THE FOURIER TRANSFORM IN WEIGHTED SPACES

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1. Introduction

The Fourier transform of a function f is defined by

$$(Ff)(x) = \hat{f}(x) = \int_{-\infty}^{\infty} e^{-2\pi ixy} f(y) dy, \quad x \in \mathbb{R},$$

whenever the integral exists. The size and estimate of the Fourier transform of a given function in some function space plays an important role in harmonic analysis and many applications. For example, it is a trivial observation that the Fourier transform of an L^1 -function is bounded. Less trivial, but well known, is the Riemann–Lebesgue lemma, which asserts that if $f \in L^1$, then $\lim_{y \rightarrow \infty} \hat{f}(y) = 0$, while Plancherel's theorem shows that the Fourier transform is an isometry from L^2 onto itself, with $\|\hat{f}\|_2 = \|f\|_2$. From this and the L^1 boundedness of the transform one obtains at once from the Riesz–Thorin interpolation theorem the Hausdorff–Young inequality $\|\hat{f}\|_{p'} \leq \|f\|_p$, $1 < p \leq 2$, where here and in the sequel the conjugate index p' of p is defined by $p' = p/(p-1)$ if $p \neq 1$ and $p' = \infty$ if $p = 1$. It is noteworthy that there exists a significant refinement of this inequality due to K. I. Babenko [2] (for p' an even integer) and W. Beckner [3] (for the remaining values of p), namely that

$$\|\hat{f}\|_{p'} \leq [p^{1/p}(p')^{-1/p'}]^{1/2} \|f\|_p.$$

This optimal form of the Hausdorff–Young inequality may be utilized to establish an entropy inequality which in turn implies the Heisenberg–Weyl uncertainty inequality [3], [16], [18]. The Fourier estimates discussed below involves constants, denoted by C which are not sharp and may be different at different occurrences. To establish sharp or even good constants requires new proofs which are not available.

In 1978 B. Muckenhoupt [24] posed the problem to characterize the non-negative weight functions u and v defined on $\mathbb{R}$, such that for $1 < p \leq q < \infty$

$$(1.1) \quad \left\{ \int_{-\infty}^{\infty} |\hat{f}(y)|^q u(y) dy \right\}^{1/q} \leq C \left\{ \int_{-\infty}^{\infty} |f(x)|^p v(x) dx \right\}^{1/p}.$$

The object of this paper is to recount some of the recent results in this direction. For a discussion of this problem we also refer to [6] where also an excellent historical survey is presented, and to [15] where related operators and the higher dimensional cases are discussed.

In the next section the estimate (1.1) is discussed, not only for the case $1 < p \leq q < \infty$ but also when $1 < q < p < \infty$. If $0 < p \leq 1$, inequalities of the form (1.1) are not in general possible unless the L^p -spaces in the domain of F are replaced by the corresponding Hardy spaces. Then one has the estimate

$$\int_0^{\infty} |\hat{f}(y)|^p |y|^{p-2} dy \leq C \|f\|_{H^p}^p, \quad 0 < p \leq 1,$$

which together with a weighted extension is discussed in Section 3. By means of the weighted atomic Hardy spaces the weighted extensions allow weights to be in some (Muckenhoupt) A_p class. The final section contains some additional results. Specifically it is shown that the majorant property of the Fourier transform of functions in weighted L^p -spaces fails to hold and a weighted Heisenberg–Weyl inequality is given.

2. Weighted L^p -estimates

The first weighted result is due to Hardy and Littlewood [11].

THEOREM 2.1. (a) *If $1 < p \leq 2$, then for all $f \in L^p$*

$$\int_{-\infty}^{\infty} |\hat{f}(y)|^p |y|^{p-2} dy \leq C \int_{-\infty}^{\infty} |f(x)|^p dx.$$

(b) *If $q > 2$, then*

$$\int_{-\infty}^{\infty} |\hat{f}(y)|^q dy \leq C \int_{-\infty}^{\infty} |f(x)|^q |x|^{q-2} dx.$$

Clearly these inequalities are of the form (1.1) and are now very easily established. For example, if one observes that the operator T defined by $(Tf)(x) = x\hat{f}(x)$, satisfies

$$\int_{-\infty}^{\infty} |(Tf)(x)|^2 d\mu(x) = \int_{-\infty}^{\infty} |f(x)|^2 dx$$

by Plancherel's theorem, with μ defined by $\mu(E) = \int_E x^{-2} dx$ for $E \subset \mathbb{R} \setminus \{0\}$, and also $\mu(\{x \in \mathbb{R}: |(Tf)(x)| > \lambda\}) \leq 2\|f\|_1/\lambda$ for any $\lambda > 0$, then the inequality of (a) follows from the Marcinkiewicz interpolation theorem. The inequality of (b) follows at once by duality.

