ON ERROR ESTIMATES FOR INTERPOLATING SPLINES WITH MINIMAL DEFECT

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1. Introduction and basic notation

The problems of spline approximation have recently been studied very intensively. One of the main aspects of this research is connected with error estimates, and interpolating splines with minimal defect are of particular interest here. For such splines, a number of precise results have been obtained for traditional problems of approximation of function classes; it has also turned out that in many cases these results are sharp in the broadest possible sense, i.e. in the sense of realizing Kolmogorov's diameters. Note that for interpolating polynomials the problem of sharp, or at least asymptotically sharp error estimates remains open for a long time now.

Progress in error estimates for spline approximation was achieved, in the first place, by using the specific character of splines (see e.g. [3]). Here we will show that in the case where the error is estimated by the norm of the m th derivative of the interpolated function, the sharp results are consequences of some general comparison theorems.

Below, $C[a, b]$ and $L_p[a, b]$ ($1 \leq p \leq \infty$) denote respectively the spaces of continuous, integrable with exponent p ($1 \leq p < \infty$) and essentially bounded ($p = \infty$) functions f on the interval $[a, b]$, with the usual norms:

$$\begin{aligned} \|f\|_{C[a,b]} &= \max_{a \leq t \leq b} |f(t)|, & \|f\|_{L_\infty[a,b]} &= \operatorname{ess\,sup}_{a \leq t \leq b} |f(t)|, \\ \|f\|_{L_p[a,b]} &= \left(\int_a^b |f(t)|^p dt \right)^{1/p} & (1 \leq p < \infty). \end{aligned}$$

The corresponding spaces of 2π -periodic functions defined on the whole real line will be denoted by C and L_p ; for brevity we will write for $f \in L_p$

$$\|f\|_p = \|f\|_{L_p} = \|f\|_{L_p[0, 2\pi]} \quad (1 \leq p \leq \infty).$$

Together with the space $C^m[a, b]$ of m times continuously differentiable functions on $[a, b]$, $m = 1, 2, \dots$, we will consider the set $L_p^m[a, b]$ of those functions f for which $f^{(m-1)}$ is absolutely continuous on $[a, b]$ and $f^{(m)} \in L_p[a, b]$. In the periodic case the corresponding classes will be denoted by C^m and L_p^m ; one may say that C^m and L_p^m are the sets of m th periodic integrals of functions g from C and L_p respectively such that $\int_0^{2\pi} g(t) dt = 0$. In relations of general character, the symbols $C^0[a, b]$ and $L_p^0[a, b]$ are understood as $C[a, b]$ and $L_p[a, b]$, and similarly for C^0 and L_p^0 .

Further, for $K > 0$, $m = 1, 2, \dots$, $1 \leq p \leq \infty$, let

$$KW_p^m[a, b] = \{f: f \in L_p^m[a, b], \|f^{(m)}\|_{L_p[a, b]} \leq K\},$$

$$W_p^m[a, b] = 1W_p^m[a, b],$$

and similarly in the periodic case:

$$KW_p^m = \{f: f \in L_p^m, \|f^{(m)}\|_p \leq K\}, \quad W_p^m = 1W_p^m.$$

Note that $KW_p^{m+1}[a, b]$ coincides with the class $W^m KH^1[a, b]$ of functions $f \in C^m[a, b]$ such that for $t', t'' \in [a, b]$

$$(1) \quad |f^{(m)}(t') - f^{(m)}(t'')| \leq K |t' - t''|.$$

Analogously, KW_∞^{m+1} coincides with the class $W^m KH^1$ of 2π -periodic functions $f \in C^m$ satisfying (1) for all t', t'' .

2. Comparison theorems

In proving Theorems 1 and 2 below, we will have to use some considerations analogous to Rolle's theorem, but in a more general setting — for integrals of, generally speaking, discontinuous functions. To simplify the reasoning, we introduce the notions of separated zeros and essential sign changes (for details see [3, §1.2]).

Suppose $f \in C[a, b]$ and $f(x_1) = f(x_2) = 0$, where $a \leq x_1 < x_2 \leq b$. The zeros x_1 and x_2 are called *separated* if f is not identically zero on the interval (x_1, x_2) . Saying that a continuous function has n separated zeros in an interval (open, closed or half-closed) will mean that the zeros are pairwise separated.

Moreover, we will say that a piecewise continuous function f *essentially changes sign* n times on (a, b) if there exist $n+1$ points z_j , $a < z_0 < z_1 < \dots < z_n < b$, such that f is continuous at each z_j and

$$\operatorname{sgn} f(z_{j-1}) = -\operatorname{sgn} f(z_j) \neq 0 \quad (j = 1, \dots, n).$$

For continuous 2π -periodic functions, the number of separated zeros is computed on any period $[a, a+2\pi)$. The number of essential sign changes of

a piecewise continuous 2π -periodic function is computed on any interval $(a, a+2\pi)$, where $f(a) \neq 0$; this number is always even.

We now turn to comparison theorems, starting from the periodic case. Fix a partition

$$(2) \quad \Delta_N^t: \quad t_0 < t_1 < \dots < t_N = t_0 + 2\pi;$$

we will think of it as being extended 2π -periodically to the whole real line. Denote by $KH_{\Delta_N^t}^1$ the set of 2π -periodic functions f satisfying the Lipschitz condition with constant K on each interval (t_{i-1}, t_i) , $i = 1, \dots, N$:

$$(3) \quad |f(t') - f(t'')| \leq K |t' - t''| \quad (t_{i-1} < t', t'' < t_i).$$

A function $f \in KH_{\Delta_N^t}^1$ may have discontinuities at the knots t_k of the partition (2); for definiteness one can assume that $f(t_k) = f(t_k - 0)$.

Further, let $W^m KH_{\Delta_N^t}^1$ ($m = 1, 2, \dots$) be the set of m th periodic integrals of functions $f \in KH_{\Delta_N^t}^1$ such that $\int_0^{2\pi} f(t) dt = 0$; $W^0 KH_{\Delta_N^t}^1 = KH_{\Delta_N^t}^1$. Note that for any partition (2), $W^m KH_{\Delta_N^t}^1$ contains the class $KW_{\infty}^{m+1} = W^m KH^1$, but is not a subset of C^m since it contains functions with discontinuous m th derivative.

If $t_0 = 0$ and the knots are equally spaced, i.e. $t_k = 2k\pi/N$ ($k = 0, 1, \dots, N$), we write $W^m KH_N^1$ instead of $W^m KH_{\Delta_N^t}^1$. Furthermore, we put $W^m H_{\Delta_N^t}^1 = W^m 1 H_{\Delta_N^t}^1$, $W^m H_N^1 = W^m 1 H_N^1$.

THEOREM 1. Suppose that $\varphi \in W^{m-1} H_{\Delta_N^t}^1$ satisfies

$$(4) \quad \pm \varphi^{(m)}(t) = (-1)^i, \quad t_{i-1} < t < t_i, \quad i = 1, \dots, N,$$

and that φ has N zeros in a period:

$$\varphi(\tau_j) = 0 \quad (j = 1, \dots, N), \quad \tau_1 < \dots < \tau_N < \tau_{N+1} = \tau_1 + 2\pi.$$

If $f \in W^{m-1} KH_{\Delta_N^t}^1$ and $f(\tau_j) = 0$ ($j = 1, 2, \dots$), then for all t (for all $t \neq \tau_j$ if $m = 1$)

$$(5) \quad |f(t)| \leq K |\varphi(t)|.$$

Proof. For $m = 1$ this is easy, since then $|f'(t)| \leq K$ and $\pm \varphi'(t) = (-1)^i$ on each interval (t_{i-1}, t_i) , and $f(\tau_k) = \varphi(\tau_k) = 0$ for some $\tau_k \in (t_{i-1}, t_i)$.

