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THE CENTRALIZER OF ERGODIC THEORY GROUP EXTENSIONS

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For an ergodic automorphism $T\colon (X, \mathscr{B}, \mu) \to (X, \mathscr{B}, \mu)$ with pure point spectrum and its ergodic G-extension T_φ (G is a compact metric abelian group) we prove that the set $X_0 \subset X$ of those x's which lift to the centralizer of T_φ is of Haar measure zero. Some related problems for weakly mixing two-fold simple automorphisms are considered.

Introduction

Let $T\colon (X, \mathscr{B}, \mu) \to (X, \mathscr{B}, \mu)$ be an ergodic automorphism of a Lebesgue space. By the centralizer $C(T)$ of T we mean the semigroup of all endomorphisms $S\colon (X, \mathscr{B}, \mu) \to (X, \mathscr{B}, \mu)$ commuting with T, i.e. such that $ST = TS$.

Let $\hat{T}\colon (\hat{X}, \hat{\mathscr{B}}, \hat{\mu}) \to (\hat{X}, \hat{\mathscr{B}}, \hat{\mu})$ be another ergodic automorphism. Assume that $\mathscr{B} \subset \hat{\mathscr{B}}$, $\hat{\mu}|_{\mathscr{B}} = \mu$, $\hat{T}|_{\mathscr{B}} = T$ and, besides, for every $\hat{S} \in C(T)$

$$\hat{S}^{-1}\mathscr{B} = \mathscr{B}. \tag{1}$$

Then it is well-known that $\hat{S}$ can be represented as a skew product over an $S \in C(T)$, i.e.

$$\hat{X} = X \times Z \quad \text{and} \quad \hat{S}(x, z) = (Sx, S_x z).$$

In this case it is natural to say that $S \in C(T)$ can be lifted to the centralizer of $\hat{T}$.

The problem we deal with in this paper is to answer the question, how large is the set of such S's.

To ensure that condition (1) holds we will restrict our attention to the following classes of automorphisms:

$\mathscr{K}$: the clas of automorphisms with pure point spectra,
$\mathscr{S}_2$: the class of weakly mixing 2-fold simple automorphisms.

Moreover, we will assume that the automorphism $\hat{T}$ is the so called group extension of T (over G), where G is a metric compact abelian group, i.e. $\hat{T} = T_\varphi$, φ: $X \to G$ is measurable and

$$T_\varphi: (X \times G, \tilde{\mu}) \to (X \times G, \tilde{\mu})$$
$$T_\varphi(x, g) = \big(Tx, \varphi(x)+g\big). \tag{2}$$

Here $\tilde{\mu} = \mu \times \mu_G$ and μ_G is the Haar measure on G.

If $T \in \mathscr{K}$ then we can identify it with an ergodic translation σ_{x_0}: $(X, \mu_X) \to (X, \mu_X)$, $\sigma_{x_0}(x) = x+x_0$, where X is a compact, monothetic group, μ_X is its Haar measure and x_0 is a cyclic generator of X. Then

$$C(T) = C(\sigma_{x_0}) = \{\sigma_{x_1}\colon x_1 \in X\} \approx X. \tag{3}$$

When T_φ is a G-extension of T we let X_0 denote the set of such x_1's that σ_{x_1} can be lifted to the centralizer of T_φ. In view of (3) we can use Haar measure to estimate how big the set X_0 is. The main result of the paper is the following

THEOREM 1. *If T_φ is an ergodic G-extension of a pure point spectrum automorphism T: $(X, \mu_X) \to (X, \mu_X)$ then X_0 is measurable and either*

$$\mu_X(X_0) = 0 \quad or \tag{4}$$

$$X_0 = X. \tag{5}$$

Moreover (5) *holds iff T_φ has also pure point spectrum.*

The case $T \in \mathscr{S}_2$ seems to be more complicated. In Section IV we explain some difficulties and distinguish some subclass of $\mathscr{S}_2$-automorphisms for which a result analogous to Theorem 1 holds.

I. Definitions and remarks

Throughout the paper all automorphisms are assumed to be ergodic (unless otherwise state) and acting on a Lebesgue space.

