] and [2].

Given some arbitrary positive numbers M, α , let $\mathfrak{V}_M^\alpha$ be the set of all complex-valued functions f satisfying the growth-condition

$$(1) \quad |f(t)| \leq Mt^{\alpha} \quad (0 \leq t < \infty).$$

For $f \in \mathfrak{V}_M^\alpha$ and $x \in (0, \infty)$ the Szász-Mirakyan operator is defined by

$$S_n[f](x) = \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) p_k(nx) \quad (n \in \mathbb{N}),$$

where $p_k(t) = e^{-t} t^k / k!$.

Using the modulus of variation, we shall present some quantitative estimates of the rate of convergence

$$(2) \quad \lim_{n \rightarrow \infty} S_n[f](x) = \frac{1}{2} \{f(x+0) + f(x-0)\}$$

for any function $f \in \mathfrak{V}_M^\alpha$, at every point $x \in (0, \infty)$ for which the one-sided

limits $f(x \pm 0)$ exist. As corollaries, for the special subclasses of $\mathfrak{P}_M$, the corresponding results of [3] and [4] will be deduced.

The symbols $c_j, c_j(p, \dots)$ ($j = 1, 2, \dots$) occurring below will mean some positive absolute constants or positive constants depending only on the indicated parameters $p, \dots$. The integral part of a real number u will be signified by $[u]$.

2. Main results

Let us begin with the following

THEOREM 1. *Suppose that the function f of class $\mathfrak{P}_M$ is continuous at a fixed point $x \in (0, \infty)$. Let A, B be arbitrary positive numbers satisfying the inequalities $x \leq A, 2A - x \leq B$. Then, putting*

$$\mu = [x\sqrt{n/A}], \quad m = [(2A - x)\sqrt{n/b}],$$

we have

$$\begin{aligned} (3) \quad |S_n[f](x) - f(x)| &\leq \frac{3M}{\sqrt{nA}} (4A)^{4A\alpha} \left(\frac{x}{2A} \exp\left(1 - \frac{x}{2A}\right) \right)^{2nA} \\ &\quad + c_1(A) \left\{ \sum_{j=1}^{\mu-1} \frac{v(j, f; x - jA/\sqrt{n}, x)}{j^3} + \frac{v(\mu-1, f; 0, x)}{(\mu-1)^2} \right\} \\ &\quad + c_1(B) \left\{ \sum_{j=1}^{m-1} \frac{v(j, f; x, x + jB/\sqrt{n})}{j^3} + \frac{v(m-1, f; x, 2A)}{(m-1)^2} \right\} \end{aligned}$$

whenever $nx^2 \geq 4A^2$, $n(2A - x)^2 \geq 4B^2$ and $n \geq N(A, \alpha)$, where $N(A, \alpha)$ denotes some sufficiently large positive number depending on A and α .

Proof. Clearly,

$$\begin{aligned} S_n[f](x) - f(x) &= \left(\sum_{k \geq 2nA} + \sum_{nx \leq k < 2nA} + \sum_{k < nx} \right) \left\{ f\left(\frac{k}{n}\right) - f(x) \right\} p_k(nx) \\ &= \sum_1 + \sum_2 + \sum_3, \quad \text{say.} \end{aligned}$$

In view of inequality (1),

$$|\sum_1| \leq 2M \sum_{k \geq 2nA} \left(\frac{k}{n} \right)^{\alpha k/n} p_k(nx).$$

Arguing similarly as in [4] (p. 397) we easily get

$$(4) \quad |\sum_1| \leq \frac{3M}{\sqrt{nA}} (4A)^{4A\alpha} \left(\frac{x}{2A} \exp\left(1 - \frac{x}{2A}\right) \right)^{2nA}$$

for all integers $n \geq N(A, \alpha)$, with

$$N(A, \alpha) = \max \left\{ \alpha, \frac{1}{2A}, \frac{\alpha \log 2eA}{\log(4/3)} \right\}.$$