Let $m \geq 2$. Suppose, on the contrary, that $|f(\xi)| > K |\varphi(\xi)|$ for some ξ , $\tau_1 < \xi < \tau_{N+1}$, $\xi \neq \tau_j$. There exists λ with $0 < |\lambda| < 1$ such that $\lambda f(\xi) = K \varphi(\xi)$, and so the difference

$$(6) \quad \delta(t) = K \varphi(t) - \lambda f(\xi)$$

satisfies $\delta(\tau_j) = 0$ ($j = 1, \dots, N$), $\delta(\xi) = 0$, i.e. δ has $N+1$ distinct zeros in the period $[\tau_1, \tau_{N+1})$.

Consider now the $(m-1)$ th derivative of (6):

$$\delta^{(m-1)}(t) = K\varphi^{(m-1)}(t) - \lambda f^{(m-1)}(t),$$

which exists for all $t \neq t_i$. If $t_{i-1} < t' < t'' < t_i$, then

$$|\varphi^{(m-1)}(t') - \varphi^{(m-1)}(t'')| = |t' - t''|$$

and (since $|\lambda| < 1$)

$$|\lambda f^{(m-1)}(t') - \lambda f^{(m-1)}(t'')| \leq K|\lambda||t' - t''| < K|t' - t''|.$$

Therefore

$$|\lambda f^{(m-1)}(t') - \lambda f^{(m-1)}(t'')| < K|\varphi^{(m-1)}(t') - \varphi^{(m-1)}(t'')|.$$

This fact, together with the assumptions of Theorem 1, has two important consequences (the reader is advised to draw a picture here):

1) The derivative $\delta^{(m-1)}$, and hence also the difference (6), is identically zero on no interval $[\alpha, \beta]$; it follows that all zeros of δ are separated.

2) On each interval (t_{i-1}, t_i) between two consecutive points of the partition (2) the derivative $\delta^{(m-1)}$ can change sign only once, and if it essentially changes sign at t_i , then it has constant sign in (t_{i-1}, t_i) or in (t_i, t_{i+1}) .

By 2), $\delta^{(m-1)}$ can have at most N essential sign changes on a period, and hence the continuous function $\delta^{(m-2)}$ cannot have more than N separated zeros on a period. However, the assumption that the inequality (5) fails at $t = \xi \neq \tau_j$ has led us to the conclusion that δ has $N+1$ separated zeros on a period; by Rolle's theorem, the continuous derivative $\delta^{(m-2)}$ then also has $N+1$ separated zeros. This contradiction proves (5). Note that in the proof we have used the scheme of reasoning from [7], where the uniform error by spline interpolation on W_∞^m is estimated.

COROLLARY 1. Let $f \in KW_\infty^m$ and suppose, under the assumptions of Theorem 1, that $f(\tau_j + \alpha) = 0$ ($j = 1, \dots, N$) for some $\alpha \in \mathbf{R}$. Then

$$|f(t + \alpha)| \leq K|\varphi(t)| \quad \text{for all } t \quad \text{and} \quad \|f\|_p \leq K\|\varphi\|_p \quad (1 \leq p \leq \infty).$$

If the partition (2) satisfies

$$(7) \quad \sum_{i=1}^N (-1)^i (t_i - t_{i-1}) = 0,$$

then the derivative $\varphi^{(m)}$, defined by (4), has zero average on a period, and we can assume φ to be the m th periodic integral of $\varphi^{(m)}$. In this case $\varphi \in C^{m-1}$ and φ is a perfect periodic spline of order m corresponding to the knots (2), and uniquely determined by these knots up to sign.

In the case $N = 2n$ and $t_k = k\pi/n$ ($k = 0, 1, \dots, 2n$), condition (7) is obviously satisfied, and in Theorem 1 we can take for φ the perfect Euler

spline $\varphi_{n,m}$ defined by

$$\varphi_{n,0}(t) = \operatorname{sgn} \sin nt, \quad \varphi_{n,m}(t) = \int_{\gamma_m}^t \varphi_{n,m-1}(u) du \quad (m = 1, 2, \dots),$$

where $\gamma_m = [1 - (-1)^m] \pi / (4n)$. The zeros of the spline $\varphi_{n,m}$ on the period $(0, 2\pi]$ are

$$\tau_{m,j} = j\pi/n - [1 - (-1)^m] \pi / (4n) \quad (j = 1, \dots, 2n).$$

Clearly, $K\varphi_{n,m} \in KW_\infty^m$.

Theorem 1 also immediately implies

COROLLARY 2. Suppose $f \in W^{m-1}KH_{2n}^1$ (in particular, $f \in KW_\infty^m$) and $f(\tau_{m,j}) = 0$ ($j = 1, \dots, 2n$). Then $|f(t)| \leq K|\varphi_{n,m}(t)|$ for all t (for all $t \neq k\pi/n$ if $m = 1$). Consequently,

$$\begin{aligned} \sup \{ \|f\|_p : f \in W^{m-1}KH_{2n}^1, f(\tau_{m,j}) = 0 \ (j = 1, \dots, 2n) \} \\ = \sup \{ \|f\|_p : f \in KW_\infty^m, f(\tau_{m,j}) = 0 \ (j = 1, \dots, 2n) \} \\ = K \|\varphi_{n,m}\|_p \quad (1 \leq p \leq \infty). \end{aligned}$$

Let us consider analogous problems for functions defined on a finite interval $[a, b]$. Here we have to take into account, together with the zeros of the reference function φ , also the zeros of its derivatives at a and b .

Fix a partition

$$(8) \quad \Delta_N[a, b]: \quad a = t_0 < t_1 < \dots < t_N = b$$

and denote by $KH_{\Delta_N[a,b]}^1$ ($1H_{\Delta_N[a,b]}^1 = H_{\Delta_N[a,b]}^1$) the set of functions f defined on $[a, b]$ and satisfying (3) on each interval (t_{i-1}, t_i) ($i = 1, \dots, N$). Also, denote by $W^m KH_{\Delta_N[a,b]}^1$ ($W^m 1H_{\Delta_N[a,b]}^1 = W^m H_{\Delta_N[a,b]}^1$) the set of m th integrals on $[a, b]$ of functions $g \in KH_{\Delta_N[a,b]}^1$, i.e. the set of all $f \in C^{m-1}[a, b]$ whose m th derivative exists on each interval (t_{i-1}, t_i) and satisfies (1) for $t', t'' \in (t_{i-1}, t_i)$. We will assume that for $f \in W^m KH_{\Delta_N[a,b]}^1$, hold $f^{(v)}(a) = f^{(v)}(a+0)$, $f^{(v)}(b) = f^{(v)}(b-0)$ for all $v = 0, 1, \dots, m-1$ (as usual, $f^{(0)}(t)$ means $f(t)$).

