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# SUMMABILITY OF GENERALIZED FOURIER SERIES AND THEIR CONJUGATE SERIES

K. S. KAZARIAN

*Institute of Mathematics, Academy of Sciences of the Armenyan S.S.R., Erevan, U.S.S.R.*

We will say that a weight function $w(x) \geqslant 0$ *has a singularity* at $x_0$ if $w(x)^{-1}$ is Lebesgue integrable in no neighbourhood of $x_0$. The singularity will be called *of finite order* if $|x-x_0|^n w(x)^{-1}$ is Lebesgue integrable in some neighbourhood of $x_0$ for some $n > 0$. If $w$ has a singularity at one point at least, then clearly every $L^p(w)$ space, $1 \leqslant p < \infty$, contains a function which is not Lebesgue integrable. Therefore, as a rule, the study of the usual integral operators or families of operators generated by some kernels is impossible in such weighted spaces. In [5] we announced the results which show that for weight functions with singularities of finite order in a finite number of points, it is possible to modify the well-known classical kernels in such a way that the new kernels often play the same role in the $L^p$ spaces with those weights as do the classical kernels in the usual $L^p$ spaces. This idea arose in the study of the basisness, in various senses, of subsystems of the classical orthonormal systems in weighted $L^p$ spaces. The point is that if we are given a function $w$ with a finite number of singularity points of finite order, then the classical orthonormal systems are no more minimal in $L^p(w)$ spaces, but we may obtain a closed minimal system in $L^p(w)$, $1 \leqslant p < \infty$, by deleting a finite number of functions from the given system. Also, it is very important that we can write down the unique biorthogonal system. Now, considering the biorthogonal expansion of an $L^p(w)$ function and forming the means for various summability methods, we arrive at the truncated kernels mentioned above.

Another approach is also possible: we forget about the weighted functions, the closedness and minimality of subsystems, etc. and we just assume that there is given a measurable function $f$ which may not be Lebesgue integrable, but whose product $f \cdot T$ with some trigonometric polynomial $T$ is integrable. Further, basing on the zeros of $T$ and their multiplicities, we define the coefficients of the expansion of the given function to be equal to the coefficients of the biorthogonal expansion described above. All the necessary precise formulations and the notation are given in [6]. We also consider there the problem of Abel–Poisson summability of such generalized Fourier series with respect to the trigonometric system as well as the almost everywhere convergence of generalized Fourier–Haar systems. Note that in [6] the generalized Fourier–Haar series for functions with infinitely many singularity points were also considered.

The term "generalized Fourier series" is taken from A. Yu. Petrovich's work [9],(1) where a generalized Fourier series is defined for functions with a singularity at a single point, and where the problem of $(C, \alpha)$ summability of such series is considered. Observe that earlier R. Boas [1] studied the coefficients defined by

$$a_n = \pi^{-1} \int_{-\pi}^{\pi} f(t)[\cos nt - 1]\,dt, \qquad b_n = \pi^{-1} \int_{-\pi}^{\pi} f(t)\sin nt\,dt$$

for $f$ odd and even respectively. A. Yu. Petrovich [9], basing on R. Boas' works, subtracted, in defining the coefficients, from $\cos nt$ and $\sin nt$ the first terms of their Taylor series at $x_0 = 0$. In this connection, let us also mention M. I. D'yachenko's papers [2], [3].

We now turn to the necessary formulations and definitions. Let $X = \{x_j\}_{j=1}^{s}$ be an arbitrary collection of distinct points in $[-\pi, \pi)$, and let $\alpha = (\alpha_1, \ldots, \alpha_s)$ be a vector of positive integers. We let $A = \sum_{j=1}^{s} \alpha_j$ and, for $X$ and $\alpha$ given, we define the collection of fundamental interpolating polynomials $T_{j,\lambda}(x)$ $(1 \leqslant j \leqslant s,\ 0 \leqslant \lambda \leqslant \alpha_j - 1)$ as follows:

If $A = 2m+1$ $(m = 0, 1, \ldots)$, then $T_{j,\lambda}$ is the trigonometric polynomial of order at most $m$ which satisfies

$$T_{j,\lambda}^{(h)}(x_i) = \delta_{ij}\delta_{h\lambda} \qquad (0 \leqslant h,\ \lambda \leqslant \alpha_j - 1,\ 1 \leqslant i, j \leqslant s), \tag{1}$$

where $T^{(h)}$ denotes the $h$th derivative of $T$, $T^{(0)} = T$ and $\delta_{ij}$ is the Kronecker symbol.

If $A = 2m$ $(m = 1, 2, \ldots)$, then in order to get the uniqueness of the $T_{j,\lambda}$, we impose, besides (1), the following condition: either the order of $T_{j,\lambda}$ is strictly less than $m$, or the ratio of the leading coefficients of these polynomials divided by the similar ratio for the polynomial

$$\omega(x) = \prod_{j=1}^{s} \sin^{\alpha_j} \tfrac{1}{2}(x - x_j) \tag{2}$$

is equal to $-1$. If the coefficient of $\cos mx$ or $\sin mx$ in $\omega(x)$ is zero, then so are, by definition, the coefficients of $\sin mx$ and $\cos mx$ respectively in $T_{j,\lambda}$.