Let T: $(X, \mathscr{B}, \mu) \to (X, \mathscr{B}, \mu)$ and τ: $(Y, \mathscr{C}, \nu) \to (Y, \mathscr{C}, \nu)$ be automorphisms. We call τ *a factor of* T if there exists a measurable map f: $X \to Y$ such that $f\mu = \nu$, $\tau f = fT$. If, besides, f is invertible then T and τ are *isomorphic*. It is well-known that if τ is a factor of T we may (and we do) identify τ with T: $(X, \mathscr{C}, \nu) \to (X, \mathscr{C}, \nu)$, where $\mathscr{C} \subset \mathscr{B}$ is a T-invariant sub-σ-algebra (we will write T: $\mathscr{C} \to \mathscr{C}$ to denote the factor).

T is said to be *coalescent* if for every T-invariant sub-σ-algebra $\mathscr{C}$, $\mathscr{C} = \mathscr{B}$ whenever T: $\mathscr{C} \to \mathscr{C}$ and T: $\mathscr{B} \to \mathscr{B}$ are isomorphic. This condition is equivalent to the following: every $S \in C(T)$ is invertible ([9], [10]).

Let $\mathscr{C} \subset \mathscr{B}$ be a T-invariant sub-σ-algebra. We call $\mathscr{C}$ *completely invariant* ([10]) if for every $S \in C(T)$, $S^{-1}\mathscr{C} = \mathscr{C}$ (i.e. every endomorphism S, $S \in C(T)$ becomes invertible on $\mathscr{C}$).

Following [10] we call T a *canonical system* if, whenever it appears as a factor of an ergodic U: $(Z, \mathscr{D}, \nu) \to (Z, \mathscr{D}, \nu)$, then there is a unique U-invariant sub-σ-algebra $\mathscr{D}'$ such that U: $\mathscr{D}' \to \mathscr{D}'$ is isomorphic to T. Using this notion D. Newton in [10] developed a method to obtain forms of isomorphisms between group extensions of canonical systems. However, as it was shown in [8] the class of canonical systems is rather small and coincides with $\mathscr{K}$. Moreover, when dealing with the centralizer $C(T_\varphi)$ of T_φ given by (2), we need merely know that the σ-algebra

$$\hat{\mathscr{B}} = \{A \times G\colon A \in \mathscr{B}\} \tag{6}$$

is completely invariant to conclude the validity of Newton's result. If this is the case then for every $\hat{S} \in C(T_\varphi)$ there exist a group endomorphism v of G, a measurable function f: $X \to G$ and an automorphism $S \in C(T)$ such that

$$\hat{S}(x, g) = (Sx, f(x) + v(g)), \tag{7}$$

$$\varphi(Sx) + f(x) = f(Tx) + v(\varphi(x)) \quad ([10]). \tag{8}$$

Actually, Newton proved something more (Th. 2.1 [10]). Namely,

$$v \text{ is a continuous epimorphism of } G. \tag{9}$$

By $J(T, T)$ we mean the space of all *2-joinings* of T, i.e. $\lambda \in J(T, T)$ if λ is a $T \times T$-invariant probability measure on $\mathscr{B}_1 \times \mathscr{B}_2$, $\mathscr{B}_i = \mathscr{B}$ $i = 1, 2$ and $\lambda|_{\mathscr{B}_i} = \mu$.

Following [4], [12] T is said to be *2-fold simple* if every ergodic $\lambda \in J(T, T)$ is either the product measure $\mu \times \mu$ or a "graph measure" μ_S, $S \in C(T)$, where

$$\mu_S(A \times B) = \mu(A \cap S^{-1}B). \tag{10}$$

It is easy to see that every 2-fold simple automorphism is coalescent. The class of all 2-fold simple automorphisms includes $\mathscr{K}$. As we will see in Section IV, (7) and (8) hold for the class of weakly mixing group extensions of automorphisms from $\mathscr{S}_2$.

II. Proof of Theorem 1

We divide the proof into 2 steps: (1) X_0 is measurable, (2) $\mu_X(X_0) = 0$.