For each $m = 1, 2, \dots$, we will choose two subsets I_a, I_b of $Q_m = \{0, 1, \dots, m-1\}$ which may be empty or coincide with Q_m . We denote by $|I_a|, |I_b|$ their cardinalities.

THEOREM 2. Suppose that $\varphi \in W^{m-1}H_{\Delta_N[a,b]}^1$, with $\Delta_N[a, b]$ defined by (8), satisfies

$$\begin{aligned} \pm \varphi^{(m)}(t) &= (-1)^i, \quad t_{i-1} < t < t_i, \quad i = 1, \dots, N, \\ (9) \quad \varphi(\tau_j) &= 0 \quad (j = 1, \dots, L), \quad a < \tau_1 < \dots < \tau_L < b, \\ \varphi^{(\mu)}(a) &= 0 \quad (\mu \in I_a), \quad \varphi^{(\kappa)}(b) = 0 \quad (\kappa \in I_b), \end{aligned}$$

where $I_a, I_b \subset Q_m$ and

$$(10) \quad L + |I_a| + |I_b| = N + m - 1.$$

Assume that $f \in W^{m-1} KH_{\Delta N[a,b]}^1$, $f(\tau_j) = 0$ ($j = 1, \dots, L$) and

$$f^{(\mu)}(a) = 0 \quad (\mu \in I_a), \quad f^{(\kappa)}(b) = 0 \quad (\kappa \in I_b).$$

Then

$$|f(t)| \leq K |\varphi(t)| \quad \text{for all } t \in [a, b] \quad (\text{for } t \neq t_j \text{ if } m = 1).$$

The proof goes, generally speaking, along the same lines as in the periodic case, with some complications due to boundary conditions. Denote by $v(g)$ the maximal number of separated zeros of $g \in C[a, b]$ on $[a, b]$ and by $\mu(g)$ the number of essential sign changes of a piecewise continuous function g on $[a, b]$.

If we assume, for $m \geq 2$, that $|f(\xi)| > K |\varphi(\xi)|$ at some $\xi \in (a, b)$, $\xi \neq \tau_j$, then for some λ , with $0 < |\lambda| < 1$, the difference $\delta(t) = K\varphi(t) - \lambda f(t)$ has at least $L+1$ distinct zeros in (a, b) . These zeros are separated, since for $t', t'' \in (t_{i-1}, t_i)$, $i = 1, \dots, N$,

$$(11) \quad |\lambda f^{(m-1)}(t') - \lambda f^{(m-1)}(t'')| < K |\varphi^{(m-1)}(t') - \varphi^{(m-1)}(t'')| = K |t' - t''|,$$

which means that neither δ nor any of its derivatives $\delta^{(k)}$ ($k = 1, \dots, m-1$) is identically zero on any interval of positive length. Hence $v(\delta) \geq L+1$, and if $0 \in I_a \cup I_b$ (i.e. $\delta(a) = 0$ or $\delta(b) = 0$) or $0 \in I_a \cap I_b$ (i.e. $\delta(a) = \delta(b) = 0$), then respectively $v(\delta) \geq L+2$ or $v(\delta) \geq L+3$.

Estimating now, with the use of Rolle's theorem, the number of separated zeros of the derivatives $\delta', \delta'', \dots, \delta^{(m-2)}$ in $[a, b]$, we observe that each differentiation decreases this number by at most one, and each of the conditions (9) increases it by one. If $m-1$ belongs neither to I_a nor to I_b , i.e. if all boundary conditions involve the derivatives up to order $m-2$, then by (10) we finally obtain

$$v(\delta^{(m-2)}) \geq L+1 - (m-2) + |I_a| + |I_b| = N+2,$$

which implies $\mu(\delta^{(m-1)}) \geq N+1$. But this is impossible, since the validity of (11) on each interval (t_{i-1}, t_i) ($i = 1, \dots, N$) yields $\mu(\delta^{(m-1)}) \leq N$. Similarly, we obtain a contradiction in the case $m-1 \in I_a \cup I_b$ (i.e. $\delta^{(m-1)}(a) = 0$ or $\delta^{(m-1)}(b) = 0$) or $m-1 \in I_a \cap I_b$ (i.e. $\delta^{(m-1)}(a) = \delta^{(m-1)}(b) = 0$).

The function φ in Theorem 2 is in $W^{m-1} H_{\Delta N[a,b]}^1$, and the continuity of the derivative $\varphi^{(m-1)}$ on $[a, b]$ is not assumed. If, however, $\varphi^{(m-1)}$ is continuous, then φ is a perfect spline of order m corresponding to the knots (8). Clearly, this cannot be guaranteed for all knots (8) and all boundary conditions.

Let us note three particular cases of Theorem 2 in each of which φ is a perfect spline. The first two corollaries formulated below concern the cases where the sets I_a and I_b are empty or coincide with Q_m ; the existence of perfect splines then follows from general facts proved e.g. in [3].

COROLLARY 1. Suppose that a perfect spline φ of order m corresponding to the knots (8) has $N+m-1$ distinct zeros in $[a, b]$. Assume that

$$f \in W^{m-1} KH_{\Delta_N[a,b]}^1 \quad \text{and} \quad f(\tau_j) = 0 \quad (j = 1, \dots, N+m-1).$$

Then

$$|f(t)| \leq K |\varphi(t)| \quad \text{for all } t \in [a, b].$$

COROLLARY 2. Suppose that $N > m$ and a perfect spline φ of order m corresponding to the knots (8) has $N-m-1$ distinct zeros in $[a, b]$ and, moreover,

$$\varphi^{(v)}(a) = \varphi^{(v)}(b) = 0 \quad (v = 0, 1, \dots, m-1).$$

Assume that $f \in W^{m-1} KH_{\Delta_N[a,b]}^1$, $f(\tau_j) = 0$ ($j = 1, \dots, N-m-1$) and $f^{(v)}(a) = f^{(v)}(b) = 0$ ($v = 0, 1, \dots, m-1$). Then

$$|f(t)| \leq K |\varphi(t)| \quad \text{for all } t \in [a, b].$$

In the third case the reference function for $[a, b] = [0, \pi]$ is the restriction of the perfect spline $\varphi_{N,m}$ to $[0, \pi]$. The partition $\Delta_N[0, \pi]$ is determined by the points $t_k = k\pi/N$ ($k = 0, 1, \dots, N$), the τ_j are zeros of $\varphi_{N,m}$ in $(0, \pi)$, and the sets I_0, I_π are determined by the zeros of $\varphi_{N,m}$ and of the derivatives $\varphi_{N,m}^{(v)}$ ($v = 1, \dots, m-1$) at $t = 0$ and $t = \pi$.

COROLLARY 3. Suppose that $f \in W^{m-1} KH_N^1[0, \pi]$ and, for m even

$$f(j\pi/N) = 0 \quad (j = 0, 1, \dots, N),$$

$$f^{(2v)}(0) = f^{(2v)}(\pi) = 0 \quad (v = 1, 2, \dots, m/2-1),$$

while for m odd

$$f((2j-1)\pi/(2N)) = 0 \quad (j = 1, \dots, N),$$

$$f^{(2v-1)}(0) = f^{(2v-1)}(\pi) = 0 \quad (v = 1, \dots, (m-1)/2).$$

Then

$$|f(t)| \leq K |\varphi_{N,m}(t)| \quad \text{for all } t \in [0, \pi].$$

3. Interpolation of periodic functions

We will say that a linear manifold $\mathfrak{N}$ of 2π -periodic splines is *interpolating* at the system of points τ_j : $\tau_1 < \dots < \tau_N < \tau_1 + 2\pi$ if for each $f \in C$ there is a unique spline $\sigma(t) = \sigma(f, t) \in \mathfrak{N}$ satisfying $\sigma(f, \tau_j) = f(\tau_j)$ ($j = 1, \dots, N$).

