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INVARIANT MEASURES AND ERGODIC PROPERTIES OF NUMBERTHEORETICAL ENDOMORPHISMS

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§ 1. Introduction

1.1. DEFINITION. Let B be a set and T: $B \to B$ a map. We will call (B, T) a *fibered system* if there is a set of "digits" I and a map k: $B \to I$ with the following property: Let $B(i) = \{x \in B\colon k(x) = i\}$ be a cylinder of the time-1-partition, then the restriction of T to $B(i)$ is injective for any $i \in I$.

1.2. Let $\mathcal{F}$ be a σ-algebra of subsets of B. Then we require:

(1) T: $B \to B$ is measurable

(2) Every set $B(i)$ is measurable

1.3. Let λ: $\mathcal{F} \to \boldsymbol{R} \cup \{\infty\}$ be a finite or σ-finite measure, then we additionally suppose

(3) If $\lambda(E) = 0$ then $\lambda(T^{-1}E) = 0$

1.4. The theory of fibered systems has its roots in the so-called metrical number theory. Classical papers in this area are Ryll–Nardzewski 1951, Rényi 1957, Fischer 1972 and Adler 1973. We further mention Schweiger 1975a and Schweiger 1975b. A survey of some results appeared in Schweiger 1981b.

1.5. The following problems are typical for this area:

(E) Is T *ergodic* with respect to the given measure λ?

(C) Is T *conservative* with respect to the given measure λ?

Note that if T is ergodic and there is a finite invariant measure $\mu \ll \lambda$, then T is conservative with respect to μ.

(M) Does T admit a finite or σ-finite *invariant measure* $\mu \ll \lambda$?

(D) Let T admit a finite or σ-finite invariant measure $\mu \ll \lambda$. Can one calculate the *density* $h(x) = \dfrac{d\mu}{d\lambda}(x)$?

1.6. We remark that there is a vast literature on the Kuzmin (–Frobenius–Perron) operator A defined by duality as follows

$$\int_{T^{-1}E} f d\lambda = \int_E (Af)\, d\lambda.$$

Then h is an invariant density if and only if $Ah = h$.

1.7. We now give some standard examples. In all cases λ is Lebesgue measure.

(1) *g-adic map* ($g \geqslant 2$, integral)

$$B = [0, 1], \qquad Tx = gx \bmod 1,$$
$$k(x) = [gx];$$

T is ergodic and Lebesgue measure λ is invariant.

(2) *Continued fractions*

$$B = [0, 1], \qquad Tx = \frac{1}{x} \bmod 1,$$
$$k(x) = \left[\frac{1}{x}\right];$$

T is ergodic.

The invariant density is given as:

$$h(x) = \frac{1}{1+x}.$$

(3) *Boole's transformation*

$$B = \boldsymbol{R}, \qquad Tx = x - \frac{1}{x},$$
$$k(x) = 0 \text{ for } x < 0, \qquad k(x) = 1 \text{ for } x > 0;$$

T is ergodic (Adler–Weiss 1973, Schweiger 1975a) and Lebesgue measure λ is invariant.

(4) *Tangent map*

$$B = \boldsymbol{R}, \qquad Tx = \tan x,$$
$$k(x) = z \quad \text{if } -\frac{\pi}{2} + z\pi < x < \frac{\pi}{2} + z\pi;$$

T is ergodic (Schweiger 1978, Aaronson 1978).

The invariant density is

$$h(x) = \frac{1}{x^2}.$$

(5) *Series expansions of Balkema–Oppenheim type* (see Galambos 1976; Perron 1960)

$$B = [0, 1], \quad k(x) = k \text{ for } \frac{1}{k+1} < x \leqslant \frac{1}{k},$$

$$Tx = b(k)\left(x - \frac{1}{k+1}\right) \quad \text{on } B(k).$$

The following choices of $b(k)$ lead to classical series expansions:

(5.1) $b(k) = k(k+1)$: *Lüroth's series* (see Jager–de Vroedt 1969)
T is ergodic in this case and Lebesgue measure λ is invariant.

(5.2) $b(k) = k+1$: *Engel's series* (may be also called *Sierpiński's series*; Sierpiński 1911);
T is ergodic, but not conservative.

(5.3) $b(k) = 1$: *Sylvester's series*;
T is not ergodic (Vervaat 1972).

