SINAI-BILLIARD IN POTENTIAL FIELD. ERGODIC COMPONENTS

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Foreword

We consider the present work by A. Vetier to be a didactical introduction to the theory of hyperbolic dynamical systems with singularities. Vetier's system is the simplest one which brings together the features of the geodesic flows on compact Riemannian manifolds and those of the Sinai-billiard. The clear, elementary and self-contained (modulo the absolute continuity of the fibrations) treatment helps the reader to enter into the prosperous theory developed by Pesin, Katok, Strelcyn and Wojtkowski (cf. [6], [7] and [8]). Vetier's system is a paradigm revealing the simple geometric and measure theoretical ideas leading to the deep machinery of this theory.

Indeed, discussions during this semester convinced us of the necessity to get Vetier's work published and in a forthcoming paper we present a proof of the K -property of this system based on Vetier's approach.

Budapest, February 1987

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From the point of view of ergodic theory the Sinai-billiard in potential field is equivalent to a billiard on some Riemannian surface without any potential (see [3]). Thus, in fact billiards on Riemannian surfaces are examined. It is known that the hyperbolicity of the classical Sinai-billiard is the consequence of the dispersing effect of the scatterers. The Gaussian curvature of the Riemannian surface corresponding to the potential can be positive, which can give rise to a contractive effect. We show that under some conditions (see conditions (1)–(3) below) the scattering effect overcomes the contractive one which makes possible to prove that almost every ergodic component of this billiard has positive measure.

1. Introduction

Let us consider a two-dimensional, three times continuously differentiable, compact closed, oriented Riemannian surface. A connected, open subset of the surface will be called a *scatterer* if its boundary is a three times continuously differentiable curve with positive curvature from inside. Let us have a finite number of scatterers. The complement of the union of the scatterers will be called the *billiard table*, denoted by Q . The boundary of the billiard table, denoted by ∂Q , consists of the union of the boundaries of the scatterers.

Let us imagine that a particle is moving on the billiard table with unit velocity along geodesics, and it is reflected elastically by the scatterers. The state of the particle is determined by its place and velocity. So a state x means a place q and a direction v together: $x = (q, v)$. The state x can be identified with a unit vector at the tangent plane at the point q . The set of all states is the phase space, denoted by M . The projection from M to Q is the mapping $\pi: M \rightarrow Q$ for which $\pi(x) = q$ if $x = (q, v)$.

The motion of the particle defines some dynamics $\{S^t\}_{-\infty < t < \infty}$ on the phase space: if the particle is in the state x at time 0 then $S^t x$ means its state at time t ($-\infty < t < \infty$). The states $S^t x$ ($-\infty < t < \infty$) together constitute the trajectory of the state x .

The Riemannian metric of the surface induces an area measure μ_Q on the billiard table. The set $\pi^{-1}(q)$ of all unit vectors of the tangent plane at a point q can be obviously identified with the boundary of the unit circle in this plane. The length measure on the boundary of this circle corresponds to a measure σ^q on $\pi^{-1}(q)$ by this identification. Since the set M is the union of the sets $\pi^{-1}(q)$ ($q \in Q$), the measure μ_Q and the measures σ^q ($q \in Q$) together define a measure μ on M :

$$\mu(X) := \int_Q \sigma^q(X \cap \pi^{-1}(q)) \mu_Q(dq) \quad (X \subset M).$$

It is known that the measure μ is invariant under the dynamics $\{S^t\}_{-\infty < t < \infty}$. So we get a dynamical system $(M, \{S^t\}, \mu)$ which we call the *billiard on the billiard table*.