(1) Earlier this term was used by A. Zygmund ([13], vol. 1, p. 84).

We are now in a position to define the generalized coefficients in the form of some Lebesgue integrals, the only assumption on $f$ being that the integrals appearing below exist. We put

$$
(3)\qquad
\begin{aligned}
a_n(f) &= \pi^{-1} \int_{-\pi}^{\pi} f(t)\Big[\cos nt - \sum_{j=1}^{s} \sum_{\lambda=0}^{\alpha_j-1} (\cos nt)^{(\lambda)}_{t=x_j} T_{j,\lambda}(t)\Big]\, dt,\\
b_n(f) &= \pi^{-1} \int_{-\pi}^{\pi} f(t)\Big[\sin nt - \sum_{j=1}^{s} \sum_{\lambda=0}^{\alpha_j-1} (\sin nt)^{(\lambda)}_{t=x_j} T_{j,\lambda}(t)\Big]\, dt.
\end{aligned}
$$

It is easy to observe that $a_n = b_n = 0$ for $0 \leqslant n \leqslant [A/2]$, where $[y]$ is the integer part of $y$. If $X$ is an empty set, we recover the classical notation, where for the Fourier series and its conjugate series we have the classical theorems of M. Riesz [10] and A. Zygmund [11] respectively; therefore in the sequel we assume $X$ to be nonempty. Since in that case always $a_0 = b_0 = 0$, we will write the generalized Fourier series of a function $f$ in the form

$$
(4)\qquad \sum_{n=1}^{\infty} [a_n(f)\cos nx + b_n(f)\sin nx].
$$

In [6] it was shown that the series (4) is almost everywhere summable by the Abel–Poisson method to $f$, i.e.

$$
\sum_{n=1}^{\infty} [a_n(f)\cos nx + b_n(f)\sin nx]\, r^n \to f(x) \text{ a.e.} \quad \text{as } r \to 1-.
$$

Note that in [6] we could have obtained the results describing the behaviour of the series (4) at individual points. Since the formulations of these results are analogous to the classical case (with the exceptional points $x_j$, $1 \leqslant j \leqslant s$, being of course taken into account), also in the present paper we do not give all possible formulations.

Along with the trigonometric series (4) it is natural to consider its conjugate series

$$
(5)\qquad \sum_{n=1}^{\infty} [a_n(f)\sin nx - b_n(f)\cos nx].
$$

In this paper we obtain the results on the summability almost everywhere of the series (4) and (5) by the Cesàro $(C, \beta)$ methods, $\beta > 0$, and we prove these results to be final with respect to the scale of Cesàro summability methods. Especially we would like to draw attention to the explicit form of the limit of a conjugate series, which will be called the *conjugate function*.

Recall the definition of the $(C, \beta)$ means of a series $\sum_{j=0}^{\infty} u_j$ (see [13], Vol. 1, p. 130):

$$
(6)\qquad \sigma_n^\beta = \frac{S_n^\beta}{A_n^\beta} = \frac{1}{A_n^\beta} \sum_{\nu=0}^{n} A_{n-\nu}^{\beta} u_\nu = \frac{1}{A_n^\beta} \sum_{\nu=0}^{n} A_{n-\nu}^{\beta-1} S_\nu,
$$

where the numbers $A_n^\beta$ $(n = 0, 1, \ldots)$ are determined by the formula

$$(7) \qquad \sum_{n=0}^{\infty} A_n^\beta x^n = (1-x)^{-\beta-1}, \quad A_n^\beta = \binom{n+\beta}{n} = \frac{n^\beta}{\Gamma(\beta+1)} \quad (\beta > 0),$$

and $S_n = \sum_{k=0}^{n} u_k$. Recall also the definitions of the usual kernels of the trigonometric system and their conjugate kernels for the $(C, \beta)$ summability methods (see [13], Vol. 1, pp. 157–159):

$$(8) \qquad K_n^\beta(t) = \frac{1}{A_n^\beta} \sum_{\nu=0}^{n} A_{n-\nu}^{\beta-1} D_\nu(t), \quad \text{where } D_\nu(t) = \tfrac{1}{2} + \sum_{j=1}^{\nu} \cos jt,$$

$$(9) \qquad \tilde{K}_n^\beta(t) = \frac{1}{A_n^\beta} \sum_{\nu=0}^{n} A_{n-\nu}^{\beta-1} \tilde{D}_\nu(t),$$

where

$$\tilde{D}_\nu(t) = \sum_{j=1}^{\nu} \sin jt = \frac{\cos \frac{1}{2}t - \cos(\nu+\frac{1}{2})t}{2\sin\frac{1}{2}t},$$

$$(10) \qquad \tilde{K}_n^\beta(t) = \tfrac{1}{2}\cot\tfrac{1}{2}t - \frac{1}{A_n^\beta}\sum_{\nu=0}^{n} A_{n-\nu}^{\beta-1} \frac{\cos(\nu+\frac{1}{2})t}{2\sin\frac{1}{2}t} = \tfrac{1}{2}\cot\tfrac{1}{2}t - H_n^\beta(t).$$