If one defines $h(k) = \dfrac{k(k+1)}{b(k)}$ then the sequence $\left(h(k(T^{n-1}x))\, T^n x\right)$, $n = 1, 2, 3, \ldots$, is uniformly distributed for almost every x (Schweiger 1972).

§ 2. Piecewise fractional linear maps

2.1. Definition. The map T is called *piecewise fractional linear* if

$$Tx = \frac{A_k x + B_k}{C_k x + D_k} \quad \text{for } x \in B(k),$$

$$A_k D_k - B_k C_k \neq 0.$$

2.2. The most well-known examples are maps of continued fraction type.

(1) $$B = [\alpha - 1, \alpha[, \quad 0 < \alpha \leqslant 1,$$

$$k(x) = \left[\frac{1}{x} + 1 - \alpha\right] \quad \text{on } B(k),$$

$$Tx = \frac{1}{x} - k(x) \quad \text{on } B(k).$$

Tanaka and Ito 1981 have found the following invariant measures for the case $1/2 \leqslant \alpha \leqslant (\sqrt{5}-1)/2$. The density function still is unknown for the case $(\sqrt{5}-1)/2 < \alpha < 1$. The result of Tanaka–Ito uses very long calculations. The following subcases have to be distinguished:

$$\text{(i)} \qquad \frac{1}{2} \leqslant \alpha \leqslant 2-\sqrt{2},$$

$$\text{(ii)} \qquad 2-\sqrt{2} < \alpha \leqslant \frac{7+\sqrt{13}}{18},$$

$$\text{(iii)} \qquad \frac{7+\sqrt{13}}{18} < \alpha \leqslant \frac{10-\sqrt{2}}{14},$$

$$\text{(iv)} \qquad \frac{10-\sqrt{2}}{14} < \alpha \leqslant \frac{\sqrt{5}-1}{2},$$

In each subcase the density is given piecewise. For sake of illustration we give the complete result for case (i) which is actually the most simple one. Let

$$\frac{1}{2} \leqslant \alpha < 2-\sqrt{2} \quad \text{and} \quad \beta = \frac{\sqrt{5}-1}{2}.$$

Then

$$h(x) = \begin{cases} \dfrac{1}{x+\beta+2} - \dfrac{1}{x-\beta-1} & \text{if } \alpha-1 \leqslant x \leqslant \dfrac{1}{\alpha-1}+2, \\ \dfrac{1}{x+\beta+2} - \dfrac{1}{x-\beta-2} & \text{if } \dfrac{1}{\alpha-1}+2 < x \leqslant \dfrac{1}{\alpha}-2, \\ \dfrac{1}{x+\beta+1} - \dfrac{1}{x-\beta-2} & \text{if } \dfrac{1}{\alpha}-2 < x < \alpha. \end{cases}$$

$$\text{(2)} \qquad B = [\alpha-1, \alpha[,$$

$$k(x) = \left[\left|\frac{1}{x}\right| + 1 - \alpha\right], \qquad 0 < \alpha \leqslant 1,$$

$$Tx = \frac{1}{x} - k(x) \quad \text{on } B(k).$$

Nakada 1981 succeeded in determining the invariant measures:

(i) If $\dfrac{1}{2} \leqslant \alpha \leqslant \dfrac{\sqrt{5}-1}{2}$, then (with $\beta = \dfrac{\sqrt{5}-1}{2}$):

$$h(x)=\begin{cases}\dfrac{1}{x+\beta+2} & \text{if } \alpha-1\leqslant x\leqslant\dfrac{1}{\alpha}-2,\\[2mm] \dfrac{1}{x+2} & \text{if } \dfrac{1}{\alpha}-2<x<\dfrac{1\alpha-1}{1-\alpha},\\[2mm] \dfrac{1}{x+\beta+1} & \text{if } \dfrac{2\alpha-1}{1-\alpha}<x<\alpha.\end{cases}$$

(ii) If $\dfrac{\sqrt{5}-1}{2}<\alpha\leqslant 1$, then

$$h(x)=\begin{cases}\dfrac{1}{x+2} & \text{if } \alpha-1\leqslant x\leqslant\dfrac{1}{\alpha}-1,\\[2mm] \dfrac{1}{x+1} & \text{if } \dfrac{1}{\alpha}-1<x<\alpha.\end{cases}$$

The case $\alpha=\frac{1}{2}$ ("continued fractions to the nearest integer") was studied in detail by Rieger 1978.