We suppose that the position of the scatterers guarantees that

(1) the distance of any two scatterers is positive,
 (2) the free path of the particle is limited, that is to say there exists a finite value $\tau_{\max}$ such that the length of any geodesic on the billiard table cannot be longer than $\tau_{\max}$,

(3) the Gaussian curvature of the billiard table and the free path are small enough compared to the curvature of the boundary of the scatterers, which precisely means the following. Let $k_{\min} > 0$ denote the minimum of the curvature of the boundary of the billiard table, and let $K_{\max}$ denote the

maximum of the Gaussian curvature of the billiard table. Our condition is that

$$K_{\max} \leq 0$$

or

$$K_{\max} > 0, \quad \text{and} \quad \sqrt{K_{\max}} \tau_{\max} < \frac{\pi}{2} \quad \text{and}$$

$$\sqrt{K_{\max}} \tau_{\max} < \arctan(k_{\min} K_{\max}^{-1/2}).$$

The purpose of this paper is to prove the following theorem. After the formulation of the theorem we shall immediately see that the theorem implies that almost every ergodic component of the billiard has positive measure.

THEOREM. *If conditions (1)–(3) hold, then for almost every $x \in M$ there exists $E = E(x) \subset M$ for which*

(a) *the set E is almost everywhere a subset of some ergodic component of the billiard, and*

(b) *at the element x the density of the set E is not equal to zero,*

$$\liminf_{\varepsilon \rightarrow 0} \mu(E \cap B_\varepsilon) / \mu B_\varepsilon > 0,$$

where B_ε denotes the closed sphere in M with center x and radius ε .

First let us show that this Theorem really implies that almost every ergodic component of the billiard has positive measure. Let F denote the union of the ergodic components of positive measure. We must prove that $\mu(M \setminus F) = 0$. By the Lebesgue density theorem it is enough to prove that for almost every $x \in M$ the density of the set F is not equal to zero. This really follows from the Theorem, because

$$\liminf_{\varepsilon \rightarrow 0} \mu(F \cap B_\varepsilon) / \mu(B_\varepsilon) \geq \liminf_{\varepsilon \rightarrow 0} \mu(E \cap B_\varepsilon) / \mu(B_\varepsilon) > 0.$$

The proof of the Theorem consists of four parts. First we shall show that the phase space can be covered by a countably infinite number of “regular” sets (Section 2), which permits later the application of the Hopf method. Then, for almost every element of each regular set, we shall define the set E (Section 3). Then we shall prove by the Hopf method that the set E is almost everywhere the subset of some ergodic component of the billiard (Section 4), and finally we shall prove that the density of the set E is not equal to zero at the element for which it was defined (Section 5).

We shall use some results: the existence of stable and unstable fibers and the so-called *absolute continuity*, which are proved in [3], [4] and [5]. These results are stated in Lemmas 2.1, 2.2, and 4.1. The conditions (1)–(3) were

needed in the proof of these lemmas, but they will not be explicitly used in this paper.

I would like to express my grateful thanks to Professor Ya. G. Sinai for giving me this beautiful problem and to my friends N. I. Tchernoff, A. Krámlí, and D. Szász for useful discussions and advices.

2. Construction of regular sets

Let x be an element of the phase space. $H^{(s)}(x)$ will be called the *stable fiber* of x if

- (1) $H^{(s)}(x)$ is a continuously differentiable curve in the phase space,
- (2) this curve passes through x ,
- (3) for each $y \in H^{(s)}(x)$ the distance between $S^t x$ and $S^t y$ tends to zero as t tends to ∞ ,
- (4) $H^{(s)}(x)$ is the maximal curve with these properties.

The definition of an unstable fiber is similar to only difference that in condition (3) t tends to $-\infty$. (Figure 1.)

Fig. 1. Fibers and leaves

The first moment of collision in the future of the element y is

$$\tau_+(y) := \min \{t > 0: \pi(S^t y) \in \partial Q\}.$$

The last moment of collision in the past of the element y , which can be defined in an analogous way, will be denoted by $\tau_-(y)$.

