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COUNTABLY PIECEWISE EXPANDING TRANSFORMATIONS WITHOUT ABSOLUTELY CONTINUOUS INVARIANT MEASURE

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We give two examples of countably piecewise expanding transformations of an interval without any finite absolutely continuous invariant measure. For our transformations, the p-variation ($p > 1$) of the reciprocal of its derivative is finite, so we prove that Keller's result [4] cannot be generalized to countably piecewise expanding transformations. Our examples are Markov transformations and satisfy Rényi's condition, so they are counterexamples to Bowen's theorem [2, p.1].

Let I be a fixed compact subinterval of the real line. We call a transformation $f\colon I \to I$ *piecewise expanding* if there exists a countable subset A of I (containing the endpoints of I) such that for any interval $J \subset I \setminus A$, $f|_J$ is a strictly monotone C^1-mapping and $\inf_{x \in I \setminus A} |f'(x)| > 1$. We call it *finitely piecewise expanding* if A is finite and *countably piecewise expanding* if A is infinite.

For a piecewise expanding transformation $f\colon I \to I$, we define a function:

$$g_f(x) = \begin{cases} 1/|f'(x)| & \text{for } x \in I \setminus A; \\ 0 & \text{for } x \in A. \end{cases}$$

We recall the definition of the *p-variation* of a function $h\colon I \to R$, $I = [a, b]$:

$$V_p(h) = \sup_{N} \sup_{a = t_0 < t_1 < \ldots < t_N = b} \Big\{ \sum_{i=1}^{N} |h(t_i) - h(t_{i-1})|^p \Big\}^{1/p}.$$

(Some authors define p-variation without the exponent $1/p$ but this is of no importance in this note.) 1-variation is simply called *variation*.

In [5] Rychlik proved that a countably piecewise expanding transformation f with g_f of finite variation admits a finite absolutely continuous (with respect to Lebesgue measure) invariant measure (a.c.i.m.). On the other hand, Keller [4] proved that a finitely piecewise expanding transformation f with g_f of finite p-variation ($p \geqslant 1$) admits a finite a.c.i.m.

There arises the natural question: can Keller's result be generalized to countably piecewise expanding transformations? We answer this question negatively giving two examples of countably piecewise expanding transformations f with g_f of finite p-variation ($p > 1$) which have no finite a.c.i.m.

The first of them is very simple and the proof of the nonexistence of a finite a.c.i.m. is immediate. The transformation has not even infinite a.c.i.m. But its dynamical properties are not very interesting – almost all points are attracted to one fixed point.

The second example is more complicated and it is more difficult to prove the nonexistence of a finite a.c.i.m. in that case. There exists an infinite a.c.i.m. The transformation is "eventually onto", i.e. any small interval is mapped after finite number of iterations onto the whole interval I. So dynamics of the second example is far richer than the dynamics of the first one.

Both examples are Markov transformations. The transformation in the first example induces a transient Markov chain. The transformation in the second example induces an irreducible mixing Markov chain.

EXAMPLE I. Let $I = [0,1]$, $A = \{0\} \cup \{1/n\colon n = 1, 2, \ldots\}$ and $J_n = (1/(n+1), 1/n]$ for $n = 1, 2, \ldots$ We define f: for any $n = 1, 2, \ldots, f|_{J_n}$ is an increasing linear function such that $f(J_n) = (0, 1/n]$; $f(0) = 0$ (see Fig. 1). We have

$$g_f(x) = \begin{cases} 1/(n+1) & \text{for } x \in \operatorname{Int}(J_n),\ n = 1, 2, \ldots; \\ 0 & \text{for } x \in A. \end{cases}$$

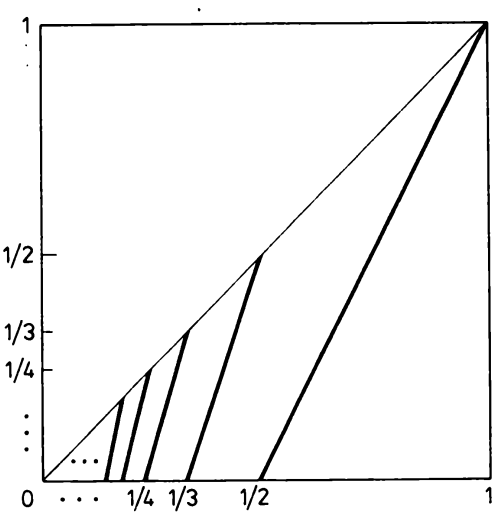

Fig. 1

It is easy to see that f is countably piecewise expanding transformation and $V_p(g_f) < \infty$ for any $p > 1$. To prove that f admits no finite a.c.i.m. it is enough to notice that almost every point of I (every point except $1/n$ for $n = 1, 2, \ldots$) tends under the iterated action of f to 0. The same argument gives the nonexistence of an infinite a.c.i.m. for this transformation. ■