(3) $$B=[0,1],\quad k(x)=\left[\frac{x}{1-x}\right],$$

$$Tx=\frac{x}{1-x} \bmod 1,$$

T is ergodic and conservative. It admits an invariant density

$$h(x)=\frac{1}{x}.$$

The conjugacy $\Psi(x)=1-x$ transforms T to $\Psi\circ T\circ\Psi$, a map which is related to a kind of continued fractions which are useful in number theory and algebraic geometry (see e.g. Zagier 1981).

(4) Rudolfer 1971 considers the maps

(i) $$Tx=\frac{bx}{1-cx} \bmod 1,$$

(ii) $$Tx=\frac{1}{ax}-\frac{b}{a} \bmod 1,$$

and finds some invariant densities.

(5) Passing reference is given to continued fractions with odd or even partial quotients (see e.g. Kalpazidou 1986).

2.3. The examples given suggest that the densities of invariant measures have a common structure. To test this hypothesis Schweiger 1983 considered a special class of piecewise fractional linear maps:

(i) The partition $\{B(0), B(1)\}$ consists of 2 intervals $[0, \alpha]$, $[\alpha, 1]$, say.
(ii) T maps $B(i)$, $i = 0, 1$, fully onto $[0, 1]$.
(iii) T has no attractive periodic point.

Then T is ergodic and the invariant density takes one of the following shapes:

$$h(x) = \frac{1}{x+A} - \frac{1}{x+B},$$

$$h(x) = \frac{1}{(x+A)^2}, \qquad h(x) = \frac{1}{x+A}, \qquad \text{or} \qquad h(x) \equiv 1.$$

2.4. A similar theorem is not longer true if the partition consists of $\geqslant 3$ intervals. No general result is known. Schweiger 1986 gives the following example: Let $N \geqslant 1$, and define

$$Tx = \frac{x}{1-(N+1)x}, \qquad 0 \leqslant x \leqslant \frac{1}{N+2},$$

$$Tx = \frac{1}{x} - k, \qquad \frac{1}{k+1} < x < \frac{1}{k}, \qquad 1 \leqslant k \leqslant N+1.$$

Then T has the invariant density

$$h(x) = \frac{1}{x} \sum_{j=0}^{\infty} \left(\frac{1}{1+j(N+1)x} - \frac{1}{1+\big(j(N+1)+1\big)x} \right).$$

2.5. *Remark.* The points of infinity in the densities correspond to indifferent fixed points of T (see Thaler 1980).

§ 3. Multidimensional continued fractions

3.1. *Bruns's algorithm.* Brun's "subtractive" algorithm (see Brun 1957) can be described as follows: Let

$$\Delta^{n+1} = \{b = (b_0, b_1, \ldots, b_n) \colon b_0 \geqslant b_1 \geqslant \ldots \geqslant b_n \geqslant 0\}.$$

Then define

$$\sigma_1 b = (b_0 - b_1, b_1, \ldots, b_n).$$

There is an index $i = i(b)$, $0 \leqslant i \leqslant n$, such that the n-tuple

$$\pi\sigma_1 b = (b_1, \ldots, b_i, b_0 - b_1, b_{i+1}, \ldots, b_n) \in \Delta^{n+1}.$$

Here π refers to an appropriate cyclic permutation. The digit map $b \mapsto i(b)$ induces a partition of Δ^{n+1} into $n+1$ cells. For the purpose of ergodic theory we use the projection (E^n denotes the n-dimensional unit cube):

$$p\colon \Delta^{n+1} \to \Delta^n \cap E^n = : B^n,$$

$$pb = \left(\frac{b_1}{b_0}, \ldots, \frac{b_n}{b_0}\right).$$

Then there is a unique map T_1 such that the diagram

$$\begin{array}{ccc} \Delta^{n+1} & \xrightarrow{\pi\sigma_1} & \Delta^{n+1} \\ p\downarrow & & \downarrow p \\ B^n & \xrightarrow[T_1]{} & B^n \end{array}$$

commutes. It is known that T_1 is ergodic and the invariant density is given by

$$h(x) = \int_0^\infty dy_1 \int_0^1 dy_2 \ldots \int_0^1 dy_n K(x, y)$$

where

$$K(x, y) = (1 + x_1 y_1 + \ldots + x_n y_n)^{-n-1}.$$

The following "multiplicative acceleration" of Brun's algorithm may be considered. Instead of using σ_1 take δ_1, namely

$$\delta_1 b = (b_0 - kb_1, b_2, \ldots, b_n), \qquad k = \left[\frac{b_0}{b_1}\right].$$

The corresponding map $\delta_1\colon B^n \to B^n$ is ergodic and has invariant density

$$h(x) = \int_{E^n} K(x, y)\, dy$$

(see Schweiger 1982).