Let us fix two partitions

$$(12) \quad \Delta_N^t: t_0 < t_1 < \dots < t_N = t_0 + 2\pi,$$

$$(13) \quad \Delta_N^\tau: \tau_0 < \tau_1 < \dots < \tau_N = \tau_0 + 2\pi,$$

and consider the corresponding classes of periodic functions $W^m KH_{\Delta_N}^{1,t_i}$ and $W^m KH_{\Delta_N}^{1,\tau_j}$ ($m = 0, 1, \dots$) defined in Section 2, as well as the subspaces $S_m(\Delta_N')$ and $S_m(\Delta_N'')$ of 2π -periodic splines of order m with defect 1 with respect to the partitions (12) and (13) respectively. Thus e.g. $s \in S_m(\Delta_N')$ means that $s \in C^{m-1}$ and $s^{(m)}(t) = c_i$ for $t \in (t_{i-1}, t_i)$ ($i = 1, \dots, N$), where the t_k are points of the partition (12).

THEOREM 3. *Suppose that the subspace $S_m(\Delta_N')$ is interpolating at the points τ_j ($j = 1, \dots, N$) of the partition (13). If $f \in W^m KH_{\Delta_N}^{1,t_i}$, then*

$$(14) \quad |f(t) - \sigma(f, t)| \leq K |\varphi(t) - \sigma(\varphi, t)|, \quad |t| < \infty,$$

where $\varphi \in W^m H_{\Delta_N}^{1,t_i}$ is determined by

$$\pm \varphi^{(m+1)}(t) = (-1)^i, \quad t_{i-1} < t < t_i, \quad i = 1, \dots, N,$$

$$\varphi^{(m)}(t) = \int_{t_0}^t \varphi^{(m+1)}(u) du + c, \quad t_0 \leq t < t_0 + 2\pi,$$

$$\int_0^{2\pi} \varphi^{(m)}(t) dt = 0.$$

If $\varphi(\tau_j) = 0$ ($j = 1, \dots, N$), then

$$(15) \quad |f(t) - \sigma(f, t)| \leq K |\varphi(t)|, \quad |t| < \infty.$$

The inequalities (14) and (15) are sharp at each t , and if the partition (12) satisfies (7), then they are even sharp on the class KW_{∞}^{m+1} .

The proof reduces to a direct application of Theorem 1. Along with f and φ , the classes $W^m KH_{\Delta_N}^{1,t_i}$ and $W^m H_{\Delta_N}^{1,t_i}$ contain the differences

$$\delta(t) = f(t) - \sigma(f, t) \quad \text{and} \quad \psi(t) = \varphi(t) - \sigma(\varphi, t)$$

respectively. And since clearly $\psi^{(m+1)}(t) = \varphi^{(m+1)}(t) = \pm(-1)^i$ for $t \in (t_{i-1}, t_i)$, Theorem 1 yields the inequality (14). If $\varphi(\tau_j) = 0$ ($j = 1, \dots, N$), then $\sigma(\varphi, t) \equiv 0$ (by the uniqueness of the interpolating spline), and we obtain (15). Finally, if (7) holds then φ is a perfect spline of order $m+1$ and $K\varphi \in KW_{\infty}^{m+1}$.

COROLLARY. *Under the assumptions of Theorem 3 and assuming that (7) holds, we have*

$$(16) \quad \sup_{f \in KW_{\infty}^{m+1}} \|f - \sigma(f)\|_p = K \|\varphi\|_p \quad (1 \leq p \leq \infty).$$

An analogous result for KW_p^{m+1} and the L_1 norm can be obtained by using the following fact.

LEMMA. Suppose that $S_m(\Delta_N^t)$ is interpolating at the points τ_j of the partition (13), and $S_m(\Delta_N^t)$ is interpolating at the points t_i of the partition (12). Denote by $\sigma(f, t)$ and $s(f, t)$ the respective splines for $f \in C$. For every $f \in C^m$ there is $g = g(f, t) \in W^m H_{\Delta_N^t}^1$ such that

$$(17) \quad \|f - \sigma(f)\|_1 = \left| \int_0^{2\pi} [g'(t) - s'(g, t)] f^{(m)}(t) dt \right|.$$

Proof. If $\delta(t) = f(t) - \sigma(f, t)$, then $\delta(\tau_j) = 0$ ($j = 1, \dots, N$) and so

$$(18) \quad \int_{\tau_{j-1}}^{\tau_j} \delta'(t) dt = 0 \quad (j = 1, \dots, N).$$

Integrating by parts over each interval (τ_{j-1}, τ_j) we obtain

$$\begin{aligned} \|\delta\|_1 &= \int_{\tau_0}^{\tau_N} \delta(t) \operatorname{sgn} \delta(t) dt = \sum_{j=1}^N \int_{\tau_{j-1}}^{\tau_j} \delta(t) \operatorname{sgn} \delta(t) dt \\ &= - \sum_{j=1}^N \int_{\tau_{j-1}}^{\tau_j} \delta'(t) l(t) dt = - \int_{\tau_0}^{\tau_N} \delta'(t) l(t) dt, \end{aligned}$$

where $l'(t) = \operatorname{sgn} \delta(t)$ for $t \in (\tau_{j-1}, \tau_j)$, so that $l \in H_{\Delta_N^t}^1$. By (18) we can assume that $\int_0^{2\pi} l(t) dt = 0$.

Let g be the m th periodic integral of l . Then $g \in W^m H_{\Delta_N^t}^1$, and if we put $\eta(t) = g(t) - s(g, t)$ we obtain $\eta(t_i) = 0$ ($i = 1, \dots, N$) and

$$(19) \quad \int_{t_{i-1}}^{t_i} \eta'(t) dt = 0 \quad (i = 1, \dots, N).$$

Since $s^{(m)}(g, t) = \operatorname{const}$ for $t \in (t_{i-1}, t_i)$, (18) shows that

$$\|\delta\|_1 = - \int_{\tau_0}^{\tau_N} \delta'(t) g^{(m)}(t) dt = - \int_{\tau_0}^{\tau_N} \delta'(t) \eta^{(m)}(t) dt = - \int_{\tau_0}^{\tau_N} \delta'(t) \eta^{(m)}(t) dt.$$

The $(m-1)$ -fold integration by parts (with the boundary terms vanishing by periodicity) yields

$$\|\delta\|_1 = (-1)^m \int_{\tau_0}^{\tau_N} \delta^{(m)}(t) \eta'(t) dt.$$

Using (19) we can now write

$$\|\delta\|_1 = (-1)^m \int_{\tau_0}^{\tau_N} f^{(m)}(t) \eta'(t) dt = \left| \int_0^{2\pi} \eta'(t) f^{(m)}(t) dt \right|,$$

which completes the proof of the lemma.