In order to evaluate the term $\sum_2$, let us introduce the points

$$t_j = x + jB/\sqrt{n} \quad (j = 1, 2, \dots, m; m = [(2A - x)\sqrt{n}/B])$$

and let us write, for $m \geq 2$,

$$\begin{aligned} \sum_2 &= \sum_{x \leq k/n < t_1} \left\{ f\left(\frac{k}{n}\right) - f(x) \right\} p_k(nx) + \sum_{j=1}^{m-1} \{f(t_j) - f(x)\} \sum_{t_j \leq k/n < t_{j+1}} p_k(nx) \\ &\quad + \sum_{j=1}^{m-1} \sum_{t_j \leq k/n < t_{j+1}} \left\{ f\left(\frac{k}{n}\right) - f(t_j) \right\} p_k(nx) \\ &\quad + \sum_{t_m \leq k/n < 2A} \left\{ f\left(\frac{k}{n}\right) - f(x) \right\} p_k(nx) \\ &= I_1 + I_2 + I_3 + I_4, \quad \text{say.} \end{aligned}$$

Obviously,

$$|I_1| \leq v(1, f; x, t_1) \sum_{k=0}^{\infty} p_k(nx) = v(1, f; x, t_1).$$

Applying twice the Abel transformation and using the inequality

$$\sum_{k \geq m} p_k(nx) \leq \frac{x}{n(t-x)^2} \quad (x < t \leq 2A)$$

(see [3], Lemma 3), we obtain

$$\begin{aligned} |I_2| &= \left| \{f(t_1) - f(x)\} \sum_{t_1 \leq k/n < t_m} p_k(nx) + \sum_{j=1}^{m-2} \{f(t_{j+1}) - f(t_j)\} \sum_{t_{j+1} \leq k/n < t_m} p_k(nx) \right| \\ &\leq \frac{x}{B^2} \left\{ |f(t_1) - f(x)| + \sum_{j=1}^{m-2} |f(t_{j+1}) - f(t_j)| \frac{1}{(j+1)^2} \right\} \\ &= \frac{x}{B^2} \left\{ |f(t_1) - f(x)| + \frac{1}{(m-1)^2} \sum_{j=1}^{m-2} |f(t_{j+1}) - f(t_j)| \right. \\ &\quad \left. + \sum_{j=1}^{m-3} \sum_{l=1}^j |f(t_{l+1}) - f(t_l)| \left(\frac{1}{(j+1)^2} - \frac{1}{(j+2)^2} \right) \right\} \\ &\leq \frac{1}{B} \left\{ v(1, f; x, t_1) + \frac{v(m-2, f; t_1, t_{m-1})}{(m-1)^2} + 2 \sum_{j=1}^{m-3} \frac{v(j, f; t_1, t_{j+1})}{(j+1)^3} \right\} \\ &\leq \frac{1}{B} \left\{ 2 \sum_{j=1}^{m-2} \frac{v(j, f; x, t_j)}{j^3} + \frac{v(m-1, f; x, 2A)}{(m-1)^2} \right\}. \end{aligned}$$

Next

$$\begin{aligned}
 |I_3| &\leq \sum_{j=1}^{m-1} v(1, f; t_j, t_{j+1}) \sum_{t_j \leq k/n < t_{j+1}} p_k(nx) \\
 &\leq \frac{x}{B^2} \sum_{j=1}^{m-1} \frac{v(1, f; t_j, t_{j+1})}{j^2} \\
 &\leq \frac{1}{B} \left\{ \frac{1}{(m-1)^2} \sum_{j=1}^{m-1} v(1, f; t_j, t_{j+1}) \right. \\
 &\quad \left. + \sum_{j=1}^{m-2} \sum_{l=1}^j v(1, f; t_l, t_{l+1}) \left(\frac{1}{j^2} - \frac{1}{(j+1)^2} \right) \right\} \\
 &\leq \frac{1}{B} \left\{ \frac{v(m-1, f; t_1, t_m)}{(m-1)^2} + 6 \sum_{j=1}^{m-2} \frac{v(j, f; t_1, t_{j+1})}{(j+1)^3} \right\} \\
 &\leq \frac{1}{B} \left\{ \frac{v(m-1, f; x, 2A)}{(m-1)^2} + 6 \sum_{j=2}^{m-1} \frac{v(j, f; x, t_j)}{j^3} \right\}.
 \end{aligned}$$