A further generalization of this result was obtained by E. M. Stein [26] and independently P. G. Rooney [25] using interpolation techniques.

THEOREM 2.2 ([25]). *If $1 < p < \infty$, $1 < r \leq \min(p, p')$ and $p \leq q \leq r'$, then*

$$\left\{ \int_{-\infty}^{\infty} |\hat{f}(y)|^q |y|^{q/r'-1} dy \right\}^{1/q} \leq C \left\{ \int_{-\infty}^{\infty} |f(x)|^p |x|^{p/r-1} dx \right\}^{1/p}.$$

I. I. Hirschman Jr. [17] obtained a different inequality. He showed that (1.1) holds with $p = q = 2$, $v(x) = |x|^\alpha$, $0 < \alpha < 1$ and u in "weak" $L^{1/\alpha}$, that is, u satisfies $\lambda^{1/\alpha} |\{x \in \mathbb{R}: u(x) > \lambda\}| \leq C$, where here and in the sequel $|E|$ denotes the Lebesgue measure of $E \subset \mathbb{R}$.

More recently N. E. Aguilera and E. O. Harboure [1] obtained inequalities of the form (1.1) with $p = q$ and $v(x) \equiv 1$ whenever u satisfies

$$\int_E u(x) dx \leq C |E|^{p-1}$$

for any measurable set $E \subset \mathbb{R}$.

To explain more general results some notation is required. If g is a Lebesgue measurable function defined on $\mathbb{R}$, then the equimeasurable decreasing rearrangement g^* of g is defined by

$$g^*(t) = \inf \{y > 0: |\{x \in \mathbb{R}: |g(x)| > y\}| \leq t\} \quad \text{where } t \geq 0.$$

The first result given next may be found in [12] (for p and q not both equal to 2), [4], [20] and for different but equivalent weight conditions in [21].

THEOREM 2.3. *If $1 \leq p \leq q < \infty$ and u, v non-negative locally integrable functions on $\mathbb{R}$ satisfying*

$$\sup_{s>0} \left\{ \int_0^s u^*(t) dt \right\}^{1/q} \left\{ \int_0^{1/s} (1/v)^*(t)^{p'-1} dt \right\}^{1/p'} < \infty,$$

then (1.1) holds.

Conversely, if (1.1) holds for all measurable functions f and u, v even weight functions, then

$$\sup_{s>0} \left\{ \int_0^s u(t) dt \right\}^{1/q} \left\{ \int_0^{1/s} v(t)^{1-p'} dt \right\}^{1/p'} < \infty.$$

Since u^* and $(1/v)^*$ are decreasing, it follows that for u and $1/v$ even and decreasing on $(0, \infty)$ this weight condition is both necessary and sufficient for (1.1).

On choosing the weight functions u and v in an obvious way, one obtains from Theorem 2.3, Theorems 2.1, 2.2 and all the previous Fourier estimates mentioned. Of course there are other inequalities that may be given. For example the functions $u(x) = |x|^{-\alpha} \ln(|x| + 1)$ and $v(x) = |x|^\alpha \ln(|x|^{-1} + 1)$, $0 < \alpha < 1$, are easily seen to satisfy the weight condition of Theorem 2.3 with $p = q = 2$. Therefore one obtains

$$(2.1) \quad \int_{-\infty}^{\infty} |y|^{-\alpha} \ln(|y| + 1) |\hat{f}(y)|^2 dy \leq C \int_{-\infty}^{\infty} |x|^\alpha \ln(|x|^{-1} + 1) |f(x)|^2 dx.$$