THEOREM 4. Suppose that $S_m(\Delta'_N)$ is interpolating at the points τ_j of the partition (13), and $S_m(\Delta_N)$ is interpolating at the points t_i of (12). Denote by $\sigma(f, t)$ and $s(f, t)$ the respective splines. For every $f \in KW_p^{m+1}$ ($1 \leq p \leq \infty$) we have

$$(20) \quad \|f - \sigma(f)\|_1 \leq K \|\varphi - s(\varphi)\|_{p'} \quad (1/p + 1/p' = 1),$$

where $\varphi \in W^m H_{\Delta_N}^{1, \tau}$ satisfies

$$\pm \varphi^{(m+1)}(t) = (-1)^j, \quad \tau_{j-1} < t < \tau_j, \quad j = 1, \dots, N,$$

$$\varphi^{(m)}(t) = \int_{\tau_0}^t \varphi^{(m+1)}(u) du + c, \quad \tau_0 \leq t < \tau_0 + 2\pi,$$

$$\int_0^{2\pi} \varphi^{(m)}(t) dt = 0.$$

If $\varphi(t_i) = 0$ ($i = 1, \dots, N$) and the average of $|\varphi(t)|^{p'-1} \operatorname{sgn} \varphi(t)$ on the period is zero (for $p > 1$), or $\max \varphi(t) = -\min \varphi(t)$ (for $p = 1$), then

$$(21) \quad \sup_{f \in KW_p^{m+1}} \|f - \sigma(f)\|_1 = K \|\varphi\|_{p'} \quad (1 \leq p \leq \infty, 1/p + 1/p' = 1).$$

Proof. Assuming that $f \in KW_p^{m+1}$ in (17), we may once again integrate by parts; using then the Hölder inequality we obtain

$$\|f - \sigma(f)\|_1 = \left| \int_0^{2\pi} [g(t) - s(g, t)] f^{(m+1)}(t) dt \right| \leq K \|g - s(g)\|_{p'}.$$

Since $g \in W^m H_{\Delta_N}^{1, \tau}$, Theorem 3 yields $\|g - s(g)\|_{p'} \leq \|\varphi - s(\varphi)\|_{p'}$, where φ is defined as in Theorem 3, but with respect to the partition (13). This finishes the proof of (20).

If $\varphi(t_i) = 0$ ($i = 1, \dots, N$), then $s(\varphi, t) \equiv 0$, and instead of (20) we get

$$(22) \quad \|f - \sigma(f)\|_1 \leq K \|\varphi\|_{p'}.$$

Under the assumptions of Theorem 3, the sharpness of the estimate (22) on the class KW_p^{m+1} is established for $p > 1$ by considering a function $f_0 \in KW_p^{m+1}$ such that

$$f_0^{(m+1)}(t) = K \|\varphi\|_{p'}^{1-p'} |\varphi(t)|^{p'-1} \operatorname{sgn} \varphi(t),$$

and for $p = 1$ by using Steklov functions (see e.g. [3, p. 198]).

The error estimates obtained above are more explicit for $N = 2n$ and the equally spaced knots if the perfect spline $\varphi_{n,m+1}$ appears as the reference function φ . For fixed $m = 0, 1, \dots$, assume the partition (12) to be formed by $t_k = k\pi/n$ ($k = 0, 1, \dots, 2n$), and the partition (13) by the zeros of the spline $\varphi_{n,m+1}$, i.e. $\tau_j = \theta_{m,j} = j\pi/n + [1 + (-1)^m] \pi/(4n)$ ($j = 0, 1, \dots, 2n$). Denote by $S_{2n,m}$ the linear manifold of 2π -periodic splines of order m with defect 1 with

respect to the knots $\{k\pi/n\}_0^{2n}$, and let $\sigma_{2n,m}(f, t) \in S_{2n,m}$ be a spline which interpolates $f \in C$ at the points $\theta_{m,j}$: $\sigma_{2n,m}(f, \theta_{m,j}) = f(\theta_{m,j})$ ($j = 1, \dots, 2n$). Such a spline exists (see e.g. [3]). Theorems 3 and 4 have the following consequences.

THEOREM 5. For every $f \in W^m KH_{2n}^1$ ($m = 0, 1, \dots$)

$$(23) \quad |f(t) - \sigma_{2n,m}(f, t)| \leq K |\varphi_{n,m+1}(t)|.$$

This estimate is sharp at each t even on the class KW_∞^{m+1} .

THEOREM 6. The following relations hold:

$$(24) \quad \sup_{f \in W^m KH_{2n}^1} \|f - \sigma_{2n,m}(f)\|_p = \sup_{f \in KW_\infty^{m+1}} \|f - \sigma_{2n,m}(f)\|_p = K \|\varphi_{n,m+1}\|_p \quad (1 \leq p \leq \infty),$$

$$(25) \quad \sup_{f \in KW_p^{m+1}} \|f - \sigma_{2n,m}(f)\|_1 = K \|\varphi_{n,m+1}\|_{p'} \quad (1/p + 1/p' = 1).$$

In particular,

$$\begin{aligned} \sup_{f \in KW_\infty^{m+1}} \|f - \sigma_{2n,m}(f)\|_C &= \sup_{f \in KW_1^{m+1}} \|f - \sigma_{2n,m}(f)\|_1 \\ &= K \|\varphi_{n,m+1}\|_C = K \mathcal{K}_{m+1} n^{-m-1}, \end{aligned}$$

where

$$\mathcal{K}_r = \frac{4}{\pi} \sum_{v=0}^{\infty} \frac{(-1)^{v(r+1)}}{(2v+1)^{r+1}}$$

is the well-known Favard constant.

Remark. The second equality in (24) and the relation (25) also mean that we have the sharp estimates

$$\begin{aligned} \|f - \sigma_{2n,m}(f)\|_p &\leq \|\varphi_{n,m+1}\|_p \|f^{(m+1)}\|_\infty, \quad f \in L_\infty^{m+1}, \\ \|f - \sigma_{2n,m}(f)\|_1 &\leq \|\varphi_{n,m+1}\|_{p'} \|f^{(m+1)}\|_p, \quad f \in L_p^{m+1}. \end{aligned}$$

Note that the inequality (23) for $f \in KW_\infty^{m+1}$ was obtained (in a different way) in [9]; in fact, it follows from the considerations concerning the estimate of $\|f - \sigma_{2n,m}(f)\|_C$ in [7]. The relation (25) was established (also by other means) in [2].

The results formulated in Theorems 5 and 6 permit a considerable progress to be achieved in solving extremal problems of best approximation.

Let X be a linear normed space, $\mathfrak{N}$ a subspace of X and $\mathfrak{M} \subset X$ a centrally symmetric subset. We introduce the quantities

$$E(f, \mathfrak{N})_X = \inf_{\psi \in \mathfrak{N}} \|f - \psi\|_X, \quad E(\mathfrak{M}, \mathfrak{N})_X = \sup_{f \in \mathfrak{M}} E(f, \mathfrak{N})_X,$$

and we define the N -dimensional Kolmogorov diameters of $\mathfrak{M}$ in X by

$$d_N(\mathfrak{M}, X) = \inf_{\mathfrak{N}_N} E(\mathfrak{M}, \mathfrak{N}_N)_X \quad (N = 1, 2, \dots),$$

where the infimum is over all subspaces $\mathfrak{N}_N \subset X$ of dimension at most N . If X is the L_p space of 2π -periodic functions, we will write $E(f, \mathfrak{N})_p$, $E(\mathfrak{M}, \mathfrak{N})_p$.