3.2. *Selmer's algorithm.* The "subtractive" algorithm of Selmer (see Selmer 1961) is similar to Brun's algorithm. For $b \in \Delta^{n+1}$ we consider

$$\sigma_2 b = (b_0 - b_n, b_1, \ldots, b_n)$$

Then there is an index $j = j(b)$, $0 \leqslant j \leqslant n$, such that

$$\pi\sigma_2 b = (b_1, b_2, \ldots, b_j, b_0 - b_n, b_{j+1}, \ldots, b_n) \in \Delta^{n+1}.$$

The map $T_2\colon B^n \to B^n$ is defined in a way such that the diagram

$$\begin{array}{ccc} \Delta^{n+1} & \xrightarrow{\pi\sigma_2} & \Delta^{n+1} \\ p\downarrow & & \downarrow p \\ B^n & \xrightarrow[T_2]{} & B^n \end{array}$$

is commutative.

The ergodic behavior of T_2 differs noticeably from that of T_1. The basic set B^n splits into a transient part A (almost every $x \in A$ leaves A forever) and a conservative part C. The restriction of T_2 to C is ergodic and has the invariant density

$$h(x) = \int_0^\infty \dots \int_0^\infty K(x, y)\,dy$$

(see Schweiger 1982).

As before one can look at the "multiplicative acceleration" defined by

$$\delta_2 b = (b_0 - kb_n, b_1, \dots, b_n), \qquad k = \left[\frac{b_0}{b_n}\right].$$

For $n \geqslant 2$ the ergodic behavior of the related map is unexplored. The difficulty is connected with the following property: For any digit k the image $TB(k)$ is not a union of cylinders of the time-1-partition.

3.3. *Jacobi–Perron algorithm.* For any $(n+1)$-tupel

$$a = (a_0, a_1, \dots, a_n), \qquad a_j \geqslant 0, \qquad 0 \leqslant j \leqslant n,$$

define

$$\delta_3 a = (a_1, a_2 - k_1 a_1, \dots, a_0 - k_n a_1),$$

$$k_1 = \left[\frac{a_2}{a_1}\right], \dots, k_n = \left[\frac{a_0}{a_1}\right].$$

Various aspects of this algorithm are considered in Peron 1907, Bernstein 1971 and Schweiger 1973. The metrical theory is obtained by considering the map T_3 which makes the diagram

$$\begin{array}{ccc} (\boldsymbol{R}^+)^{n+1} & \xrightarrow{\delta_3} & (\boldsymbol{R}^+)^{n+1} \\ \downarrow p & & \downarrow p \\ E^n & \xrightarrow[T_3]{} & E^n \end{array}$$

commutative. It is known that T_3 is ergodic and that there exists an invariant measure, but up to now all attempts have failed to calculate its density for $n \geqslant 2$.

3.4. *Güting's algorithm.* The following map was proposed by Güting 1975: Define

$$\delta_4(b_0, b_1, \dots, b_n) = (b_1, b_2, \dots, b_n, b_0 - g_1 b_1 - \dots - g_n b_n),$$

$$g_1 = \left[\frac{b_0}{b_1}\right], \qquad g_2 = \left[\frac{b_0 - g_1 b_1}{b_2}\right],$$

$$g_3 = \left[\frac{b_0 - g_1 b_1 - g_2 b_2}{b_3}\right], \dots, g_n = \left[\frac{b_0 - g_1 b_1 - \dots - g_{n-1} b_{n-1}}{b_n}\right].$$

Clearly, for questions of ergodic theory one uses T_4 as in

$$\begin{array}{ccc} \Delta^{n+1} & \xrightarrow{\delta_4} & \Delta^{n+1} \\ \downarrow p & & \downarrow p \\ B^n & \xrightarrow[T_4]{} & B^n \end{array}.$$

It is known that T_4 is ergodic and admits an invariant measure (Schweiger 1977).