The stable leaf $L^{(s)}(x)$ and the unstable leaf $L^{(u)}(x)$ are (Fig. 1):

$$L^{(s)}(x) := \{z \in M: \exists y \in H^{(s)}(x), \exists t \in (\tau_-(y), \tau_+(y)), z = S^t y\},$$

$$L^{(u)}(x) := \{z \in M: \exists y \in H^{(u)}(x), \exists t \in (\tau_-(y), \tau_+(y)), z = S^t y\}.$$

We shall say that the closed ε -neighbourhood G_ε of the point $\pi(x)$ is covered by the *stable (unstable) leaf* of the element x if

$$G_\varepsilon \subset \pi(L^{(s)}(x)) \quad (G_\varepsilon \subset \pi(L^{(u)}(x))).$$

The *character* $\chi(x) \in [0, 1]$ was defined in (3) for each $x \in M$. It was proved there that for almost every element x the character $\chi(x)$ is positive, and then the element x has a stable fiber. The constant $D_0 > 0$ was defined by Theorem 5.2 in [3]. (For the definition see Appendix, 2.)

The *stable leaf-character* of the element x is defined now by

$$\psi^{(s)}(x) := \min \left\{ D_0 \cdot \chi^2(x), \frac{1}{2} \cdot \varrho(\pi(x), \partial Q) \right\},$$

where ϱ means distance on the billiard table. The *unstable leaf-character* $\psi^{(u)}(x)$ can be defined in an analogous way. The leaf character will be useful for our purposes because $\psi^{(s)}(x), \psi^{(u)}(x) > 0$ for almost every element x , and the following lemma is valid.

LEMMA 2.1. *If the stable (unstable) leaf-character of an element x is greater than or equal to ε then the closed ε -neighbourhood of the point $\pi(x)$ is covered by the stable (unstable) leaf of the element x ($x \in M, \varepsilon > 0$).*

The proof of this lemma is based on [3], and will be given in the Appendix.

The set $D_\varepsilon(q)$ is defined for $q \in Q$ and $\varepsilon > 0$ in the following way:

$$D_\varepsilon(q) := \{x \in \pi^{-1}(q) : \psi^{(s)}(x) \geq \varepsilon\}.$$

Let $G_\varepsilon(q)$ denote the ε -neighbourhood of the point q . The set $D'_\varepsilon(q)$ is defined for $r \in G_\varepsilon(q)$ in the following way:

$$D'_\varepsilon(q) := \{z \in \pi^{-1}(r) : \exists x \in D_\varepsilon(q), z \in L^{(s)}(x)\}.$$

The projection from $D'_\varepsilon(q)$ to $D_\varepsilon(q)$ along the stable leaves is the function which maps an element $z \in D'_\varepsilon(q)$ into the element $x \in D_\varepsilon(q)$ for which $z \in L^{(s)}(x)$ ($r \in G_\varepsilon(q), \varepsilon > 0$). (Fig. 2).

Fig. 2. Projections along the stable leaves

The restriction of the measure σ^r to the set $D_\varepsilon^r(q)$ is denoted by $\sigma^r \upharpoonright D_\varepsilon^r(q)$. The following lemma is proved in [5]. A short sketch of proof can be found in [4].

LEMMA 2.2 (absolute continuity I). *Given a point $q \in Q$ and the numbers $\varepsilon > 0$, $\varphi \in (0, 1)$, there exists a positive number $\delta = \delta(q, \varepsilon, \varphi)$ such that*

(1) *for $r \in G_\delta(q)$ the measure $\sigma^r \upharpoonright D_\varepsilon^r(q)$ is equivalent to the measure $\sigma^q \upharpoonright D_\varepsilon(q)$ by the projection from $D_\varepsilon^r(q)$ to $D_\varepsilon(q)$, and*

(2) *the density function of $\sigma^r \upharpoonright D_\varepsilon^r(q)$ with respect to $\sigma^q \upharpoonright D_\varepsilon(q)$ according to this equivalence is greater than or equal to φ . ■*

Since $\psi^{(s)}(x) > 0$ for almost every $x \in M$, there exists such a $Q_0 \subset Q$ that

(1) $\mu_Q(Q \setminus Q_0) = 0$,

(2) $\psi^{(s)}(x) > 0$ for σ^q -almost every $x \in \pi^{-1}(q)$, that is to say $\lim_{\varepsilon \rightarrow 0} \sigma^q(D_\varepsilon(q)) = 2\pi$ for $q \in Q_0$.