Using (6), (8) and (9) it is easy to check that the $(C, \beta)$ kernels for the series (4) and (5) with coefficients defined by (3) have the following form:

$$(11) \qquad K_{n,X,\alpha}^\beta(x, t) = K_n^\beta(t-x) - \sum_{j=1}^{s}\sum_{\lambda=0}^{\alpha_j-1} (K_n^\beta(x_j - x))^{(\lambda)} T_{j,\lambda}(t),$$

$$(12) \qquad \tilde{K}_{n,X,\alpha}^\beta(x, t) = \tilde{K}_n^\beta(t-x) - \sum_{j=1}^{s}\sum_{\lambda=0}^{\alpha_j-1} (\tilde{K}_n^\beta(x_j - x))^{(\lambda)} T_{j,\lambda}(t).$$

The $(C, \beta)$ means of these series are

$$(13) \qquad \sigma_{n,X,\alpha}^\beta(f, x) = \pi^{-1} \int_{-\pi}^{\pi} f(t) K_{n,X,\alpha}^\beta(x, t)\, dt,$$

$$(14) \qquad \tilde{\sigma}_{n,X,\alpha}^\beta(f, x) = \pi^{-1} \int_{-\pi}^{\pi} f(t) \tilde{K}_{n,X,\alpha}^\beta(x, t)\, dt.$$

To obtain results which are final in the standard sense, we have to impose on $f$ certain conditions depending not on the discrete parameters $\alpha_j$ $(1 \leqslant j \leqslant s)$, where the $\alpha_j$ are positive integers, but on certain continuous parameters $\beta_j$ $(1 \leqslant j \leqslant s)$, where the $\beta_j$ are positive numbers. We will assume $f$ to be a $2\pi$-periodic function such that

$$(15) \qquad f(\cdot) \prod_{j=1}^{s} \sin^{\beta_j} \tfrac{1}{2}(\cdot - x_j) \in L_{[-\pi,\pi]}.$$

When defining the generalized Fourier series of $f$, we will take the positive integers $\alpha_j$ $(1 \leqslant j \leqslant s)$ from the conditions

$$\alpha_j - 1 < \beta_j \leqslant \alpha_j \quad (1 \leqslant j \leqslant s). \tag{16}$$

It follows from (15) and (16) that the coefficients $a_n(f)$ and $b_n(f)$ defined by (2) and (3) exist, so it makes sense to consider the summability of the series (4) and (5). We have the following

THEOREM 1. *Let $f$ be a measurable function satisfying* (15), *where $\beta_j > 0$ $(1 \leqslant j \leqslant s)$. Suppose moreover that the positive integers $\alpha_j$ $(1 \leqslant j \leqslant s)$ are determined by* (16), *and let $\beta = \max_{1 \leqslant j \leqslant s} \beta_j$. Then*

$$\sigma^{\beta}_{n,X,\alpha}(f, x) \to f(x) \quad \textit{a.e. on } [-\pi, \pi] \textit{ as } n \to +\infty,$$

$$\tilde{\sigma}^{\beta}_{n,X,\alpha}(f, x) \to \tilde{f}(x) \quad \textit{a.e. on } [-\pi, \pi] \textit{ as } n \to +\infty,$$

*where $\sigma^{\beta}_{n,X,\alpha}(f, x)$, $\tilde{\sigma}^{\beta}_{n,X,\alpha}(f, x)$ are defined by* (13), (14) (*see also* (8)–(12)) *and*

$$\tilde{f}(x) = \text{p.v.} \ -\frac{1}{\pi}\frac{1}{\omega(x)} \int_{-\pi}^{\pi} f(t) \frac{\omega(t)}{2\sin\frac{1}{2}(t-x)}\, dt \tag{17}$$

$$:= \lim_{\varepsilon \to 0+} \ -\frac{1}{\pi}\frac{1}{\omega(x)} \Big(\int_{-\pi}^{-\varepsilon} + \int_{\varepsilon}^{\pi}\Big) f(x+t) \frac{\omega(x+t)}{2\sin\frac{1}{2}t}\, dt$$

*if $\Lambda = 2m+1$ $(m = 0, 1, \ldots)$, whereas*

$$\tilde{f}(x) = \text{p.v.} \ -\frac{1}{\pi}\frac{1}{\omega(x)} \int_{-\pi}^{\pi} f(t) \frac{\omega(t)}{2\tan\frac{1}{2}(t-x)}\, dt \tag{18}$$

*if $\Lambda = 2m$ $(m = 1, 2, \ldots)$. The function $\omega$ is defined by* (2).

To prove Theorem 1, we need a lemma (see [13], Vol. 2, pp. 94–96, 100, and also [4], [12]).