Step 1. In view of (3) equation (8) can be rewritten as follows

$$\varphi(x + x_1) - v(\varphi(x)) = f(x + x_0) - f(x). \tag{11}$$

Consider the set $\mathscr{M}$ of all measurable functions h: $X \to G$. Then $\mathscr{M}$

becomes a Polish space with the metric

$$(12) \qquad \varrho(h, h') = \int_X d(h(x), h'(x))\, \mu_X(dx), \qquad h, h' \in \mathcal{M}$$

where

(13) d is a rotation invariant metric on G.

Let $\mathcal{E}(G, G)$ be the set of all continuous group endomorphisms of G. If we put

$$D(v, v') = \sup_{g \in G} d(v(g), v'(g)), \qquad v, v' \in \mathcal{E}(G, G)$$

then $\mathcal{E}(G, G)$ becomes a Polish space.

Define the following maps:

$$F_1\colon X \to \mathcal{M}, \qquad F_1(x_1) = \varphi \cdot \sigma_{x_1}, \qquad x_1 \in X,$$

$$F_2\colon \mathcal{E}(G, G) \to \mathcal{M}, \qquad F_2(v) = v \cdot \varphi, \qquad v \in \mathcal{E}(G, G),$$

$$F_3\colon \mathcal{M} \to \mathcal{M}, \qquad F_3(f) = f \cdot \sigma_{x_0} - f, \qquad f \in \mathcal{M}.$$

We extend in the natural way each $F_i = 1, 2, 3$ to a map from $X \times \mathcal{E}(G, G) \times \mathcal{M}$ into $\mathcal{M}$. Let Z be the set of all elements $(x_1, v, f) \in X \times \mathcal{E}(G, G) \times \mathcal{M}$ such that

$$F_1(x_1, v, f) - F_2(x_1, v, f) = F_3(x_1, v, f).$$

Then the set X_0 is just the projection of Z on X. The projection p is continuous. We will prove that F_i, $i = 1, 2, 3$ is continuous. Since $X \times \mathcal{E}(G, G) \times \mathcal{M}$ is a Polish space and Z is a closed subset of it, $p(Z) = X_0$ is analytic, hence measurable.

We now prove that F_1 is continuous. First, suppose that $\varphi'\colon X \to G$ is continuous. Then, given $\varepsilon > 0$ we select $\delta > 0$ so that if $|x_1 - x_1'| < \delta$ then $d(\varphi'(x_1), \varphi'(x_1')) < \varepsilon/3$ and consequently

$$d(\varphi'(x_1 + x), \varphi'(x_1' + x)) < \varepsilon/3 \qquad \text{for every } x \in X.$$

Now,

$$\varrho(F(x_1), F(x_1')) = \int_X d(\varphi'(x + x_1), \varphi'(x + x_1'))\, d\mu_X(x) < \varepsilon/3.$$

The set of all continuous functions from X into G is dense in $\mathcal{M}$ with respect to ϱ. Since

$$\begin{aligned} &\varrho F(x_1), F(x_1')) \\ &\quad \leqslant \int_X [d(\varphi\sigma_{x_1}, \varphi'\sigma_{x_1}) + d(\varphi'\sigma_{x_1}, \varphi'\sigma_{x_1'}) + d(\varphi'\sigma_{x_1'}, \varphi\sigma_{x_1'})]\, \mu_X \\ &\qquad = \int_X d(\varphi, \varphi')\, \mu_X + \int_X d(\varphi'\sigma_{x_1}, \varphi'\sigma_{x_1'})\, \mu_X + \int_X d(\varphi, \varphi')\, \mu_X < \varepsilon \end{aligned}$$

whenever φ' is continuous and $\varrho(\varphi, \varphi') < \varepsilon/3$, the result follows.

Now, the map F_2 is continuous, because

$$\varrho(F_2(v), F_2(v')) \leqslant \int_X \sup_{g \in G} d(v(g), v'(g))\, \mu_X(dx) = D(v, v').$$

Since d satisfies (13), F_3 is also continuous.