For a fixed point $q \in Q_0$ and a fixed number $\varphi \in (0, 1)$ let us choose the number $\varepsilon > 0$ in such a way that $\sigma^q(D_\varepsilon(q)) > 2\pi\varphi$. Then let us choose the number δ according to Lemma 2.2. The regular set constructed at the point q according to the level φ is defined in the following way:

$$P_\varphi(q) := \bigcup_{x \in D_\varepsilon(q)} L^{(s)}(x) \cap \pi^{-1}(G_\delta(q)).$$

The set $L^{(s)}(x) \cap \pi^{-1}(G_\delta(q))$ will be called a *leaf-piece*. Let us notice that a regular set is the union of some leaf-pieces.

LEMMA 2.3. *Given a level $\varphi \in (0, 1)$ there exists a finite number of points $q_1, \dots, q_n$ that the relative measure of the union of the regular sets constructed at these points according to the level φ is greater than or equal to φ^3 :*

$$\mu\left(\bigcup_{i=1}^n P_\varphi(q_i)\right) \geq \varphi^3 \cdot \mu(M).$$

Proof. The numbers ε and δ in the definition of $P_\varphi(q)$ were chosen in such a way that property (2) of Lemma 2.2 implies that

$$\sigma^r(D_\varepsilon^r(q)) \geq \varphi \cdot \sigma^q(D_\varepsilon(q)) \geq 2\pi\varphi^2 \quad (r \in G_\delta(q), q \in Q_0).$$

Since $\mu_Q(Q_0) = \mu_Q(Q)$, there exists such a compact $F \subset Q_0$ that $\mu_Q(F) \geq \varphi_\mu(M)$. For each $q \in F$ let us consider the numbers $\varepsilon = \varepsilon(q)$ and $\delta = \delta(q)$ according to the definition of $P_\varphi(q)$. Let us choose a finite number of points $q_1, \dots, q_n \in F$ so that

$$\bigcup_{i=1}^n G_{\delta(i)}(q_i) \supset F,$$

where $\delta(i) := \delta(q_i)$ ($i = 1, \dots, n$). Let $\varepsilon(i) = \varepsilon(q_i)$ ($i = 1, \dots, n$) and

$$\tilde{G} := \bigcup_{i=1}^n G_{\delta(i)}(q_i), \quad \tilde{D}^r := \bigcup_{i=1}^n D_{\varepsilon(i)}^r(q_i),$$

$$\tilde{P}^r := \bigcup_{i=1}^n P_{\varphi}(q_i).$$

It follows from property (2) of Lemma 2.2 that

$$\sigma^r(D_{\varepsilon(i)}^r(q_i)) \geq \varphi \sigma^q(D_{\varepsilon(i)}(q_i)) \geq 2\pi\varphi^2 \quad (r \in G_{\delta(i)}(q_i))$$

This implies that $\sigma^r(\tilde{D}^r) \geq 2\pi\varphi^2$ ($r \in \tilde{G}$). Obviously

$$\tilde{P} = \bigcup_{r \in \tilde{G}} \tilde{D}^r,$$

so

$$\mu(\tilde{P}) = \int_{\tilde{G}} \sigma^r(\tilde{D}^r) \mu_Q(dr) \geq 2\pi\varphi^2 \cdot \mu_Q(\tilde{G}) \geq \varphi^3 \mu(M). \quad \blacksquare$$

The following corollary is an obvious consequence of Lemma 2.3.