It follows from Theorem 6 that for $m = 1, 2, \dots$

$$(26) \quad E(KW_\infty^m, S_{2n,m-1})_p \leq K \|\varphi_{n,m}\|_p \quad (1 \leq p \leq \infty),$$

$$(27) \quad E(KW_p^m, S_{2n,m-1})_1 \leq K \|\varphi_{n,m}\|_{p'} \quad (1/p + 1/p' = 1).$$

It is not hard to exhibit specific functions in KW_∞^m and KW_p^m for which the best approximation by $S_{2n,m-1}$ in the L_p resp. L_1 norm coincides with the right-hand side of (26) resp. (27) for $p > 1$, or, for $p = 1$, differs from the right-hand side of (27) by as little as we please [3, §4.1]. This means that in fact equality holds in (26) and (27), i.e. in these situations the interpolating splines $\sigma_{2n,m-1}(f, t)$ realize the best approximation by $S_{2n,m-1}$ ($\dim S_{2n,m-1} = 2n$).

Much more important, however, is the fact that the inequalities (26) and (27) cannot be improved by replacing $S_{2n,m-1}$ with any other subspace of the same dimension $2n$, i.e. $S_{2n,m-1}$ is the best subspace here in the sense of Kolmogorov diameters. The proof of this fact, basing on the machinery of rearrangements and Borsuk's antipode theorem (see e.g. [8]), is given in [5], and also in [3, Ch. 6].

THEOREM 7. For all $n, m = 1, 2, \dots$

$$\begin{aligned} d_{2n}(KW_\infty^m, L_p) &= E(KW_\infty^m, S_{2n,m-1})_p = \sup_{f \in KW_\infty^m} \|f - \sigma_{2n,m-1}(f)\|_p \\ &= K \|\varphi_{n,m}\|_p = Kn^{-m} \|\varphi_{1,m}\|_p \quad (1 \leq p \leq \infty), \\ (28) \quad d_{2n}(KW_p^m, L_1) &= E(KW_p^m, S_{2n,m-1})_1 = \sup_{f \in KW_p^m} \|f - \sigma_{2n,m-1}(f)\|_1 \\ &= K \|\varphi_{n,m}\|_{p'} = Kn^{-m} \|\varphi_{1,m}\|_{p'} \\ &\quad (1 \leq p \leq \infty, 1/p + 1/p' = 1). \end{aligned}$$

Observe that (28) for $p = \infty$ was obtained in [7]. Since the interpolating splines $\sigma_{2n,m-1}(f, t)$ realize the linear projection onto $S_{2n,m-1}$, the results of Theorem 7 also yield the values of the linear and projection [3] diameters of KW_∞^m in L_p and of KW_p^m in L_1 .

4. Interpolation on the interval

As in Section 2, let $Q_m = \{0, 1, \dots, m-1\}$ ($m = 1, 2, \dots$) and $I_a, I_b \subset Q_m$. We will say that a subspace $\mathfrak{N}$ of splines defined on the interval $[a, b]$ is

interpolating at N distinct points $\tau_j \in [a, b]$ ($j = 1, \dots, N$) with the boundary conditions (I_a, I_b) if for every $f \in C^{m-1}[a, b]$ there is a unique spline $\sigma = \sigma(f, t) \in \mathfrak{N}$ such that

$$\sigma(f, \tau_j) = f(\tau_j) \quad (j = 1, \dots, N)$$

and

$$\sigma^{(\mu)}(f, a) = f^{(\mu)}(a) \quad (\mu \in I_a), \quad \sigma^{(\kappa)}(f, b) = f^{(\kappa)}(b) \quad (\kappa \in I_b).$$

Given two partitions

$$(29) \quad \Delta'_N[a, b]: \quad a = t_0 < t_1 < \dots < t_N = b,$$

$$(30) \quad \Delta'_M[a, b]: \quad a = \tau_0 < \tau_1 < \dots < \tau_M = b,$$

we will consider the corresponding classes $W^m KH_{\Delta'_N[a, b]}^1$ and $W^m KH_{\Delta'_M[a, b]}^1$ defined in Section 2, as well as the linear manifolds $S_m(\Delta'_N[a, b])$ and $S_m(\Delta'_M[a, b])$ of splines of order m with defect 1 defined on $[a, b]$ with knots at interior points of the respective partitions.

THEOREM 8. Suppose that $S_m(\Delta'_N[a, b])$ is interpolating at the interior points τ_j ($j = 1, \dots, M-1$) of the partition (30) with the boundary conditions $I = (I_a, I_b)$, $I_a, I_b \subset Q_{m+1}$. Assume that $M + |I_a| + |I_b| = N + m + 1$. Then for every $f \in W^m KH_{\Delta'_N[a, b]}^1$

$$|f(t) - \sigma(f, t)| \leq K |\psi(t) - \sigma(\psi, t)| \quad (a \leq t \leq b),$$

where $\psi \in W_{\infty}^{m+1}[a, b]$ satisfies

$$\pm \psi^{(m+1)}(t) = (-1)^i, \quad t_{i-1} < t < t_i, \quad i = 1, \dots, N.$$

In particular, if

$$\psi(\tau_j) = 0 \quad (j = 1, \dots, M-1), \quad \psi^{(\mu)}(a) = \psi^{(\kappa)}(b) = 0 \quad (\mu \in I_a, \kappa \in I_b),$$

then

$$|f(t) - \sigma(f, t)| \leq K |\psi(t)| \quad (a \leq t \leq b).$$

Theorem 8 follows immediately from Theorem 2 if we observe that the differences $\delta(t) = f(t) - \sigma(f, t)$ and $\eta(t) = \psi(t) - \sigma(\psi, t)$ belong to $W^m KH_{\Delta'_N[a, b]}^1$ and $W^m H_{\Delta'_N[a, b]}^1$ respectively, and $\eta^{(m+1)}(t) = \psi^{(m+1)}(t)$ for $t \in (t_{i-1}, t_i)$.

Boundary conditions $J = (J_a, J_b)$ will be called *complementary* to $I = (I_a, I_b)$ where $I_a, I_b \subset Q_{m+1}$ if $v \in J_a \Leftrightarrow m-v \notin I_a$ and similarly for J_b , so that $I_a \cup J_a = I_b \cup J_b = Q_{m+1}$.

THEOREM 9. Suppose that $S_m(\Delta'_N[a, b])$ is interpolating at the points τ_j ($j = 1, \dots, M-1$) of the partition (30) with the boundary conditions $I = (I_a, I_b)$, where $M + |I_a| + |I_b| = N + m + 1$ and $m \notin I_a \cup I_b$, and suppose that

$S_m(\mathcal{A}_M^1[a, b])$ is interpolating at the points t_i ($i = 1, \dots, N-1$) of the partition (29) with the boundary conditions $J = (J_a, J_b)$ complementary to I . Denote by $\sigma(f, t)$ and $s(f, t)$ the respective splines. Then for every $f \in KW_p^{m+1}[a, b]$ ($1 \leq p \leq \infty$)

$$(31) \quad \|f - \sigma(f)\|_1 \leq K \|\psi - s(\psi)\|_{p'}, \quad (1/p + 1/p' = 1),$$

where $\psi \in W_\infty^{m+1}[a, b]$ satisfies

$$\psi^{(m+1)}(t) = (-1)^j, \quad \tau_{j-1} < t < \tau_j, \quad j = 1, \dots, M.$$

In particular, if $\psi(t_i) = 0$ ($i = 1, \dots, N-1$) and $\psi^{(\mu)}(a) = \psi^{(\kappa)}(b) = 0$ ($\mu \in J_a, \kappa \in J_b$), then $s(\psi, t) \equiv 0$ and

$$(32) \quad \|f - \sigma(f)\|_1 \leq K \|\psi\|_{p'}.$$

The estimates (31) and (32) are sharp on the class $KW_p^{m+1}[a, b]$.