Finally,

$$|I_4| \leq v(1, f; x, 2A) \sum_{t_m \leq k/n < 2A} p_k(nx) \leq \frac{1}{B} \frac{v(m-1, f; x, 2A)}{(m-1)^2}.$$

Collecting the results and putting $c_1(B) = (2 + \max\{B, 6\})/B$ we get

$$\begin{aligned}
 (5) \quad |\Sigma_2| &\equiv \left| \sum_{nx \leq k < 2nA} \left\{ f\left(\frac{k}{n}\right) - f(x) \right\} p_k(nx) \right| \\
 &\leq c_1(B) \sum_{j=1}^{m-1} \frac{v(j, f; x, x+jB/\sqrt{n})}{j^3} + \frac{3}{B} \frac{v(m-1, f; x, 2A)}{(m-1)^2},
 \end{aligned}$$

provided that $x \leq A$, $n(2A-x)^2 \geq 4B^2$, $2A-x \leq B$.

Analogously, it can be verified that

$$|\Sigma_3| \leq c_1(A) \sum_{j=1}^{\mu-1} \frac{v(j, f; x-jA/\sqrt{n}, x)}{j^3} + \frac{3}{A} \frac{v(\mu-1, f; 0, x)}{(\mu-1)^2}$$

whenever $x \leq A$, $nx^2 \geq 4A^2$.

Thus, the proof is completed.

The right-hand side of the inequality (3) converges to zero as $n \rightarrow \infty$.

Indeed, for an arbitrary $r \in \mathbb{N}$,

$$(6) \quad \left(\frac{x}{2A} \exp\left(1 - \frac{x}{2A}\right) \right)^{2nA} \leq c_2(A, r) \frac{1}{n^r} \quad (0 < x \leq A)$$

(cf. [4], p. 397). Moreover,

$$\frac{v(\mu-1, f; 0, x)}{(\mu-1)^2} + \frac{v(m-1, f; x, 2A)}{(m-1)^2} \leq 2M(2A)^{2A\alpha} \left(\frac{1}{\mu-1} + \frac{1}{m-1} \right),$$

$$\sum_{j=1}^{\mu-1} \frac{v(j, f; x-jA/\sqrt{n}, x)}{j^3} \leq \sum_{j=1}^{\mu-1} \frac{v(1, f; x-jA/\sqrt{n}, x)}{j^2}$$

$$\leq \frac{4}{\mu} \sum_{k=1}^{\mu} v\left(1, f; x - \frac{x}{k}, x\right)$$

and

$$\sum_{j=1}^{m-1} \frac{v(j, f; x, x+jB/\sqrt{n})}{j^3} \leq \frac{4}{m} \sum_{k=1}^m v\left(1, f; x, x + \frac{2A-x}{k}\right).$$

Since f is continuous at x , we have

$$\lim_{\delta \rightarrow 0+} v(1, f; x-\delta, x) = 0, \quad \lim_{\delta \rightarrow 0+} v(1, f; x, x+\delta) = 0,$$

and the desired assertion follows.

Now, we shall present an estimate of the rate of convergence of $S_n[f](x)$ at discontinuity points of f , mainly.