COROLLARY 2.4. *There exist countably many regular sets the union of which contains almost every element of the phase space. ■*

Provided that $q \in Q_0$ and $\varphi \in (0, 1)$ are fixed we shall examine now the regular set $P_{\varphi}(q)$. Since q and φ are fixed, the numbers ε and δ in the definition of $P_{\varphi}(q)$ are fixed, too. In order to simplify our notations let us put P, D, G, D^r ($r \in \tilde{G}$) instead of $P_{\varphi}(q), D_{\varepsilon}(q), G_{\delta}(q), D_{\varepsilon}^r(q)$.

As it was noticed, the set P is the union of some leaf-pieces. Obviously the projection π is a one-to-one mapping from each leaf-piece to the set G .

LEMMA 2.5. *The conditional measures of the measure $\mu \upharpoonright P$ on the leaf-pieces are equivalent to the measure $\mu_Q \upharpoonright G$ by the projection π .*

Proof. The set P can obviously be regarded as the product of the sets G and D : $P = G \times D$. An element $z \in P$ corresponds to the pair $(r, x) \in G \times D$ for which $z \in \pi^{-1}(r) \cap L^{(s)}(x)$ (Fig. 3).

Fig. 3. $P = G \times D$

The projection $(r, x) \rightarrow r$ is the restriction of the projection π , and the projection $(r, x) \rightarrow x$ is the projection along the stable leaves.

The conditional measures of the measure μ on the sets $\pi^{-1}(q)$ are the measures σ^q ($q \in Q$). This implies that the conditional measures of the measure $\mu \upharpoonright P$ on the sets D^r are the measures $\sigma^r \upharpoonright D^r$ ($r \in G$). Since these measures are equivalent to the measure $\sigma^q \upharpoonright D$ by the projection along the stable leaves (see Lemma 2.2), the conditional measures of the measure $\mu \upharpoonright P$ on the leaf-pieces are equivalent to the projection $(\mu \upharpoonright P)_Q$ of the measure $\mu \upharpoonright P$ by the projection π . But obviously $(\mu \upharpoonright P)_Q$ is equivalent to $\mu_Q \upharpoonright G$, because $\sigma^r(D^r) > 0$ ($r \in G$). ■

The intersection of a stable fiber and the set P will be called a *fiber-piece*. The projection of a fiber-piece by π is obviously a curve or at most a countably infinite number of curves on the billiard table. The arc length on these curves corresponds to some measures on the fiber-pieces, which will be called *arc measures on the fiber pieces*.

A simple consequence of Lemma 2.5 is the following

COROLLARY 2.6. *The conditional measures of the measure $\mu \upharpoonright P$ on the fiber-pieces are equivalent to the arc-measures on the fiber-pieces.*

Proof. The conditional measures of $\mu \upharpoonright P$ on the fiber-pieces can be derived from the conditional measures of $\mu \upharpoonright P$ on the leaf-pieces. The conditional measures on the leaf-pieces are equivalent to the measure $\mu_Q \upharpoonright G$ by π . The projections of the stable fibers on the billiard table are smooth curves, which implies that the conditional measures of any measure equivalent to the measure $\mu_Q \upharpoonright G$ are equivalent to the arc measure on the projection of the stable fibers. From these facts Corollary 2.6 follows. ■

3. Definition of E

If $f \in L_1(M, \mu)$, then the limits

$$f^+(x) := \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(S^t x) dt$$

and

$$f^-(x) := \lim_{T \rightarrow -\infty} \frac{1}{|T|} \int_T^0 f(S^t x) dt$$

exist and are equal to each other for almost every $x \in M$, and the functions $f^+, f^-: M \rightarrow \mathbb{R}^1$ are constant along the trajectories of the dynamics $\{S^t\}_{-\infty < t < \infty}$.

Let us choose and fix some continuous functions $f_1, f_2, \dots$ in such a way that they form a dense system in $L_2(M, \mu)$.

By Corollary 2.4 it is enough to define the set E for almost every element of each regular set P . Therefore, we choose and fix a regular set P .