LEMMA 1. *Let $K^{\beta}_n(t)$ and $H^{\beta}_n(t)$, $\beta > 0$, be defined by* (8) *and* (10), *and let $r$ be a nonnegative integer with $0 \leqslant \beta \leqslant r+1$. Then*

$$\left|\frac{d^r}{dt^r} K^{\beta}_n(t)\right| \leqslant \frac{Cn^{r+1}}{(1+nt)^{\beta+1}} \quad (0 \leqslant t \leqslant \pi), \tag{19}$$

$$\left|\frac{d^r}{dt^r} H^{\beta}_n(t)\right| \leqslant \frac{Cn^{r+1}}{(1+nt)^{\beta+1}} \quad (0 \leqslant t \leqslant \pi). \tag{20}$$

We have the following

PROPOSITION 1. *Let $r$ be a nonnegative integer. Then for arbitrary $\delta > 0$ we have for $\beta > r$*

$$\max\left\{\frac{d^r}{dt^r} K^{\beta}_n(t), \frac{d^r}{dt^r} H^{\beta}_n(t)\right\} \to 0 \ \textit{uniformly on } [-\pi, \pi]\setminus(-\delta, \delta)$$

$$\textit{as } n \to +\infty,$$

*and if $\beta > r+1$, then*

$$(21)\qquad n\left|\frac{d^r}{dt^r}K_n^\beta(t)\right| \leqslant C, \quad n\left|\frac{d^r}{dt^r}H_n^\beta(t)\right| \leqslant C \quad \textit{for } t\in[-\pi,\pi]\setminus(-\delta,\delta),$$

*where $C = C(\delta, \beta, r)$ is independent of $n$.*

The first assertion of the proposition follows immediately from Lemma 1. To show (21), we apply the proof of that lemma (see [13], Vol. 2, pp. 94–95) to obtain

$$\max\left[\left|\frac{d^r}{dt^r}K_n^\beta(t)\right|, \left|\frac{d^r}{dt^r}H_n^\beta(t)\right|\right]$$
$$\leqslant \frac{C}{n^\beta}\left[\sum_{j=1}^{l}\frac{n^{\beta-j}}{t^{j+1+r}} + \sum_{\mu=0}^{r}\frac{n^\mu}{t^{\beta+1+r-\mu}} + \sum_{\mu=0}^{r}\frac{n^{\beta+\mu-l}}{t^{s+r-\mu+1}}\right],$$

where $l$ is a certain positive integer satisfying only $l > \beta+r$. Hence (21) follows easily.

We will also need the following lemma.

LEMMA 2. *In the notation of the present paper, the following identities hold*:

$$(22)\qquad \cot\tfrac{1}{2}(t-x) - \sum_{j=1}^{s}\sum_{\lambda=0}^{\alpha_j-1}\frac{d^\lambda}{dt^\lambda}\cot\tfrac{1}{2}(t-x)\Big|_{t=x_j} T_{j,\lambda}(t)$$
$$= \begin{cases}\omega(x)^{-1}\omega(t)\cot\frac{1}{2}(t-x) & \textit{for } A = 2m\ (m=1,2,\ldots),\\ \omega(x)^{-1}\omega(t)[\sin\frac{1}{2}(t-x)]^{-1} & \textit{for } A = 2m+1\ (m=0,1,\ldots).\end{cases}$$

*Proof of Theorem* 1. We set

$$(23)\qquad I_\delta = \bigcup_{l=-1}^{1}\bigcup_{j=1}^{s}(x_j+2\pi l-\delta, x_j+2\pi l+\delta)\cap[-\pi,\pi), \qquad I_\delta^c = [-\pi,\pi)\setminus I_\delta.$$

Write $f = f_\delta + f_\delta^c$, where

$$f_\delta(x) = \begin{cases} f(x) & \text{for } x\in I_\delta^c,\\ 0 & \text{for } x\in I_\delta.\end{cases}$$

Clearly, Theorem 1 will be proved as soon as we show the convergence of $\sigma_{n,X,\alpha}^\beta(f,x)$ and $\tilde{\sigma}_{n,X,\alpha}^\beta(f,x)$ almost everywhere on $I_{2\delta}^c$ for sufficiently small $\delta > 0$. By M. Riesz's theorem [10], from (11), (13) and Proposition 1 we obtain

$$(24)\qquad \sigma_{n,X,\alpha}^\beta(f,x)\to f_\delta(x) \quad \text{a.e. on } I_\delta^c \text{ as } n\to+\infty.$$

For the conjugate series we have to apply A. Zygmund's theorem [11] (see also [13], Vol. 1, pp. 157–159), Proposition 1, the equalities (10), (12), (14), and Lemma 2. It then follows immediately that

$$\tilde{\sigma}^{\beta}_{n,X,\alpha}(f, x) \to \tilde{f}_{\delta}(x) \quad \text{a.e. on } I^c_{\delta} \text{ as } n \to +\infty,$$

where $\tilde{f}_{\delta}$ is defined by (17) or (18), with $f$ replaced by $f_{\delta}$, according as $\Lambda$ is even or odd. Since

$$\tilde{f}_{\delta}(x) \to \tilde{f}(x) \quad \text{a.e. on } [-\pi, \pi) \text{ as } \delta \to 0+,$$

the proof of Theorem 1 will be complete if we show that

$$\sigma^{\beta}_{n,X,\alpha}(f^c_{\delta}, x) \to 0 \quad \text{uniformly on } I^c_{2\delta} \text{ as } n \to +\infty \tag{25}$$

and for any fixed sufficiently small $\delta_0 > 0$

$$\tilde{\sigma}^{\beta}_{n,X,\alpha}(f^c_{\delta}, x) \to 0 \quad \text{uniformly on } I^c_{\delta_0} \text{ as } n \to +\infty, \delta \to 0+. \tag{26}$$