THEOREM 2. Let $f \in \mathfrak{B}_M^{\alpha}$ and let at a fixed $x \in (0, \infty)$ the one-sided limits $f(x \pm 0)$ exist. Write

$$g_x(t) = \begin{cases} f(t) - f(x-0) & \text{when } 0 \leq t < x, \\ 0 & \text{when } t = x, \\ f(t) - f(x+0) & \text{when } x < t < \infty. \end{cases}$$

Then, for every integer $n \geq \max\{4, N(x, \alpha)\}$,

$$(7) \quad |S_n[f](x) - \frac{1}{2}\{f(x+0) + f(x-0)\}|$$

$$\leq \frac{3M}{\sqrt{nx}} (4x)^{4\alpha} \left(\frac{e}{4}\right)^{nx}$$

$$+ c_1(x) \left\{ \sum_{j \neq 1}^{m-1} \frac{v(j, g_x; x-jx/\sqrt{n}, x) + v(j, g_x; x, x+jx/\sqrt{n})}{j^3} \right.$$

$$\left. + \frac{v(m-1, g_x; 0, x) + v(m-1, g_x; x, 2x)}{(m-1)^2} \right\}$$

$$+ \frac{c_3}{\sqrt{nx}} \left\{ |f(x+0) - f(x-0)| + x \left| f(x) - \frac{f(x+0) + f(x-0)}{2} \right| \right\},$$

with $m = [\sqrt{n}]$; the number x is equal to 1 in the case nx is an integer and is equal to 0 otherwise.

Proof. It is easy to observe that

$$\begin{aligned} S_n[f](x) - \frac{1}{2} \{f(x+0) + f(x-0)\} \\ = S_n[g_x](x) + H + \frac{1}{2} (f(x+0) - f(x-0)) \left\{ \sum_{k > nx} p_k(nx) - \sum_{k < nx} p_k(nx) \right\}, \end{aligned}$$

where the term H equals $(f(x) - \frac{1}{2} \{f(x+0) + f(x-0)\}) p_{nx}(nx)$ if nx is an integer and 0 otherwise; in either case, by Stirling's formula, we have

$$|H| \leq |f(x) - \frac{1}{2} \{f(x+0) + f(x-0)\}| (2\pi nx)^{-1/2} \quad (n \in \mathbb{N}).$$

Further, in view of Lemma 2 in [3],

$$\left| \sum_{k > nx} p_k(nx) - \sum_{k < nx} p_k(nx) \right| \leq c_4 \sqrt{nx}.$$

Thus, our assertion follows by applying Theorem 1 (with $A = B = x$) to the term

$$S_n[g_x](x) = S_n[g_x](x) - g_x(x).$$

Since the function g_x is continuous at the point x , the right-hand side of the inequality (7) converges to zero as $n \rightarrow \infty$, and this gives (2).

3. Corollaries

Suppose that a function f of class $\mathfrak{B}_M^r$ is continuous on the interval $\langle 0, \infty \rangle$. Denote by $\omega_{2A}(\delta; f)$, $\delta \geq 0$, the modulus of continuity of f in the interval $\langle 0, 2A \rangle$. It can easily be observed (see [1], [5]) that for every interval $\langle a, b \rangle \subset \langle 0, 2A \rangle$,

$$(8) \quad v(j, f; a, b) \leq 2j\omega_{2A}((b-a)/j; f) \quad (j \in \mathbb{N}).$$

Applying the inequalities (6), (8) and the well-known properties of the modulus of continuity we get from Theorem 1 (with $B = 2A$) the estimate

$$|S_n[f](x) - f(x)| \leq c_5(A, \alpha) \left\{ \omega_{2A}\left(\frac{A}{\sqrt{n}}; f\right) + \frac{M}{n} \right\}$$

for sufficiently large n and all x such that $2A/\sqrt{n} \leq x \leq A$. In the case $0 \leq x \leq 2A/\sqrt{n}$ we have

$$\begin{aligned} |S_n[f](x) - f(x)| &\leq \omega_{2A}\left(\frac{2A}{\sqrt{n}}; f\right) + \left| \sum_{k > nx} \left\{ f\left(\frac{k}{n}\right) - f(x) \right\} p_k(nx) \right| \\ &\leq c_6(A, \alpha) \left\{ \omega_{2A}\left(\frac{A}{\sqrt{n}}; f\right) + \frac{M}{n} \right\}, \end{aligned}$$

by the inequalities (4), (5), (6) and (8).