An element $x \in P$ will be called *bad* if for some $i = 1, 2, \dots$ $f_i^+(x) \neq f_i^-(x)$. The set of bad elements will be denoted by R . As mentioned above, $\mu(R) = 0$. From this fact and Corollary 2.6 it is easy to show that the following proposition is true.

PROPOSITION 3.1. *For almost every $x \in P$ the arc measure of the set of bad elements on the fiber piece containing the element x is zero. ■*

Let $HD(x)$ denote the fiber-piece containing the element x ($x \in P$), and let

$$N_\infty(x) := \left\{ y \in HD(x) : \psi^{(u)}(y) \geq \frac{1}{2} \psi^{(u)}(x) \right\} \quad (x \in P).$$

Since $\psi^{(u)} > 0$ almost everywhere and the conditional measures of the measure $\mu \upharpoonright P$ are equivalent to the arc measures on the fiber-pieces, the following proposition is also true.

PROPOSITION 3.2. *For almost every $x \in P$ the density of the set $N_\infty(x)$ by the arc measure on the fiber-piece containing the element x is 1. ■*

Let us choose and fix an element $x \in P$ for which $\psi^{(u)}(x) > 0$ and Propositions 3.1 and 3.2 hold. In order to simplify the notations let us put N_∞ and G_δ instead of $N_\infty(x)$ and $G_\delta(\pi(x))$.

The set N_δ is defined in the following way:

$$N_\delta := N_\infty \cap \pi^{-1}(G_\delta) \quad (\delta > 0).$$

It is easy to check that G_δ is covered by the unstable leaf $L^{(u)}(y)$ if $\delta < \frac{1}{4} \psi^{(u)}(x)$ and $y \in N_\delta$. Let us choose and fix the number $\Delta = \Delta(x) > 0$ in such a way that

$$\Delta < \min \left\{ \frac{1}{4} \psi^{(u)}(x), \varrho(\pi(x), \partial Q) \right\}.$$

In order to simplify our notations let us put N and G instead of N_Δ and G_Δ .

Now we define the set E in the following way

$$E := \bigcup_{y \in N} L^{(u)}(y) \cap \pi^{-1}(G).$$

4. Application of the Hopf method

In this section the property (a) will be shown for the set E .

Let the set N^r be defined in the following way:

$$N^r := \{z \in \pi^{-1}(r) : \exists y \in N, z \in L^{(u)}(y)\} \quad (r \in G).$$

The projection of N^r to N along the unstable leaves is the function which maps an element $z \in N^r$ into the element $y \in N$ for which $z \in L^{(u)}(y)$ (Fig. 4).

Fig. 4. Projections along the unstable leaves

Let i denote the arc measure on the fiber piece $HD(x)$. The following lemma is proved in [5]. A short sketch of proof can be found in [4].

LEMMA 4.1 (absolute continuity II). *For each $r \in G$*

(1) *The measure $\sigma^r \upharpoonright N^r$ is equivalent to the measure $i \upharpoonright N$ by the projection along the unstable leaves, and*

(2) *there exists such a constant $a = a(x, \Delta) > 0$, not depending on r , that the density function of $\sigma^r \upharpoonright N^r$ with respect to $i \upharpoonright N$ according to this equivalence is greater than or equal to a . ■*

The use of this lemma can be seen immediately.

LEMMA 4.2. *The projection of the measure $\mu \upharpoonright E$ to the set N along the unstable leaves is equivalent to the measure $i \upharpoonright N$.*

Proof. The set E can obviously be regarded as the product of the sets G and N . An element $z \in E$ corresponds to the pair $(r, y) \in G \times N$ for which $z \in \pi^{-1}(r) \cap L^{(u)}(y)$ (Fig. 5).