The question whether there are weight conditions for which the Fourier inequality (1.1) holds when $1 < q < p < \infty$, was until recently open. A result in this direction was given in [13] with different weight conditions. We now state the following result:

THEOREM 2.4 ([4]). *If $1 < q < p < \infty$ and the two conditions*

$$(2.2) \quad \left\{ \int_0^\infty \left[\left(\int_0^{1/x} u^*(t) dt \right)^{1/q} \left(\int_0^x (1/v)^*(t)^{p'-1} dt \right)^{1/q'} \right]^r (1/v)^*(x)^{p'-1} dx \right\}^{1/r} < \infty,$$

$$\left\{ \int_0^\infty \left[\left(\int_{1/x}^\infty t^{-q/2} u^*(t) dt \right)^{1/q} \left(\int_x^\infty t^{-p'/2} (1/v)^*(t)^{p'-1} dt \right)^{1/q'} \right]^r \right.$$

$$\left. \times x^{-p'/2} (1/v)^*(x)^{p'-1} dx \right\}^{1/r} < \infty$$

hold, where $1/r = 1/q - 1/p$, then (1.1) holds.

Recently, G. Sinnamon (verbal communication) has shown that Theorem 2.4 also holds in the case $0 < q < p$, $p \geq 1$.

It is easy to see that no power weights satisfy the conditions (2.2). Nevertheless there are many functions which will satisfy the conditions. For example $u(x) = \chi_{(-a,a)}(x)$ and $v(x) = |x|^\alpha$, $0 < \alpha < p-1$, is such a pair for $1 < q < p \leq 2$.

A familiar weight class of functions is the A_p , $1 < p < \infty$, class of Muckenhoupt. This class consists of all non-negative locally integrable functions on $\mathbf{R}$, such that, for all intervals $(a, b) \in \mathbf{R}$

$$\left[\int_a^b w(t) dt \right]^{1/p} \left[\int_a^b w(t)^{1-p'} dt \right]^{1/p'} \leq C(b-a),$$

where $C > 0$ is a constant.

If in Theorem 2.3, $q = p$, $1 < p \leq 2$, $u(t) = |t|^{p-2} w(1/t)$ and $v(t) = w(t)$, where w is even and non-decreasing on $(0, \infty)$, then (1.1) takes the form

$$(2.3) \quad \int_{-\infty}^{\infty} |\hat{f}(y)|^p |y|^{p-2} w(1/y) dy \leq C \int_{-\infty}^{\infty} |f(x)|^p w(x) dx,$$

and this holds whenever

$$\sup_{s>0} \left[\int_0^s t^{p-2} w(1/t) dt \right]^{1/p} \left[\int_0^{1/s} w(t)^{1-p'} dt \right]^{1/p'} < \infty.$$

By a result of Hunt, Muckenhoupt and Wheeden ([19], Lemma 1) this supremum is finite if $w \in A_p$. It seems surprising that the converse is also true.

THEOREM 2.5 ([5]). *Let w be an even function on $\mathbf{R}$, non-decreasing on $(0, \infty)$. For $1 < p \leq 2$, (2.3) holds if and only if $w \in A_p$.*

It follows from this result with $p = 2$ and the Fourier inequality (2.1) that $|x|^\alpha \ln(|x|^{-1} + 1) \in A_2$ for $0 < \alpha < 1$.

In the same paper ([5]) a number of other Fourier estimates with A_p -weights were given, of which the following Hausdorff–Young inequality is a special case:

THEOREM 2.6 ([5]). *Let w be an even function on $\mathbf{R}$, non-decreasing on $(0, \infty)$. If $1 < p \leq 2$, then*

$$\left[\int_{-\infty}^{\infty} |\hat{f}(y)|^{p'} w(1/y)^{1-p'} dy \right]^{1/p'} \leq C \left[\int_{-\infty}^{\infty} |f(x)|^p w(x) dx \right]^{1/p}$$

holds, if and only if $w^{p'-1} \in A_2$.