3.5. *Skew products*. Recently Shunji Ito has investigated several types of 2-dimensional skew products with applications to number theory (Ito 1984). An n-dimensional generalization is the following algorithm:

$$\delta_5(a_0, a_1, \dots, a_4) = (a_1, a_0 - k_1 a_1, a_2 - k_2 a_1, \dots, a_n - k_n a_1),$$

$$k_1 = \left[\frac{a_0}{a_1}\right], \quad k_2 = \left[\frac{a_2}{a_1}\right], \dots, k_n = \left[\frac{a_n}{a_1}\right].$$

In inhomogeneous terms

$$T_5(x_1, x_2, \dots, x_n) = \left(\frac{1}{x_1} - k_1, \frac{x_2}{x_1} - k_2, \dots, \frac{x_n}{x_1} - k_n\right).$$

Let $\pi = \pi(x)$ be a permutation such that

$$x_{\pi 1} \leqslant x_{\pi 2} \leqslant \dots \leqslant x_{\pi n}.$$

Then put

$$E_j = \{x = (x_1, x_2, \dots, x_n)\colon \pi_j = 1\}.$$

The invariant density is given piecewise as follows:

$$h(x) = \int_0^1 \frac{(1+y_1)^{j-1}}{(1+x_1 y_1)^{n+1}} dy_1, \qquad x \in E_j, \qquad 1 \leqslant j \leqslant n.$$

Since this result has not been published, we will sketch the proof. The digits are subject to the following conditions:

(a) $k_1(x) \geqslant k_j(x) \geqslant 0$, $1 \leqslant j \leqslant n$,
(b) $k_1(x) \geqslant 1$,
(c) If $k_1(x) = k_j(x)$ for $j \geqslant 2$ then $k_j(Tx) = 0$.

Condition (c) implies:

(c′) If $k_j(x) \geqslant 1$ for $j \geqslant 2$, then for any z with $Tz = x$ the condition $k_1(z) > k_j(z)$ holds.

Let $x \in E_j$. Then without loss of generality we may assume

$$x_2 \leqslant \dots \leqslant x_j \leqslant x_1 \leqslant x_{j+1} \leqslant \dots \leqslant x_n.$$

Hence

$$k_2 = \ldots = k_j = 0, \quad k_{j+1} \geqslant 1, \ldots, k_{n+1} \geqslant 1.$$

If $Tz = x$ then

$$z = \left(\frac{1}{g_1+x_1}, \frac{g_2+x_2}{g_1+x_1}, \ldots, \frac{g_n+x_n}{g_1+x_1}\right).$$

By conditions (a) and (c′) we have

$$0 \leqslant g_2 < g_1, \ldots, 0 \leqslant g_j < g_1,$$

$$0 \leqslant g_{j+1} \leqslant g_1, \ldots, 0 \leqslant g_n \leqslant g_1.$$

If $z \in E_i$ then exactly i digits out of $g_2, g_3, \ldots, g_n$ are zero. Therefore, if $z \in E_i$, then

$$h\left(\frac{1}{g_1+x_1}, \frac{g_2+x_2}{g_1+x_1}, \ldots, \frac{g_n+x_n}{g_1+x_1}\right)\frac{1}{(g_1+x_1)^{n+1}} = \int_0^1 \frac{(1+y_1)^{i-1}}{(g_1+x_1+y_1)^{n+1}} dy.$$

Let $A(j, g_1; i)$ denote the number of all $(n-1)$-tuples $(g_2, \ldots, g_n)$ which are admissible for $z \in E_i$ (given the value of g_1). Then

$$\sum_{Tz=x} h(z) z_1^{n+1} = \sum_{g_1=1}^{\infty} \sum_{i=1}^{n} A(j, g_1; i) \int_0^1 \frac{(1+y_1)^{i-1}}{(g_1+x_1+y_1)^{n+1}} dy_1$$

$$= \sum_{g_1=1}^{\infty} \int_0^1 \frac{(g_1+1+y_1)^{j-1}(g_1+y_1)^{n-j}}{(g_1+x_1+y_1)^{n+1}} dy_1$$

$$= \sum_{g_1=1}^{\infty} \int_{g_1}^{g_1+1} \frac{(z_1+1)^{j-1} z_1^{n-j}}{(z_1+x_1)^{n+1}} dz_1$$

$$= \int_1^{\infty} \frac{(z_1+1)^{j-1} z_1^{n-j}}{(z_1+x_1)^{n+1}} dz_1 = \int_0^1 \frac{(1+w_1)^{j-1}}{(1+x_1 w_1)} dw_1.$$

3.6. *Remarks.* (1) A good account on arithmetic properties of several multidimensional continued fractions is Brentjes 1981.