By (14), (12), (10) and Lemma 2 we can write for $x \in I^c_{\delta_0}$

$$\tilde{\sigma}^{\beta}_{n,X,\alpha}(f^c_{\delta}, x) = \pi^{-1} \int_{I^c_{\delta}} H^{\beta}_{n,X,\alpha}(x, t) f(t)\, dt + \tilde{f}^c_{\delta}(x), \tag{27}$$

$$H^{\beta}_{n,X,\alpha}(x, t) = H^{\beta}_{n}(x-t) - \sum_{j=1}^{s} \sum_{\lambda=0}^{\alpha_j - 1} (H^{\beta}_{n}(x-x_j))^{(\lambda)}\, T_{j,\lambda}(t), \tag{28}$$

and $\tilde{f}^c_{\delta}(x)$ is defined by (18) or (19) according as $\Lambda$ is even or odd, with $f$ in the integrand replaced by $f^c_{\delta}$. From the definition of $\tilde{f}^c_{\delta}(x)$ and the absolute continuity of the Lebesgue integral it easily follows that $\tilde{f}^c_{\delta}(x) \to 0$ uniformly on $I^c_{\delta_0}$ as $\delta \to 0+$. Hence (27) shows immediately that the proof of (26) will be complete as soon as we show that

$$\int_{I^c_{\delta}} f(t) H^{\beta}_{n,X,\alpha}(x, t)\, dt \to 0 \quad \text{uniformly on } I^c_{2\delta} \text{ as } n \to +\infty. \tag{29}$$

The proofs of (25) and (29) being completely analogous, we deal with (29) only. Let $\varepsilon > 0$ be arbitrary and let $x \in I^c_{2\delta}$. We will show the existence of a $\gamma > 0$ such that for $n$ sufficiently large

$$\Big| \int_{I^c_{\gamma/n} \cap I_{\delta}} f(t) H^{\beta}_{n,X,\alpha}(x, t)\, dt \Big| < \varepsilon/2 \quad \text{for } x \in I^c_{2\delta}. \tag{30}$$

We use the obvious identity (see [8])

$$\sum_{j=1}^{s} T_{j,0}(t) \equiv 1.$$

Hence by (12) and (10) we immediately obtain

$$\begin{aligned} H^{\beta}_{n,X,\alpha}(x, t) &= \sum_{j=1}^{s} \Big[ H^{\beta}_{n}(x-t)\, T_{j,0}(t) - \sum_{\lambda=0}^{\alpha_j - 1} (H^{\beta}_{n}(x-x_j))^{(\lambda)}\, T_{j,\lambda}(t) \Big] \\ &= \sum_{j=1}^{s} H^{\beta}_{n,X,\alpha}(x, t)_j. \end{aligned} \tag{31}$$

Consequently, by the well-known results (see [13], Vol. 1, pp. 132, 127), (30) will be established if we show, for every $j$ $(1 \leqslant j \leqslant s)$, the existence of a $\gamma_j > 0$ such that for $n$ sufficiently large

$$(32) \qquad \Big| \int_{I^c_{\gamma_j/n} \cap I_\delta} f(t) H^{\beta_j}_{n,X,\alpha}(x, t)_j \, dt \Big| < \varepsilon/(2j) \quad \text{for } x \in I^c_{2\delta}.$$

By (1), Taylor's formula gives

$$(33) \qquad H^{\beta_j}_{n,X,\alpha}(x, t)_j + \big(H^{\beta_j}_n(x-x_j)\big)^{(\alpha_j-1)} T_{j,\alpha_j-1}(t)$$
$$= \Big\{\big[H^{\beta_j}_n(x-\theta(t))\, T_{j,0}(\theta(t))\big]^{(\alpha_j-1)}$$
$$- \sum_{\lambda=0}^{\alpha_j-2} \big(H^{\beta_j}_n(x-x_j)\big)^{(\lambda)} T^{(\alpha_j-1)}_{j,\lambda}(\theta(t))\Big\}(t-x_j)^{\alpha_j-1},$$

where $|x_j-\theta(t)| < |x_j-t|$ and $t \in (x_j-\delta, x_j+\delta)$. Consequently, using Lemma 1 we obtain from (1) and (21) for $n > \delta^{-1}$ and $x \in I^c_{2\delta}$, writing $W_j = I^c_{\gamma_j/n} \cap (x_j-\delta, x_j+\delta)$,