Hence the proof of Step 1 is complete.

Step 2. First, let us notice that $X_0 = X$ iff T_φ has pure point spectrum. This statement folows from [4]. Indeed, each $T \in \mathscr{K}$ is 2-fold simple and moreover if every element of $C(T)$ can be lifted to $C(T_\varphi)$ then T_φ is also 2-fold simple (see the proof of Th. 5.4 in [4]). Furthermore, if T_φ is 2-fold simple and has some point spectrum then it has pure point spectrum ([4]). We have proved that $X = X_0$ implies that T_φ has discrete spectrum. The reverse implication is more or less trivial.

Next, note that X_0 is a semigroup (this is a consequence of (7)). Hence the set $X_0 \cap X_0^{-1}$ is a group.

If T_φ has no discrete spectrum then by the assertion we have just proved there is an element $x_1 \in X$ such that $x_1 \notin X_0$. Observe that $x_0 \in X_0$, so $TX_0 = \sigma_{x_0} X_0 = X_0$. But σ_{x_0} acts ergodically and therefore $\mu_X(X_0) = 0$ or 1. Assume $\mu_X(X_0) = 1$. Then $\mu_X(X_0 \cap X_0^{-1}) = 1$. But this is impossible because $X_0 \cap X_0^{-1}$ is a subgroup and $x_1 \notin X_0 \cap X_0^{-1}$. We have shown that $\mu_X(X_0) = 0$ and the proof of Theorem 1 is complete.

III. Examples

Now, we deliver some examples which show that the set X_0 can be as small as possible, i.e.

$$X_0 = \{x_0^n \colon n \in Z\}. \tag{14}$$

EXAMPLE 1. Consider some subclass of quasi-discrete spectrum transformations on the 2-torus $[0, 1] \times [0, 1)$ given by the formula

$$U(x, y) = (x + \alpha, nx + y + \beta)$$

where $n \in Z \setminus \{0\}$ and the set $\{1, \alpha, \beta\}$ is rationally independent. Putting $Tx = x + \alpha$, $\varphi(x) = nx + \beta$ we see that U is precisely a $[0, 1)$-extension of the rotation T. Now U is coalescent (this is proved in [2]). Let $S \in C(U)$. By (7) we get

$$S(x, y) = (x + \alpha', f(x) + v(y))$$

where $f\colon [0, 1) \to [0, 1)$ is measurable and v is a group automorphism of $[0, 1)$. Now (9) implies immediately that $v = \mathrm{id}$ (the possibility $v = -\mathrm{id}$ is excluded). Furthermore from (8) it follows that

$$n\alpha' + f(x) = f(x + \alpha)$$

or

$$\exp(2\pi \mathrm{i} n\alpha')\exp(2\pi \mathrm{i} f(x)) = \exp(2\pi \mathrm{i} f(x+\alpha)).$$

Therefore $\exp(2\pi \mathrm{i} n\alpha')$ is a proper value of the irrational rotation on α. That means that $n\alpha' = k\alpha$ for some integer k. In other words the set X_0 consists of all n-roots of $k\alpha$, $k = 0, \pm 1, \pm 2, \ldots$ If $n = 1$ then (14) hoods.

EXAMPLE 2. Let $x = b^0 \times b^1 \times \ldots$ be a Morse sequence (see [5], [7]). Consider the shift transformation τ on $\mathcal{O}_x$ (the closure of the trajectory of x via τ). When x is regular and the lengths of the $b^{t'}s$ are bounded then

$$C(\tau) = \{\tau^n \sigma^j \colon\ n \in Z, j = 0, 1\} \quad ([7]), \tag{15}$$

where $(\sigma z)[i] = 1 - z[i]$, $z \in \mathcal{O}_x$. The shift transformation τ on $\mathcal{O}_x$ is isomorphic to a Z_2-extension of $T = \sigma_{x_0}$, where σ_{x_0} is an ergodic translation on a group X of some n_t-adic integers (for details see [5]). Consequently by (15) we get that (14) holds.