Since $u \in A_2$ if and only if $u(1/x) \in A_2$ for even u , non-decreasing on $(0, \infty)$ ([5]), it follows (since $u \in A_2$ if and only if $1/u \in A_2$) from Theorem 2.6 that $u(y) = w(1/y)^{1-p'} \in A_2$ if $w^{p'-1} \in A_2$. Therefore the weighted Hausdorff–Young inequality above takes the form

$$\left[\int_{-\infty}^{\infty} |\hat{f}(y)|^{p'} u(y) dy \right]^{1/p'} \leq C \left[\int_{-\infty}^{\infty} |f(x)|^p u(1/x)^{1-p} dx \right]^{1/p}$$

if and only if $u \in A_2$.

Unfortunately the constant C in the above result is not sharp. A sharp constant, similarly to the one obtained by K. I. Babenko and W. Beckner might lead to weighted entropy and sharp Heisenberg–Weyl uncertainty inequalities.

3. Weighted H^p -estimates

It is well known that the Fourier inequality of Theorem 2.1 (a) extends to the range $0 < p \leq 1$, provided the L^p -spaces in the domain of F are replaced by the Hardy spaces H^p . It is natural then to expect that weighted results in this situation are also possible. To explain these results one must define the weighted atomic Hardy spaces.

The classical Hardy spaces with weight w consist of those holomorphic functions f on the upper half plane for which

$$\sup_{y>0} \left\{ \int_{-\infty}^{\infty} |f(x+iy)|^r w(x) dx \right\}^{1/r}, \quad r > 0$$

is finite. For arbitrary weights such functions may not have non-tangential limits, however, if w is in some Muckenhoupt weight class, say A_s , $s > 1$, then ([10], Theorem II.1.1) non-tangential limits of f exist a.e. for $x \in \mathbb{R}$ and

$$\lim_{y \rightarrow 0+} \int_{-\infty}^{\infty} |f(x+iy)|^r w(x) dx = \int_{-\infty}^{\infty} |f(x)|^r w(x) dx.$$

One may therefore define the weighted Hardy spaces H_w^r , $0 < r \leq 1$, with $w \in A_s$, $s > 1$, to be the boundary distributions of the real parts of functions in the classical Hardy spaces. These spaces (as indeed the classical weighted Hardy spaces) permit a characterization in terms weighted (r, s) atoms. To describe this, recall that $w \in A_q$, $q > 1$, implies that $w \in A_r$ for some $r < q$ ([23]). The number q_0 defined by $q_0 = \inf \{r: r < q\}$ is called the *critical exponent* for w .

DEFINITION 3.1. Let $w \in A_s$, $s > 1$, with critical exponent q_0 . A *weighted (r, s) atom*, $0 < r \leq 1$, with weight w is a real valued function a , defined on $\mathbb{R}$, if

- (i) a is supported in an interval $I \subset \mathbb{R}$.
- (ii) $[\int_I |a(x)|^s w(x) dx]^{1/s} \leq w(I)^{-1/r+1/s}$ if $s < \infty$ and $\|a\|_{\infty} \leq w(I)^{-1/r}$ if $s = \infty$. Here $w(I) = \int_I w(x) dx$.
- (iii) $\int_{-\infty}^{\infty} x^k a(x) dx = 0$, $k = 0, 1, \dots, [q_0/r] - 1$, where $[\cdot]$ denotes the greatest integer function.

The *weighted atomic Hardy spaces H_w^r -atomic*, $0 < r \leq 1$, with weight $w \in A_s$, $s > 1$, are those distributions f of the form $\sum \lambda_j a_j$ for which

$$\|f\|_{H_w^r}^r := \inf \left\{ \sum_{j=0}^{\infty} |\lambda_j|^r : f = \sum \lambda_j a_j \right\}$$

is finite, where a_j are weighted (r, s) atoms and λ_j real numbers. J. Garcia-Cuerva ([10], Theorem II.3.7) proved that $f \in H_w^r$, $0 < r \leq 1$, $w \in A_s$, $s > 1$, if and only if there exist real numbers λ_j and weighted (r, s) atoms a_j , $j = 0, 1, \dots$, such that

$$f(x) = \sum_{j=0}^{\infty} \lambda_j a_j(x)$$

and

$$A \|f\|_{H_w^r}^r \leq \sum_{j=0}^{\infty} |\lambda_j|^r \leq B \|f\|_{H_w^r}^r$$

for A and B some constants independent of f .