(2) The known densities are given in the form of integrals. This representation has its explanation in the theory of dual (or backward) algorithms (Schweiger 1979, Ito 1984).

§ 4. The Parry–Daniels map

4.1. Daniels' 1962 and Parry 1962 consider the following map. Let

$$\Delta = \{x \in E^n \colon \sum_{i=1}^{n} x_i = 1\}.$$

Denote by π a permutation such that

$$x_{\pi 1} \leqslant x_{\pi 2} \leqslant \ldots \leqslant x_{\pi n}.$$

Then define $T\colon \Delta \to \Delta$ as

$$T(x_1, x_2, \ldots, x_n) = \left(\frac{x_{\pi 1}}{x_{\pi n}}, \frac{x_{\pi 2} - x_{\pi 1}}{x_{\pi n}}, \ldots, \frac{x_{\pi n} - x_{\pi(n-1)}}{x_{\pi n}}\right).$$

4.2. The restriction of the measure with density

$$h(x) = \frac{1}{x_1 (x_1 + x_2) \ldots (x_1 + x_2 + \ldots + x_{n-1})}$$

to Δ is an invariant measure (see Daniels 1962).

4.3. The case $n = 2$ can be reduced to continued fractions (Parry 1962). Therefore in this case T is ergodic.

4.4. It is unknown if T is ergodic for $n \geqslant 3$. However Schweiger 1981 a proves that T is not conservative for $n = 3$. The proof consists in constructing a Cantor set F with $TF = F$ but $\lambda(F) > 0$. Unfortunately the method breaks down for $n \geqslant 4$.

§ 5. Final remarks

The aim of this survey is to show that there are examples of maps which arise in more or less natural contexts. But their ergodic behavior is not easy to determine or leads to problems which are unsolved up to now.

References

J. Aaronson, 1978: *Ergodic theory for inner functions of the upper half plane*, Ann. Inst. H. Poincaré, Sect. B, 14, pp. 233–255.

R. L. Adler, 1973: *F-expansions revisited*, in: *Recent Advances in Topological Dynamics*, Lectures Notes in Math. 318, pp. 263–278.

R. L. Adler and B. Weiss, 1973: *The ergodic infinite measure preserving transformation of Boole*, Israel J. Math. 16, pp. 263–278.

L. Bernstein, 1971: *The Jacobi–Perron algorithm. Its theory and application*, Lecture Notes in Math. 207. Springer-Verlag. Berlin–Heidelberg–New York.

A. J. Brentjes, 1981: *Multi-dimensional continued fraction algorithms*, Mathematical Centre Tracts 145, Mathematisch Centrum Amsterdam.

V. Brun, 1957: *Algorithmes euclidiens pour trois et quatre nombres*, in: *13 ième Congr. Math. Scand.*, pp. 45–64. Helsinki.

H. E. Daniels, 1962: *Processes generating permutation expansions*, Biometrika 49, pp. 139–149.

R. Fischer, 1972: *Ergodische Theorie von Ziffernentwicklungen in Wahrscheinlichkeitsräumen*, Math. Z. 128, pp. 217–230.

J. Galambos, 1976: *Representations of real numbers by infinite series*, Lecture Notes in Math. 502. Springer-Verlag, Berlin–Heidelberg–New York.

R. Güting, 1975: *Zur Verallgemeinerung des Kettenbruchalgorithmus*, I, J. Reine Angew. Math. 278/279, pp. 165–173.

Sh. Ito, 1984: *Number theoretic expansions, algorithms and metrical observations*, Séminaire de Théorie des Nombres de Bordaux Année 1984–85 exposé no. 3.

Sh. Ito and S. Tanaka, 1981: *On a family of continued fraction transformations and their ergodic properties*, Tokyo J. Math. 4, pp. 153–176.