$$\Big| \int_{W_j} f(t) \big[H^{\beta_j}_{n,X,\alpha}(x, t)_j + \big(H^{\beta_j}_n(x-x_j)\big)^{(\alpha_j-1)} T_{j,\alpha_j-1}(t)\big] dt\Big|$$
$$\leqslant C \int_{W_j} |f(t)|\, |t-x_j|^{\alpha_j-1} dt \sum_{\lambda=0}^{\alpha_j-1} \big|\big(H^{\beta_j}_n(\delta)\big)^{(\lambda)}\big|$$
$$\leqslant C \Big(\frac{n}{\gamma_j}\Big)^{\beta_j-\alpha_j+1} \sum_{\lambda=0}^{\alpha_j-1} \big|\big(H^{\beta_j}_n(\delta)\big)^{(\lambda)}\big| \int_{-\pi}^{\pi} |f(t)| \prod_{j=1}^{s} |\sin \tfrac{1}{2}(t-x_j)|^{\beta_j} dt$$
$$\leqslant C \Big(\frac{1}{\gamma_j}\Big)^{\beta_j-\alpha_j+1} \int_{-\pi}^{\pi} |f(t)| \prod_{j=1}^{s} |\sin \tfrac{1}{2}(t-x_j)|^{\beta_j} dt.$$

Hence choosing $\gamma_j > 1$ sufficiently large we obtain

$$(34) \qquad \Big| \int_{W_j} f(t) \big[H^{\beta}_{n,X,\alpha}(x, t)_j + \big(H^{\beta_j}_n(x-x_j)\big)^{(\alpha_j-1)} T_{j,\alpha_j-1}(t)\big] dt\Big| \leqslant \varepsilon/(4j),$$
$$n > \gamma_j/\delta.$$

From (1), (15) and (16) we clearly have

$$\int_{x_j-\delta}^{x_j+\delta} |f(t)\, T_{j,\lambda}(t)|\, dt < +\infty \qquad (0 \leqslant \lambda \leqslant \alpha_j-1).$$

Consequently, by Proposition 1 and (34), there is a positive integer $N_j > \gamma_j/\delta$ such that (32) holds for $n > N_j$. Since $\beta \geqslant \beta_j$, we hence conclude by standard arguments (see [13], Vol. 1, pp. 132, 127) that (30) holds for $\gamma = \max_{1 \leqslant j \leqslant s} \gamma_j$, with $n > N_0 = \max_{1 \leqslant j \leqslant s} N_j$.

We now prove that

$$(35) \qquad \Big| \int_{I_{\gamma/n}} f(t) H^{\beta}_{n,X,\alpha}(x, t)_j \, dt\Big| < \varepsilon/(2j) \quad \text{for } x \in I^c_{2\delta},\ n > N,$$

for some $N > 0$. Since for any fixed $x \in I^c_{2\delta}$ and each $j$ $(1 \leqslant j \leqslant s)$, $H^{\beta_j}_{n,X,\alpha}(x, t)_j$ has a zero of order $\alpha_j - 1$ at $x_j$, applying Taylor's formula we get from (31) and Proposition 1

$$\Big|\int_{I_{\gamma/n}} f(t) H^{\beta_j}_{n,X,\alpha}(x, t)_j \, dt\Big| = \Big|\sum_{i=1}^{s} \int_{x_i-\gamma/n}^{x_i+\gamma/n} f(t)\big[H^{\beta_j}_{n,X,\alpha}(x, \theta(t))_j\big]^{(\alpha_j)} (t-x_j)^{\alpha_j} \, dt\Big|$$

$$\leqslant C_\delta \sum_{i=1}^{s} \int_{x_i-\gamma/n}^{x_i+\gamma/n} |f(t)|\,|t-x_i|^{\beta_i} \, dt < \varepsilon/(2j) \quad \text{for } x \in I^c_{2\delta},\ n > N,$$

where the last inequality follows by a suitable choice of $N > 0$. This shows (35), which together with (30) yields (29). The proof of Theorem 1 is complete.

The following theorem holds ([7]):

THEOREM 2. *Let $\{x_j\}_{j=1}^s$ be distinct points of $[-\pi, \pi)$, let $\{\beta_j\}_{j=1}^s$ be any positive numbers and put $\beta = \max_{1 \leqslant j \leqslant s} \beta_j$. Moreover, suppose $V$ is a $2\pi$-periodic even nonnegative function which is monotonous on $[0, \pi]$ and satisfies $\lim_{x \to 0} V(x) = 0$. Then there is a function $f$ such that*

$$\prod_{j=1}^{s} V(\cdot - x_j) f(\cdot) |\sin \tfrac{1}{2}(\cdot - x_j)|^{\beta_j} \in L_{[-\pi,\pi]} \tag{36}$$

*and if $\alpha_j$ $(1 \leqslant j \leqslant s)$ are any positive integers satisfying*

$$\alpha_j \geqslant \beta_j \quad (1 \leqslant j \leqslant s), \tag{37}$$

*then the coefficients of $f$ defined by (3) are not $o(n^\beta)$.*

We will prove a stronger assertion:

THEOREM 3. *Under the assumptions of Theorem 2, for every $i$ $(1 \leqslant i \leqslant s)$ there is a function $f$ such that*

$$V(\cdot - x_i) f(\cdot) \prod_{j=1}^{s} |\sin \tfrac{1}{2}(\cdot - x_j)|^{\beta_j} \in L_{[-\pi,\pi]} \tag{38}$$

*and such that for any $\alpha_j$ $(1 \leqslant j \leqslant s)$ satisfying (37) the coefficients*

$$a_{n,i}(f) = \pi^{-1} \int_{-\pi}^{\pi} f(t)\Big[\cos nt \cdot T_{i,0}(t) - \sum_{\lambda=0}^{\alpha_i-1} \frac{d^\lambda}{dt^\lambda} \cos nt \Big|_{t=x_i} T_{i,\lambda}(t)\Big] dt, \tag{39}$$

$$b_{n,i}(f) = \pi^{-1} \int_{-\pi}^{\pi} f(t)\Big[\sin nt \cdot T_{i,0}(t) - \sum_{\lambda=0}^{\alpha_i-1} \frac{d^\lambda}{dt^\lambda} \sin nt \Big|_{t=x_i} T_{i,\lambda}(t)\Big] dt \tag{40}$$

*are not $o(n^{\beta_i})$.*

*Proof.* Without loss of generality we can assume that $x_i = 0$. If $\alpha_i - 1 \geqslant \beta_i$, then the absolute value of the $(\alpha_i - 1)$th derivative at zero of $\cos nt$ or $\sin nt$ is $n^{\alpha_i - 1}$ according as $\alpha_i - 1$ is even or odd. Therefore by the linear independence of the functions $T_{i,\lambda}$ $(0 \leqslant \lambda \leqslant \alpha_i - 1)$ on $[-\pi, \pi)$ we find a trigonometric polynomial of order $m$, i.e. of the same order as the $T_{j,\lambda}$, which is orthogonal on $[-\pi, \pi]$ to the polynomials $T_{i,\lambda}$ $(0 \leqslant \lambda \leqslant \alpha_i - 2)$, but not to $T_{i,\alpha_i - 1}$. Obviously, this trigonometric polynomial satisfies the assertion of Theorem 3. Consequently, it remains to consider the case where

$$\alpha_i - 1 < \beta_i \leqslant \alpha_i. \tag{41}$$

For $\beta_i < \alpha_i$ it is natural to consider a function $V(\cdot)$ such that

$$V(x) \geqslant |x|^{\alpha_i - \beta_i} \quad (-\pi \leqslant x \leqslant \pi). \tag{42}$$

Suppose $\alpha_i$ is even. We estimate from below the expression $\cos nt \cdot T_{i,0}(t) - \sum_{\lambda=0}^{\alpha_i - 1} (d^\lambda/dt^\lambda) \cos nt \,|_{t=0}\, T_{i,\lambda}(t)$ in some neighbourhood of $x_i = 0$. Using Taylor's formula for $\cos nt$ and $T_{i,\lambda}$ $(0 \leqslant \lambda \leqslant \alpha_i - 1)$ and noting that $(\cos nt)^{(\alpha_i - 1)}|_{t=0} = 0$, we easily obtain

$$\cos nt \cdot T_{i,0}(t) - \sum_{\lambda=0}^{\alpha_i - 1} \frac{d^\lambda}{dt^\lambda} \cos nt \Big|_{t=0} T_{i,\lambda}(t) \geqslant C(nt)^{\alpha_i} \tag{43}$$

$$\text{for } |nt| \leqslant 1/m_1,\ n \geqslant m_1,$$

where $C > 0$ is independent of $n$, and $m_1$ is a positive integer.

We will define by induction a sequence $\boldsymbol{m} = \{m_k\}_{k=1}^\infty$ of positive integers such that

$$m_{k+1} > 2m_1 m_k, \qquad V(1/m_k) < 1/2^k \qquad (k = 1, 2, \ldots). \tag{44}$$

For sufficiently large $m_k$ this is satisfied by the condition that $V(x) \to 0$ as $x \to 0$.

To each $\boldsymbol{m}$ satisfying (44) we assign the function

$$f_{\boldsymbol{m}}(t) = \begin{cases} t^{-(\beta_i + 1)} & \text{for } t \in E, \\ 0 & \text{for } t \in [0, \pi] \setminus E, \end{cases} \tag{45}$$

where $E = \bigcup_{k=0}^\infty [1/(2m_1 m_k), 1/m_k]$, and $f_{\boldsymbol{m}}(-t) = f_{\boldsymbol{m}}(t)$ for $t \in [0, \pi]$. To check (38) it is enough to observe that the upper bound for $\int_0^\pi f_{\boldsymbol{m}}(t) V(t) t^{\beta_i} dt$ depends on $m_1$ only, namely

$$\int_0^\pi f_{\boldsymbol{m}}(t) V(t) t^{\beta_i} dt \leqslant 4(2m_1 - 1). \tag{46}$$