Fig. 5. $E = G \times N$

The projection $(r, y) \rightarrow y$ is the projection along the unstable leaves. The conditional measures of the measure $\mu \upharpoonright E$ on the sets N^r are the measures $\sigma^r \upharpoonright N^r$ ($r \in G$). Since these measures are equivalent to the measure $i \upharpoonright N$ by the projection along the unstable leaves (see Lemma 4.1), the same fact holds for $\mu \upharpoonright E$. ■

The good and bad parts of E are defined in the following way:

$$E_{\text{good}} := \bigcup_{y \in N \setminus R} L^{(u)}(y) \cap \pi^{-1}(G),$$

$$E_{\text{bad}} := \bigcup_{y \in N \cap R} L^{(u)}(y) \cap \pi^{-1}(G).$$

LEMMA 4.3. $\mu(E_{\text{bad}}) = 0$.

Proof. The projection of the set E_{bad} on the set N by the unstable leaves in $N \cap R$. The i -measure of this set is zero (see Proposition 3.1), which implies that the i -measure of the set E_{bad} is zero (see Lemma 4.2). ■

LEMMA 4.4. *Each of the functions $f_1, f_2, \dots$ is equal to some constant on the set E_{good} .*

Proof. If y_1 and y_2 belong to the same stable fiber, then the distance between $S^t y_1$ and $S^t y_2$ tends to zero ($t \rightarrow \infty$), and $f(S^t y_1) - f(S^t y_2) \rightarrow 0$ ($t \rightarrow \infty$) for every continuous function f , which implies that $f^+(y_1) = f^+(y_2)$.

Analogously $f^-(y) = f^-(z)$ if y and z belong to the same unstable fiber or unstable leaf.

Let $z_k \in E_{\text{good}}$ ($k = 1, 2$), and let $y_k \in N \setminus R$ be such that $z_k \in L^{(u)}(y_k)$. Since $y_k \notin R$ we have $f_i^-(y_k) = f_i^+(y_k)$ ($i = 1, 2, \dots$), so $f_i^-(z_1) = f_i^-(y_1) = f_i^+(y_1) = f_i^+(y_2) = f_i^-(y_2) = f_i^-(z_2)$. ■

Since the functions $f_1, f_2, \dots$, are equal to constants almost everywhere on E , and they form a dense system in $L_2(M, \mu)$, every function $f \in L_2(M, \mu)$

is equal to some constant almost everywhere on E . This shows that the set E is almost everywhere a subset of some ergodic component. Property (a) of the Theorem is proved.

5. The density of E at x

In this section the property (b) will be shown for the set E .

The sets N_δ ($\delta > 0$) and the number Δ were defined in Section 3. Now we define the set E_δ for $0 < \delta \leq \Delta$ in the following way:

$$E_\delta := \bigcup_{y \in N_\delta} L^{(u)}(y) \cap \pi^{-1}(G_\delta).$$

It is clear that $E_\delta \subset E_\Delta = E$ ($0 < \delta \leq \Delta$).

LEMMA 5.1. $\liminf_{\delta \rightarrow 0} \mu(E_\delta) \cdot \delta^{-3} > 0$.

Proof. Obviously, $E_\delta = G_\delta \times N_\delta$ ($0 < \delta \leq \Delta$) (Fig. 6).

Fig. 6. $E_\delta = G_\delta \times N_\delta$

Like the sets N^r ($r \in G$) defined in Section 4, now the sets N_δ^r will be defined for $0 < \delta \leq \Delta$, $r \in G_\delta$

$$N_\delta^r := \{z \in \pi^{-1}(r) : \exists y \in N_\delta, z \in L^{(u)}(y)\}.$$

Property (2) of Lemma 4.1 implies that $\sigma^r(N^r) \geq a \cdot i(N_\delta)$ ($r \in G_\delta$), and

$$\mu(E_\delta) = \int_{G_\delta} \sigma^r(N_\delta^r) \mu_Q(dr) \geq a \cdot i(N_\delta) \cdot \mu_Q(G_\delta).$$

Proposition 3.2 guarantees that $i(N_\delta) \sim 2\delta$ ($\delta \rightarrow 0$). Since $\mu_Q(G_\delta) \sim \pi\delta^2$ ($\delta \rightarrow 0$), the lemma is proved. ■