In the rest of this section we consider therefore only weighted atomic Hardy spaces which are also denoted by H_w^r . Note that since $w \in A_s$, $s > 1$, the spaces also depend on s .

Let K be the integral operator defined by

$$(Kf)(x) = \int_{-\infty}^{\infty} K(x, y) f(y) dy, \quad x \in \mathbb{R},$$

where $K(x, y)$ is homogeneous of degree -1 . Then the weighted Fourier inequality in this setting follows from the following result:

THEOREM 3.2 ([14]). *Suppose*

- (i) $K(1, y)$ has bounded derivatives of order n , $n \geq 1$,
- (ii) $w \in A_{(n+1)p}$, $1 < (n+1)p \leq 2$, w even, non-decreasing on $(0, \infty)$ and has critical exponent $q_0 > np$,
- (iii) for each weighted $(p, (n+1)p)$ atom, with weight w

$$\|Ka\|_{(n+1)p, w} \leq C \|a\|_{(n+1)p, w} \quad (\text{here } \|g\|_{r, w} = \|w^{1/r} g\|_r).$$

Then

$$\left[\int_0^{\infty} |(Kf)(x)|^p w(x) dx \right]^{1/p} \leq C \|f\|_{H_w^p}.$$

Let $(Kf)(x) = \hat{f}(1/x)/x$, then $K(x, y) = x^{-1} e^{-2\pi i y/x}$, which is homogeneous of degree -1 and $K(1, y)$ has bounded derivatives of any order. Moreover, if $q = (n+1)p$, then $1 < q \leq 2$ so that condition (iii) with $a = g$ takes the form

$$\begin{aligned} \int_{-\infty}^{\infty} |\hat{g}(1/x)/x|^q w(x) dx &= \int_{-\infty}^{\infty} |\hat{g}(y)|^q |y|^{q-2} w(1/y) dy \\ &\leq C \int_{-\infty}^{\infty} |g(x)|^q w(x) dx. \end{aligned}$$

But this is precisely inequality (2.3) which holds by Theorem 2.5. Hence this result and Theorem 3.2 imply:

THEOREM 3.3. *Let w be even, non-decreasing on $(0, \infty)$. If for fixed n , $n \geq 1$, $w \in A_{(n+1)p} \setminus A_{np}$, where $1 < (n+1)p \leq 2$, then*

$$\int_0^{\infty} |\hat{f}(y)|^p |y|^{p-2} w(1/y) dy \leq C \|f\|_{H_w^p}^p.$$

If in particular $n = 1$, w even, non-decreasing on $(0, \infty)$, then $w \in A_{2p}$, $1/2 < p \leq 1$, implies

$$\int_0^{\infty} |\hat{f}(y)|^p |y|^{p-2} w(1/y) dy \leq C \|f\|_{H_w^p}^p.$$

Note that $w(x) = |x|^\alpha$ with $0 \leq \alpha < 2p-1$ is in A_{2p} .

4. Additional results

Fourier norm estimates are intimately connected with physical phenomena. For example the mathematical formulation of the Heisenberg uncertainty principle is expressed in the inequality $\|f\|_2^2 \leq 4\pi \|xf\|_2 \|y\hat{f}\|_2$, where $f \in L^2(\mathbf{R})$. Weighted Heisenberg–Weyl inequalities, but without sharp constants were obtained in [7], [16] and [18]. Here we prove such an inequality involving A_2 weights using Theorem 2.5 with $p = 2$.

THEOREM 4.1. *Let w be an even function, non-decreasing on $(0, \infty)$. If $w \in A_2$ then $\bar{w}(x) = w(1/x)$ is in A_2 and*

$$\|f\|_2^2 \leq C \left[\int_{-\infty}^{\infty} w(y) |y\hat{f}(y)|^2 dy \right]^{1/2} \left[\int_{-\infty}^{\infty} \bar{w}(x) |xf(x)|^2 dx \right]^{1/2}$$

for each $f \in \mathcal{S}$, the Schwarz class of slowly increasing functions.