A more detailed description of the centralizer of continuous Morse sequences is given in [6]. It is proved that the set X_0 is always of the form

$$X_0 = \bigcap_{p=1}^{\infty} \bigcup_{n=0}^{\infty} \bigcap_{k=1}^{\infty} A_{n,k,p} \tag{16}$$

where $A_{n,k,p}$ is a finite union of cylinder sets. Hence $A_{n,k,p}$ is simultaneously closed and open. It follows from (16) that $X \setminus X_0$ contains a G_δ-set. Now, let $x_1 \in X \setminus X_0$. Then the set $\{x_0^n x_1\}$ is dense in X and $\{x_0^n x_1\} \subset X \setminus X_0$ because X_0 is a subgroup of X. We conclude that $X \setminus X_0$ is residual and hence X_0 is of the first category.

We conjecture that this result is true for every ergodic group extension (with partly continuous spectrum) of automorphisms from $\mathscr{K}$.

IV. Centralizer of weakly mixing group extensions of $\mathscr{S}_2$-automorphisms

We start with the following

PROPOSITION 1. *Let $T\colon (X, \mathscr{B}, \mu) \to (X, \mathscr{B}, \mu)$ be an ergodic automorphism and let $\mathscr{C} \subset \mathscr{B}$ be a T-invariant sub-σ-algebra. Assume that $T\colon \mathscr{C} \to \mathscr{C}$ is 2-fold simple. Then for every $\hat{S} \in C(T)$ either*

$$\hat{S}^{-1}\mathscr{C} = \mathscr{C} \tag{17}$$

or

$$\hat{S}^{-1}\mathscr{C} \perp \mathscr{C}. \tag{18}$$

Proof. Take any $\hat{S} \in C(T)$ and consider the corresponding "graph measure" $\mu_{\hat{S}}$ given by (10). By the ergodicity of T this 2-joining is also ergodic and therefore $\nu = \mu_{\hat{S}}|_{\mathscr{C} \otimes \mathscr{C}}$ is ergodic. By the simplicity of T on $\mathscr{C}$ we get that either ν is the product measure (and then (18) holds) or ν is concentrated on the graph of some $S \in C(T, \mathscr{C})$, i.e. $\nu = \mu_S$. Now, given $A \in \mathscr{C}$ we have

$$(19) \qquad \mu_{\hat{S}}(\hat{S}^{-1} A \times X \vartriangle X \times A) = 0, \quad \mu_S(S^{-1} A \times X \vartriangle X \times A) = 0.$$

But $S^{-1} A \times X \vartriangle X \times A \in \mathscr{C} \otimes \mathscr{C}$, so

$$(20) \qquad \mu_{\hat{S}}(S^{-1} A \times X \vartriangle X \times A) = 0.$$

Combining (19) and (20) we get $\hat{S}^{-1} A = S^{-1} A$, $A \in \mathscr{C}$. Since S is an automorphism (we recall that 2-fold simplicity implies coalescence), $\hat{S}^{-1} \mathscr{C} = \mathscr{C}$ and the proof is complete.

The next corollary says that for some special cases the possibility (18) can be excluded.

Corollary 1. *Let T: $(X, \mathscr{B}, \mu) \to (X, \mathscr{B}, \mu)$, $T \in \mathscr{S}_2$. Let G be any compact metric abelian group. Let $\varphi: X \to G$ be measurable. If T_φ is weakly mixing then the σ-algebra (6) is completely invariant (i.e. T is a completely invariant factor of T_φ).*

Proof. By the weak mixing condition we see that $(T_\varphi \times T_\varphi, \tilde{\mu} \times \tilde{\mu})$ is an ergodic $G \times G$-extension of $(T \times T, \mu \times \mu)$. It is well-known that if this is the case and $\lambda \in J(T_\varphi, T_\varphi)$ projects into the σ-algebra $\hat{\mathscr{B}} \times \hat{\mathscr{B}}$ as $\mu \times \mu$ then $\lambda = \tilde{\mu} \times \tilde{\mu}$ ([4]).