LEMMA 5.2. *There exists such a constant $c > 0$, not depending on δ , that the set E_δ is included in the closed ball B_ε with center x and radius $\varepsilon := c \cdot \delta$: $E_\delta \subset B_\varepsilon$ ($0 < \delta \leq \Delta$).*

Proof. Any element $z \in E_\delta$ can be connected with x by three curves $\tilde{g}_1, \tilde{g}_2, \tilde{g}_3$ in the phase space so that the length of the projection g_i of $\tilde{g}_i$ ($i = 1, 2, 3$) on the billiard table should not be greater than 2δ (Fig. 7):

Fig. 7. $z \in E_\delta$ can be connected with x by $\tilde{g}_1, \tilde{g}_2, \tilde{g}_3$

$\tilde{g}_1$ is the part of a trajectory, $\tilde{g}_2$ is the part of an unstable fiber, and $\tilde{g}_3$ is the part of a stable fiber.

It follows from the of the Riemannian metrics on the billiard table and that on the phase space (see formula 2.1 in [3]) that the length of $\tilde{g}_1$ equals to the length of $\tilde{g}_1$.

The number Δ was defined in such a way that the distance of the point $\pi(x)$ from the boundary of the billiard table is greater than 2Δ . This implies that the distance of G from the boundary of the billiard table is greater than Δ . Therefore Theorem 3.2 and formula 2.1 of [3] guarantee that the lengths of $\tilde{g}_2$ and $\tilde{g}_3$ are not greater than $2\delta\sqrt{1+d^2}$, where $d := \sqrt{|K_{\min}|} \cdot \coth(\sqrt{|K_{\min}|}\Delta)$ and $K_{\min} := \min_{q \in Q} K(q)$.

This altogether means that the distance of z from x is not greater than $2\delta(1+2\sqrt{1+d^2})$, that is the constant $c := (1+2\sqrt{1+d^2})$ is suitable.

Property *b* of the Theorem easily follows from Lemmas 5.1 and 5.2. Since $E \cap B_\varepsilon \supset E_\delta$ ($\varepsilon > 0$, $\varepsilon = c \cdot \delta$), we have

$$\begin{aligned} \liminf_{\varepsilon \rightarrow 0} \mu(E \cap B_\varepsilon) / \mu(B_\varepsilon) &\geq \liminf_{\varepsilon \rightarrow 0} \mu(E_\delta) / \mu(B_\varepsilon) \\ &= \frac{3}{4\pi c^3} \liminf_{\delta \rightarrow 0} \mu(E_\delta) \delta^{-3} > 0. \end{aligned}$$

Appendix

1. Here, in the proof of Lemma 2.1 we shall use the notations of [3]. Theorem 3.14 of [3] implies that

$$P(\alpha_{1,D_0}) \subset P(\delta_1) \subset \pi(L^{(s)}(x)),$$

where the combs α_{1,D_0} and δ_1 were defined in Theorem 5.1 and after Theorem 5.5, respectively. Therefore it is enough to prove that the length of any curve g connecting the point $\pi(x)$ with the boundary of $P(\alpha_{1,D_0})$ is not less than $\psi^{(s)}(x)$. This fact, however, follows from the definition of $\psi^{(s)}(x)$, because by Corollary 3.6 the length of g is not less than

$$\frac{1}{2} W(\alpha_{1,D_0}) = D_0 \cdot \chi^2(x) \geq \psi(x). \quad \blacksquare$$

2. The definition of $\chi(y)$:

$$\chi(y) = \sup \left\{ c \in (0, 1); \text{ there is no } t \text{ such that } \pi(S^t y) \in \partial Q \text{ and } \cos(S^t y, N) \leq \frac{c}{t^2} \right\}, \text{ where } N \text{ is the normal to } \partial Q \text{ at } \pi(S^t y).$$

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