Let us assume now that

$$\int_0^\pi f_{\boldsymbol{m}}(t) \chi_{[1/(2m_1 m_k), \pi]}(t) \left[ \cos nt \cdot T_{i,0}(t) - \sum_{\lambda=0}^{\alpha_i - 1} \frac{d^\lambda}{dt^\lambda} \cos nt \Big|_{t=0} T_{i,\lambda}(t) \right] dt \tag{47}$$

$$< \frac{Cn^{\beta_i}}{4(2m_1)^{\alpha_i + 1}}$$

for large $n$ with some constant $C$ — this will be checked later. Note that condition (47) involves only the restriction of $f_m$ to $[1/(2m_1 m_k), \pi]$ which by (45) depends on $m_1, \ldots, m_k$ only. Choose now $m_{k+1}$ such that, in addition to (44), (47) holds with $n = m_{k+1}$ and with the constant $C$ from (43).

On the other hand, (43), (44) and (45) imply

$$\int_0^{(m_1 m_k)^{-1}} f_m(t)\left[\cos m_k t \cdot T_{i,0}(t) - \sum_{\lambda=0}^{\alpha_i - 1} \frac{d^\lambda}{dt^\lambda} \cos m_k t \Big|_{t=0} T_{i,\lambda}(t)\right] dt$$

$$\geqslant C m_k^{\alpha_i} \int_0^{(m_1 m_k)^{-1}} f(t) t^{\alpha_i} dt \geqslant C m_k^{\alpha_i} (m_1 m_k)^{\beta_i + 1} \left(\frac{1}{2m_1 m_k}\right)^{\alpha_i} \frac{1}{2m_1 m_k}$$

$$\geqslant C(2m_1)^{-\alpha_i - 1} m_k^{\beta_i},$$

where again $C$ comes from (43). Since $f = f_m$ is even, (43) and (47) yield the assertion of Theorem 3 for $\alpha_i$ even. It only remains to show that the left-hand side of (47) divided by $n^{\beta_i}$ tends to 0 for $k$ fixed as $n \to +\infty$. To see this, note that since $\alpha_i$ is even and $x_i = 0$,

$$\left|\cos nt \cdot T_{i,0}(t) - \sum_{\lambda=0}^{\alpha_i - 1} \frac{d^\lambda}{dt^\lambda} \cos nt \Big|_{t=x_i} T_{i,\lambda}(t)\right| \leqslant C_i n^{\alpha_i - 1} t^{\alpha_i - 1},$$

where $C_i$ is independent of $n$. Together with (41), this yields immediately the desired relation.

The proof for $\alpha_i$ odd is analogous, the main difference being that one has to consider the coefficients for $\sin nt$ instead of $\cos nt$, and the extension of $f$ from $(0, \pi]$ to $[-\pi, \pi]$ should be odd functions.

This completes the proof of Theorem 3, and hence also of Theorem 2.

## References


[1] R. P. Boas, *Integrability Theorems for Trigonometric Transforms*, Springer, 1967.

[2] M. I.' D'yachenko, *On the generalized Fourier sine series*, Mat. Zametki 29 (5) (1981), 641–657 (in Russian).

[3] —, *Integrability of functions and Fourier coefficients*, Vestnik Moskov. Univ. Ser. Mat. Mekh. 1977, 18 (in Russian).

[4] T. H. Gronwall, *Über eine Summationsmethode und ihre Anwendung aud die Fouriersche Reihe*, J. Reine Angew. Math. 147 (1916), 16–35.

[5] K. S. Kazaryan, *Uniform continuity of operators generated by truncated kernels in weighted $L^p$ spaces*, $1 \leqslant p < \infty$, Dokl. Akad. Nauk SSSR 272 (1983), 1048–1052 (in Russian).

[6] —, *Summability and convergence almost everywhere of generalized Fourier series and Fourier–Haar series*, Izv. Akad. Nauk Armyan. SSR Ser. Mat. 20 (2) (in Russian).

[7] K. S. Kazaryan, *Summability of generalized Fourier series in weighted norms and almost everywhere*, Dokl. Akad. Nauk SSSR 287 (3) (1986), 547 (in Russian).

[8] –, *Abel summability of subsystems of the trigonometric system in $L^p(d\mu)$, $1 \leqslant p < \infty$, the Dirichlet problem in weighted norm, and weighted $H^p$ spaces, $p > 1$*, Anal. Math. 13 (1987) (in Russian).

[9] A. Yu. Petrovich, *On the summability of generalized Fourier series*, ibid. 4 (4) (1978), 303–311 (in Russian).

[10] M. Riesz, *Sur les fonctions conjuguées*, Math. Z. 27 (1927), 218–244.

[11] A. Zygmund, *Sur la sommation des séries conjuguées aux séries de Fourier*, Bull. Int. Acad. Polon. Sci. 1924, 251–258.

[12] –, *Sur un théorème de M. Gronwall*, ibid. 1925, 207–217.

[13] –, *Trigonometric Series*, Vols. 1, 2, Mir, Moscow 1965 (in Russian).

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