Now, fix an ergodic $T \in \mathscr{S}_2$. Due to Corollary 1 we can develop a theory which is parallel to the case $T \in \mathscr{K}$. However there is an essential difference between these two cases. The reason is that for the weakly mixing case $C(T)$ need not be even locally compact. So, the method we have used to prove that the set $X_0 \subset C(T)$ does not work in general. It would be interesting to know how big the set X_0 is with respect to the weak topology.

The following result is implicitly contained in [4] (Th. 5.4 anf 3.2).

Proposition 2. *Let $T \in \mathscr{S}_2$ and φ: $X \to G$ be measurable. If T_φ is weakly mixing then $X_0 = C(T)$ iff $T_\varphi \in \mathscr{S}_2$.*

Now, let τ: $(Y, \mathscr{C}, \nu) \to (Y, \mathscr{C}, \nu)$ be 2-fold simple and

$$C(\tau) = \{\tau^i \colon i \in Z\}.$$

Let H be a compact metric abelian group. Fix φ: $Y \to H$ such that $T = \tau_\varphi$, T: $(X, \tilde{\nu}) \to (X, \tilde{\nu})$, $X = Y \times H$, $\tilde{\nu} = \nu \times \mu_H$ and T is weakly mixing. Then $T \in \mathscr{S}_2$ and moreover

$$C(T) = \{T^n \sigma_h \colon n \in Z, h \vee H\} \qquad ([4]).$$

We will consider G-extensions, say T_f, of T. By ignoring powers of T we can transfer Haar measure μ_H from H into the centralizer of T. In other words if we denote $X_0 = \{h \in H \colon \sigma_h$ can be lifted to $C(T_f)\}$ and assume that T_f is weakly mixing but is not in $\mathscr{S}_2$ (Proposition 2), the question is whether or not

$$\mu_H(H_0) = 0. \tag{21}$$

Let us notice that X_0 is Borel because the method we have used in Step 1 works well in this situation. We will deal with some special $G = S^1$-extensions for which (21) is satisfied.

Fix a character χ_0: $H \to S^1$ and define the S^1-extension of T by putting

$$T_{\chi_0}((y, h), z) = (T(y, h), \chi_0(h)z). \tag{22}$$

PROPOSITION 3. *T_{χ_0} is weakly mixing provided χ_0 is not of finite order.*

Proof. The proof is a slight modification of Glasner's proof in [1], Prop. 1.7.

Now we are able to prove the following

PROPOSITION 4. *If χ_0 is not of finite order then T satisfies* (21).

Proof. First of all we show that

$$X_0 = \ker \chi_0 = \{h \in H: \chi_0(h) = 1\}. \tag{23}$$

Let $h_0 \in X_0$. Then by (7), (8) and (9) we get that there are a map f: $X \to S^1$ and a continuous epimorphism v: $S^1 \to S^1$ such that

$$\hat{\sigma}_{h_0}((y, h), z) = (\sigma_{h_0}(y, h), f(y, h) v(z)). \tag{24}$$

It follows from (24) and (8) that

$$\chi_0(h+h_0) f(y, h) = f(T(y, h))[\chi_0(h)]^n,$$

where $v(z) = z^n$. We convert this equality to the following

$$\frac{f(T(y, h))}{f(y, h)} = \chi_0(h_0)[\chi_0(h)]^{-n+1}$$

Write

$$\varphi(y, h) = \chi_0(h), \quad p \in \hat{S}^1, \quad p(z) = z^{-n+1}, \quad c = \chi_0(h_0).$$

We get

$$\frac{f(T(y, h))}{f(y, h)} = cp(\varphi(y, h)).$$

Take the function $\tilde{f}(y, h, z) = f(y, h)/p(z)$. Easy calculation shows that $\tilde{f} T_{\chi_0} = c\tilde{f}$. Thus, by the weak mixing of T_{χ_0} we conclude that $c = 1$ and $\tilde{f}$ is constant a.e. This implies $n = 1$ and hence (23) holds.

If $\mu_H(X_0)$ were strictly positive then the group H/X_0 would have to be finite. This means that the image of χ_0 is a group of roots of unity of a fixed, say k, degree. Hence χ_0^k is the trivial character and we get a contradiction.

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