H. Jager and C. de Vroedt, 1969: *Lüroth series and their ergodic properties*, Indag. Math. 31, pp. 31–42.

S. Kalpazidou, 1986: *On a problem of Gauss–Kuzmin type for continued fraction with odd partial quotients*, Pacific J. Math. 123, pp. 103–114.

H. Nakada, 1981: *Metrical theory for a class of continued fraction transformations and their natural extensions*, Tokyo J. Math. 4, pp. 399–426.

W. Parry, 1862: *Ergodic properties of some permutation processes*, Biometrika 49, pp. 151–154.

O. Perron, 1907: *Grundlagen für eine Theorie des Jacobischen Kettenbruchalgorithmus*, Math. Ann. 64, pp. 1–76.

– 1960: *Irrationalzahlen*, Walter de Gruyter & Co., Berlin.

W. Philipp, 1967: *Some metrical results in number theory*, Pacific J. Math. 20, pp. 109–127.

A. Rényi, 1957: *Representations for real numbers and their ergodic properties*, Acta Math. Acad. Sci. Hungar. 8, pp. 477–493.

H. G. Rieger, 1978: *Ein Gauss–Kusmin–Lévy-Satz für Kettenbrüche nach nächstem Ganzen*, Manuscripta Math. 24, pp. 437–448.

S. D. Rudolfer, 1971: *Ergodic properties of linear fractional transformations mod one*, Proc. London Math. Soc. (3) 23, pp. 515–531.

C. Ryll-Nardzewski, 1951: *On the ergodic theorems* (*Ergodic theory of continued fractions*), Studia Math. 12, pp. 74–79.

F. Schweiger, 1972: *Metrische Sätze über Oppenheimentwicklungen*, J. Reine Angew. Math. 254, pp. 152–159.

– 1973: *The metrical theory of Jacobi-Perron algorithm*, Lecture Notes in Math. 334. Springer-Verlag, Berlin–Heidelberg–New York.

– 1975a: *Numbertheoretical endomorphisms with σ-finite invariant measure*, Israel J. Math. 21, pp. 308–318.

–, 1975b: *Some remarks on ergodicity and invariant measures*, Michigan Math. J. 22, pp. 181–187.

–, 1977: *Über einen Algorithmus von R. Güting*, J. Reine Angew. Math. 293/294, pp. 263–270.

–, 1978: $\tan x$ *is ergodic*, Proc. Amer. Math. Soc. 71, pp. 54–56.

–, 1979: *Dual algorithms and invariant measures*, Arbeitsberichte Inst. Math. Salzburg 3/1979, pp. 17–29 (mimeographed).

–, 1981a: *On the Parry–Daniels transformation*, Analysis 1, pp. 171–175.

–, 1981b: *Ergodic properties of fibered systems*, in: *Proc. Sixth Conference Probability Theory 1979, Braşov, Romania*, Editura Acad. Rep. Soc. Rom., Bucureşti, pp. 221–228.

–, 1982: *Ergodische Eigenschaften der Algorithmen von Brun und Selmer*, S. = Ber. Öster. Akad. Wiss. Math. naturw. Kl. Abt. II, Math. Phys. Techn. Wiss. 191, pp. 325–329.

–, 1983: *Invariant measures for piecewise linear fractional maps*, J. Austral. Math. Soc. (Series A) 34, pp. 55–59.

–, 1986: *Piecewise fractional linear maps with explicit invariant measure*, S. = Ber. Öster. Akad. Wiss. Math. naturw. Kl. Abt. II, Math. Phys. Techn. Naturw. 195, pp. 171–174.

E. Selmer, 1961: *Om flerdimensjonal kjede brøk*, Nordisk Mat. Tidskr. 9, pp. 37–43.

W. Sierpiński, 1911: *Sur quelques algorithmes pour développer les nombres réels en séries*, in: *Oeuvres choisies*. Tome I Warszawa 1974, pp. 236–254.

M. Thaler, 1980: *Estimates of the invariant densities of endomorphisms with indifferent fixed points*, Israel J. Math. 37, pp. 303–314.

W. Vervaat, 1972: *Success epochs in Bernoulli trials (with applications in number theory)*, Math. Centre Tracts 42. Math Centrum Amsterdam.

D. Zagier, 1981: *Zetafunktionen und quadratische Körper*, Springer-Verlag, Berlin–Heidelberg–New York.