The proof follows the same lines as in the periodic case; in the outline given below, we stress the points connected with boundary conditions. Let $f \in KW_p^{m+1}[a, b]$ and $\delta(t) = f(t) - \sigma(f, t)$. Then

$$(33) \quad \delta(\tau_j) = 0 \quad (j = 1, \dots, M-1), \quad \delta^{(\mu)}(a) = \delta^{(\kappa)}(b) = 0 \quad (\mu \in I_a, \kappa \in I_b).$$

Take a function $g \in W_\infty^{m+1}[a, b]$ satisfying $g^{(m+1)}(t) = \operatorname{sgn} \delta(t)$ ($a \leq t \leq b$). If $\eta(t) = g(t) - s(g, t)$, then

$$(34) \quad \eta(t_i) = 0 \quad (i = 1, \dots, N-1), \quad \eta^{(\mu)}(a) = \eta^{(\kappa)}(b) = 0 \quad (\mu \in J_a, \kappa \in J_b),$$

and since $m \notin I_a \cup I_b$, we have $0 \in J_a \cap J_b$, i.e. $\eta(a) = \eta(b) = 0$.

If $0 \in I_a \cap I_b$ and hence $\delta(a) = \delta(b) = 0$, we integrate by parts over $[a, b]$ using (33) and the fact that $s^{(m)}(g, t) = \operatorname{const}$ for $t \in (\tau_{j-1}, \tau_j)$ to obtain

$$\|\delta\|_1 = \int_a^b \delta(t) g^{(m+1)}(t) dt = - \int_a^b \delta'(t) \eta^{(m)}(t) dt.$$

On the other hand, if $0 \notin I_a$ ($0 \notin I_b$) then $m \in J_a$ ($m \in J_b$) and then $\eta^{(m)}(a) = 0$ ($\eta^{(m)}(b) = 0$); and since $\eta^{(m)}(t) = g^{(m)}(t)$ for $t \in (\tau_{j-1}, \tau_j)$, we obtain the same result by integrating by parts over each interval (τ_{j-1}, τ_j) .

The $(m-1)$ -fold integration by parts over $[a, b]$ now gives

$$\|\delta\|_1 = \sum_{v=1}^{m-1} (-1)^v [\delta^{(v)}(b) \eta^{(m-v)}(b) - \delta^{(v)}(a) \eta^{(m-v)}(a)] + (-1)^m \int_a^b \delta^{(m)}(t) \eta'(t) dt.$$

The boundary terms on the right-hand side vanish by the complementarity of the boundary conditions I and J , and since $\sigma^{(m)}(f, t) = \operatorname{const}$ for $t \in (t_{i-1}, t_i)$, we obtain for $f \in KW_p^{m+1}[a, b]$

$$\|\delta\|_1 = \left| \int_a^b f^{(m)}(t) \eta'(t) dt \right| = \left| \int_a^b f^{(m+1)}(t) \eta(t) dt \right| \leq K \|\eta\|_{p'}.$$

But $\eta \in W^m H_{\Delta_M^1[a,b]}^1$ and therefore $\|\eta\|_{p'} \leq \|\psi - s(\psi)\|_{p'}$ by Theorem 8, which completes the proof of the inequality (31).

The sharpness of (31) and (32) for $1 < p \leq \infty$ is easily established by considering a function $f_0 \in W_p^{m+1}[a, b]$ with

$$f_0^{(m+1)}(t) = \|\eta_0\|_{p'}^{1-p'} |\eta_0(t)|^{p'-1} \operatorname{sgn} \eta_0(t),$$

where $\eta_0(t) = \psi(t) - s(\psi, t)$; it suffices to observe that

$$\operatorname{sgn}[f_0(t) - \sigma(f_0, t)] = \pm \operatorname{sgn} \psi^{(m+1)}(t) \quad (a \leq t \leq b).$$

For $p = 1$, the proof of the sharpness of (31) and (32) with the use of Steklov functions is standard.

The assertion of Theorem 8 is contained in [1]; Theorem 9 is proved in [4].

Let us note the most important particular cases of Theorems 8 and 9, corresponding to Corollaries 1–3 of Theorem 2.

1) Let $\Psi_{N,m}(t) = \Psi_{N,m}(p, t)$ be a perfect spline with minimal $L_p[a, b]$ norm among all perfect splines of order m with at most $N-1$ knots in (a, b) . Such a spline exists [6]; it has $N-1$ knots and $N+m+1$ distinct zeros in (a, b) . If the partition $\Delta_N^*[a, b]: a = t_0 < t_1 < \dots < t_N = b$ is formed by the knots of $\Psi_{N,m}$, then the subspace $S_{m-1}(\Delta_N^*[a, b])$ of splines of order $m-1$ with defect 1 is interpolating at the zeros of $\Psi_{N,m}$ (see e.g. [3, Ch. 1]) without boundary conditions.

THEOREM 10. For $f \in C[a, b]$, let $s(f, t) \in S_{m-1}(\Delta_N^*[a, b])$ be a spline satisfying $s(f, \tau_j) = f(\tau_j)$ ($j = 1, \dots, N+m-1$), where the τ_j are zeros of $\Psi_{N,m}(t) = \Psi_{N,m}(p, t)$ in (a, b) . If $f \in W^{m-1}KH_{\Delta_N^*[a,b]}^1$, then

$$|f(t) - s(f, t)| \leq K |\Psi_{N,m}(t)|,$$

and the estimate is sharp at each $t \in [a, b]$ on the class $KW_\infty^m[a, b]$, so that for all $N, m = 1, 2, \dots$

$$\begin{aligned} (35) \quad & \sup_{f \in W^{m-1}KH_{\Delta_N^*[a,b]}^1} \|f - s(f)\|_p \\ &= \sup_{f \in KW_\infty^m[a,b]} \|f - s(f)\|_p = K \|\Psi_{N,m}\|_p \quad (1 \leq p \leq \infty), \end{aligned}$$

where $\|\cdot\|_p = \|\cdot\|_{L_p[a,b]}$.