Proof. Following the proof of the Heisenberg–Weyl inequality [9] for the unweighted case, then integration by parts and Schwarz's inequality yields

$$\begin{aligned} \|f\|_2^2 &\leq 2 \int_{-\infty}^{\infty} |xf(x) f'(x)| \bar{w}(x)^{1/2-1/2} dx \\ &\leq 2 \left[\int_{-\infty}^{\infty} \bar{w}(x) |f'(x)|^2 dx \right]^{1/2} \left[\int_{-\infty}^{\infty} \bar{w}(x) |xf(x)|^2 dx \right]^{1/2} \\ &\leq C \left[\int_{-\infty}^{\infty} w(y) |y\hat{f}(y)|^2 dy \right]^{1/2} \left[\int_{-\infty}^{\infty} \bar{w}(x) |xf(x)|^2 dx \right]^{1/2}. \end{aligned}$$

The last inequality follows from Theorem 2.5 with $p = 2$ and properties of the Fourier transform. Note that by [5], Proposition 3.3, $w \in A_2$ if and only if $\bar{w} \in A_2$ (for w even, non-decreasing on $(0, \infty)$).

Other weighted Heisenberg–Weyl inequalities may be deduced from Theorems 2.3 and 2.4 in the same way, but the constant in all such weighted estimates is not optimal.

As noted, the Riemann–Lebesgue lemma shows that $f \in L^1(\mathbf{R})$ implies $\lim_{y \rightarrow \infty} \hat{f}(y) = 0$. A result of A. Kolmogoroff extended to the real line by P. C. Curtis and A. Figa-Talamanca [8] shows that the Fourier transform of an L^1 -function can tend to zero at an arbitrary slow rate. More precisely, given $\varphi \in C_0$, there exists an $f \in L^1$, such that $|\hat{f}| > |\varphi|$. Although there is a trivial version of Kolmogoroff's result for L^2 , using the Riesz–Fischer and Plancherel theorems, a corresponding result for L^p , $1 < p < 2$, is not true as was shown by W. R. Bloom [6]. His result, namely that for $1 < p < 2$, there exists $g \in L^{p'}$, such that $|\hat{f}| > |g|$ holds for no $f \in L^p$ can be extended to the

weighted case by using Theorem 2.5 and a minor modification of his argument.

THEOREM 4.2. *Let $1 < p < 2$ and w be an even function, non-decreasing on $(0, \infty)$. If $w \in A_p$, then there exists a function g , $g \in L_w^{p'}$, where $u(x) = w(1/x)^{p'-1}$, such that $|\hat{f}| > |g|$ for no $f \in L_w^p$.*

Proof. Define g on $\mathbb{R}$ by

$$g(y) = \sum_{n=1}^{\infty} n^{-1/p'} \log^{-1/p}(n+1) w(1/y)^{-1/p} \chi_{(n, n+1)}(|y|)$$

and zero otherwise. Then

$$\begin{aligned} \int_{-\infty}^{\infty} |g(y)|^{p'} u(y) dy &= 2 \int_0^{\infty} |g(y)|^{p'} w(1/y)^{p'-1} dy \\ &= 2 \sum_{n=1}^{\infty} n^{-1} \log^{-p'/p}(n+1) \int_n^{n+1} w(1/y)^{-p'/p} w(1/y)^{p'-1} dy < \infty. \end{aligned}$$

Now if there exists an $f \in L_w^p$, with $|\hat{f}| > |g|$, then by Theorem 2.5

$$\begin{aligned} \int_{-\infty}^{\infty} |g(y)|^p |y|^{p-2} w(1/y) dy &\leq \int_{-\infty}^{\infty} |\hat{f}(y)|^p |y|^{p-2} w(1/y) dy \\ &\leq C \int_{-\infty}^{\infty} |f(x)|^p w(x) dx. \end{aligned}$$

But

$$\begin{aligned} \int_{-\infty}^{\infty} |g(y)|^p |y|^{p-2} w(1/y) dy &= 2 \sum_{n=1}^{\infty} n^{-p/p'} \log^{-1}(n+1) \int_n^{n+1} |y|^{p-2} dy \\ &\geq 2 \sum_{n=1}^{\infty} (n+1)^{-1} \log^{-1}(n+1) = \infty \end{aligned}$$

and this contradicts the fact that $f \in L_w^p$.

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