Thus, we obtain

COROLLARY 1. If f is continuous on $\langle 0, \infty \rangle$, satisfies the condition (1), and if $f \neq \text{const.}$ on the interval $\langle 0, 2A \rangle$, then

$$S_n[f](x) - f(x) = O(\omega_{2A}(1/\sqrt{n}; f)) \quad \text{as } n \rightarrow \infty,$$

uniformly in $x \in \langle 0, A \rangle$.

This result was first stated by Gróf and Hermann ([4], Th. 2).

Let Φ be a continuous, convex and strictly increasing function on the interval $\langle 0, \infty \rangle$, such that $\Phi(0) = 0$. Suppose that the function f is of bounded Φ -variation on every finite subinterval of $\langle 0, \infty \rangle$. Then, given any $x \in (0, \infty)$, the function g_x defined in Theorem 2 is of bounded Φ -variation on the intervals $\langle 0, x \rangle$ and $\langle x, 2x \rangle$. Moreover, for every positive integer j and every interval $\langle a, b \rangle$ contained in $\langle 0, x \rangle$ or $\langle x, 2x \rangle$,

$$v(j, g_x; a, b) \leq j\Phi^{-1}\left(\frac{1}{j}V_\Phi(g_x; a, b)\right),$$

where the symbol Φ^{-1} is used for the function inverse to Φ and $V_\Phi(g_x; a, b)$ denotes the total Φ -variation of g_x on $\langle a, b \rangle$ (see [2], p. 537).

Retaining the notation $m = [\sqrt{n}]$ we easily observe that

$$\sum_{j=1}^{m-1} \frac{v(j, g_x; x - jx/\sqrt{n}, x)}{j^3} \leq \frac{9\sqrt{2}}{\sqrt{n}} \sum_{k=1}^{m-1} \frac{1}{\sqrt{k}} \Phi^{-1}\left(\sqrt{\frac{k}{n}} V_\Phi\left(g_x; x - \frac{x}{\sqrt{k}}, x\right)\right)$$

and

$$\frac{v(m-1, g_x; 0, x)}{(m-1)^2} \leq \frac{9}{\sqrt{n}} \Phi^{-1}\left(\frac{1}{\sqrt{n}} V_\Phi(g_x; 0, x)\right)$$

(cf. the considerations in [5], p. 292). Analogous estimates for two other terms in the inequality (7), corresponding to the interval $\langle x, 2x \rangle$, can be obtained, too. Hence, Theorem 2 leads to

COROLLARY 2. If a function f belonging to the class $\mathfrak{V}_M$ is of bounded Φ -variation on every finite subinterval of $\langle 0, \infty \rangle$, then, for every $x \in (0, \infty)$ and for all integers $n \geq \max\{4, N(x, \alpha)\}$, we have

$$\begin{aligned} & |S_n[f](x) - \frac{1}{2}\{f(x+0) + f(x-0)\}| \\ & \leq \frac{3M}{\sqrt{nx}} (4x)^{4\alpha} \left(\frac{e}{4}\right)^{\alpha x} + \frac{c_7(x)}{\sqrt{n}} \sum_{k=1}^n \frac{1}{\sqrt{k}} \left\{ \Phi^{-1}\left(\sqrt{\frac{k}{n}} V_\Phi\left(g_x; x - \frac{x}{\sqrt{k}}, x\right)\right) \right. \\ & \quad \left. + \Phi^{-1}\left(\sqrt{\frac{k}{n}} V_\Phi\left(g_x; x, x + \frac{x}{\sqrt{k}}\right)\right) \right\} \\ & \quad + \frac{c_3}{\sqrt{nx}} \{|f(x+0) - f(x-0)| + \alpha |f(x) - \frac{1}{2}(f(x+0) + f(x-0))|\}, \end{aligned}$$

with $c_7(x) = c_8(1+x)/x$. The right-hand side of this inequality converges to zero as $n \rightarrow \infty$.

In the special case $\Phi(u) = u$ ($u \geq 0$) the last estimate is equivalent to the one obtained by Fuhua Cheng ([3]).

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