2) Let $\Psi_{N,m}^0(t) = \Psi_{N,m}^0(p, t)$ be a perfect spline with minimal $L_p[a, b]$

norm ($1 \leq p \leq \infty$) among all perfect splines ψ of order m with at most $N+m-1$ knots in (a, b) which satisfy $\psi^{(v)}(a) = \psi^{(v)}(b) = 0$ ($v = 0, 1, \dots, m-1$). Such a spline also exists [6]; it has $N+m-1$ knots and $N-1$ zeros in (a, b) . Consider the two partitions

$$(36) \quad \Delta_{N+m}^{t,0}[a, b]: \quad a = t_0 < t_1 < \dots < t_{N+m} = b,$$

$$(37) \quad \Delta_N^{\tau,0}[a, b]: \quad a = \tau_0 < \tau_1 < \dots < \tau_N = b,$$

formed respectively by the knots t_i ($i = 1, \dots, N+m-1$) and the zeros τ_j ($j = 1, \dots, N-1$) of the spline $\Psi_{N,m}^0(t)$; let $S_{m-1}(\Delta_{N+m}^{t,0}[a, b])$ and $S_{m-1}(\Delta_N^{\tau,0}[a, b])$ be the corresponding subspaces of splines of order $m-1$ with defect 1. The known results on spline interpolation [3] permit us to conclude that $S_{m-1}(\Delta_{N+m}^{t,0}[a, b])$ is interpolating at $\tau_1, \dots, \tau_{N-1}$ with the boundary conditions $I_a = I_b = Q_m = \{0, 1, \dots, m-1\}$, and $S_{m-1}(\Delta_N^{\tau,0}[a, b])$ is interpolating at $t_1, \dots, t_{N+m-1}$ (with no boundary conditions).

THEOREM 11. For $f \in C^{m-1}[a, b]$, let $\sigma(f, t) \in S_{m-1}(\Delta_{N+m}^{t,0}[a, b])$ be a spline satisfying $\sigma(f, \tau_j) = f(\tau_j)$ ($j = 1, \dots, N-1$), where the τ_j are zeros of $\Psi_{N,m}^0$, together with the boundary conditions

$$\sigma^{(v)}(f, a) = f^{(v)}(a), \quad \sigma^{(v)}(f, b) = f^{(v)}(b) \quad (v = 0, 1, \dots, m-1).$$

Then for $f \in W^{m-1}KH_{\Delta_{N+m}^{t,0}[a,b]}^1$

$$|f(t) - \sigma(f, t)| \leq K |\Psi_{N,m}^0(t)| \quad (a \leq t \leq b),$$

and

$$\begin{aligned} \sup_{f \in W^{m-1}KH_{\Delta_{N+m}^{t,0}[a,b]}^1} \|f - \sigma(f)\|_p &= \sup_{f \in KW_{\infty}^m[a,b]} \|f - \sigma(f)\|_p \\ &= K \|\Psi_{N,m}^0\|_p \quad (1 \leq p \leq \infty), \end{aligned}$$

where $\|\cdot\|_p = \|\cdot\|_{L_p[a,b]}$.

THEOREM 12. For $f \in C[a, b]$, let $s(f, t) \in S_{m-1}(\Delta_N^{\tau,0}[a, b])$ be a spline satisfying $s(f, t_i) = f(t_i)$ ($i = 1, \dots, N+m-1$), where the t_i are knots of $\Psi_{N,m}^0(t) = \Psi_{N,m}^0(p, t)$. Then

$$(38) \quad \sup_{f \in KW_p^m[a,b]} \|f - s(f)\|_{L_1[a,b]} = K \|\Psi_{N,m}^0\|_{L_p[a,b]} \quad (1/p + 1/p' = 1).$$

3) Let $[a, b] = [0, \pi]$, let $\bar{\Delta}_N[0, \pi]$ be the partition consisting of $t_i = i\pi/N$ ($i = 0, 1, \dots, N$) and let $S_{m-1}(\bar{\Delta}_N[0, \pi])$ be the corresponding subspace of splines of order $m-1$ with defect 1 defined on $[0, \pi]$. For every $f \in C^{m-2}[0, \pi]$ there is a spline $\sigma_{m-1}(f, t) \in S_{m-1}(\bar{\Delta}_N[0, \pi])$ such that for m even

$$\sigma_{m-1}(f, j\pi/N) = f(j\pi/N) \quad (j = 0, 1, \dots, N),$$

$$\sigma_{m-1}^{(2v)}(f, 0) = f^{(2v)}(0), \quad \sigma_{m-1}^{(2v)}(f, \pi) = f^{(2v)}(\pi) \quad (v = 1, \dots, m/2-1),$$

while for m odd

$$\begin{aligned}\sigma_{m-1}(f, (2j-1)\pi/(2N)) &= f((2j-1)\pi/(2N)) \quad (j = 1, \dots, N), \\ \sigma_{m-1}^{(2v-1)}(f, 0) &= f^{(2v-1)}(0), \quad \sigma_{m-1}^{(2v-1)}(f, \pi) = f^{(2v-1)}(\pi) \\ &\quad (v = 1, \dots, (m-1)/2).\end{aligned}$$

Observe that here the interpolation points are zeros of the perfect spline $\varphi_{N,m}$.

THEOREM 13. If $f \in W^{m-1}KH_N^1[0, \pi]$, then

$$|f(t) - \sigma_{m-1}(f, t)| \leq K |\varphi_{N,m}(t)| \quad (0 \leq t \leq \pi),$$

and the estimate is sharp at each $t \in [0, \pi]$ even on the class $KW_\infty^m[0, \pi]$, so that for all $N, m = 1, 2, \dots$

$$\begin{aligned}\sup_{f \in W^{m-1}KH_N^1[0, \pi]} \|f - \sigma_{m-1}(f)\|_p &= \sup_{f \in KW_\infty^m[0, \pi]} \|f - \sigma_{m-1}(f)\|_p \\ &= K \|\varphi_{N,m}\|_p = KN^{-m} \|\varphi_{1,m}\|_p \quad (1 \leq p \leq \infty),\end{aligned}$$

where $\|\cdot\|_p = \|\cdot\|_{L_p[0, \pi]}$.

THEOREM 14. For all $N, m = 1, 2, \dots$

$$\begin{aligned}\sup_{f \in KW_p^m[0, \pi]} \|f - \sigma_{m-1}(f)\|_1 &= K \|\varphi_{N,m}\|_{p'} = KN^{-m} \|\varphi_{1,m}\|_{p'} \\ &\quad (1 \leq p \leq \infty, 1/p + 1/p' = 1),\end{aligned}$$

where $\|\cdot\|_p = \|\cdot\|_{L_p[0, \pi]}$.

Let us return to Theorems 10 and 12. Since

$$\dim S_{m-1}(\Delta_N^*[a, b]) = \dim S_{m-1}(\Delta_N^0[a, b]) = N + m - 1,$$

the relations (35) and (38) give an upper bound for the projection diameters of $KW_\infty^m[a, b]$ and $KW_p^m[a, b]$ in $L_p[a, b]$ and $L_1[a, b]$ respectively. It turns out [6], [3, Ch. 6] that the error estimates in these situations cannot be improved even if we pass to best approximations by any subspaces of the same dimension. Consequently,

$$\begin{aligned}d_{N+m-1}(KW_\infty^m[a, b], L_p[a, b]) &= K \|\Psi_{N,m}(p, \cdot)\|_p \quad (1 \leq p \leq \infty), \\ d_{N+m-1}(KW_p^m[a, b], L_1[a, b]) &= K \|\Psi_{N,m}^0(p', \cdot)\|_{p'} \quad (1/p + 1/p' = 1),\end{aligned}$$

where $\Psi_{N,m}(p, t)$ and $\Psi_{N,m}^0(p', t)$ are the perfect splines with minimal L_p and $L_{p'}$ norm respectively introduced in 1) and 2), and $\|\cdot\|_p = \|\cdot\|_{L_p[a, b]}$.

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