A piecewise expanding transformation f is called a *Markov transformation* if $f(A) \subset A$ and, for any two connected components J, J' of $I \setminus A$, if for some positive integer k we have $J \cap f^k(J') \neq \emptyset$ then $J \subset f^k(J')$. The transformation f of Example I is in this sense Markov. Below we give an example of a countably piecewise expanding transformation f with g_f of finite p-variation ($p > 1$) and without any finite a.c.i.m. which is a Markov transformation in a far stronger sense: for any connected component J of $I \setminus A$ there exists a positive integer k such that $f^k(J) = I$.

EXAMPLE II. Let $I = [0, 1]$, $A = \{0\} \cup \{1/n\colon n = 1, 2, \ldots\}$ and $J_n = (1/(n+1), 1/n]$ for $n = 1, 2, \ldots$ We define f: $f|_{J_1}(x) = 2x-1$; for any $n = 2, 3, \ldots, f|_{J_n}$ is an increasing linear function such that $f(J_n) = (0, 1/(n-1)]$; $f(0) = 0$ (see Fig. 2). We have

$$g_f(x) = \begin{cases} 1/2 & \text{for } x \in \mathrm{Int}(J_1); \\ \dfrac{n-1}{n(n+1)} & \text{for } x \in \mathrm{Int}(J_n),\ n = 2, 3, \ldots; \\ 0 & \text{for } x \in A. \end{cases}$$

It is easy to see that f is a countably piecewise expanding Markov transformation (for any positive integer n; $f^n(J_n) = (0, 1]$) and $V_p(g_f) < \infty$ for any $p > 1$. We shall prove that f has no finite a.c.i.m.

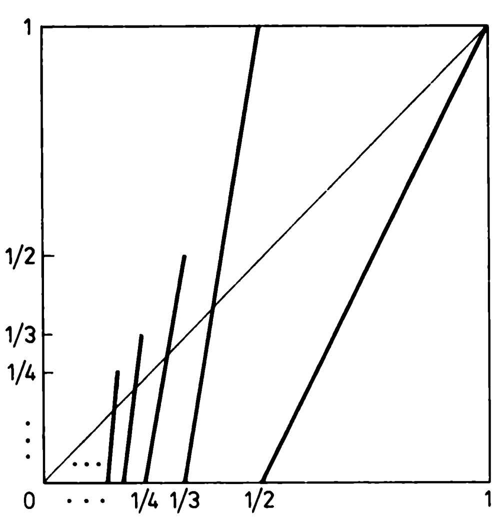

Fig. 2

We construct an infinite absolutely continuous f-invariant measure m.

We look for m whose density is constant on each J_n and we denote $m(J_n)$ by m_n, $n = 1, 2, \ldots$ Let m_1 be any positive number. The measure m will be f-invariant if and only if the numbers m_n ($n = 1, 2, \ldots$) satisfy the following equalities:

$$(*) \qquad \begin{aligned} m_1 &= (1/2)\, m_1 + (1/2)\, m_2, \\ m_n &= \left[\frac{1}{n} - \frac{1}{n+1}\right]\left[m_1 + \sum_{k=2}^{n+1} (k-1)\, m_k\right], \qquad n = 2, 3, \ldots \end{aligned}$$

(This is equivalent to the fact that, for any interval $J \subset [0, 1]$, $m(f^{-1}(J)) = m(J)$.)

The system $(*)$ is equivalent to the following one:

$$\begin{aligned} m_2 &= m_1, \\ m_{n+2} &= n(m_{n+1} - m_n) + 2m_{n+1}, \qquad n = 1, 2, \ldots \end{aligned}$$

Hence the numbers m_n can be defined so that m be f-invariant. Moreover, it is obvious that $m_{n+2} \geqslant 2m_{n+1}$, for $n = 1, 2, \ldots$, so m is infinite.

We now prove that m is an ergodic measure for f. We use the standard methods of the symbolic dynamics.

Let us consider the countable Markov chain with states J_n, $n = 1, 2, \ldots$, and transition probabilities:

$$\begin{aligned} p_{1,n} &= \frac{1}{n} - \frac{1}{n+1} && \text{for } n \geqslant 1; \\ p_{k,n} &= (k-1)\left[\frac{1}{n} - \frac{1}{n+1}\right] && \text{for } k \geqslant 2,\ n \geqslant k-1. \end{aligned}$$

This chain is irreducible and the equations $(*)$ imply that $m' = (m_1, m_2, m_3, \ldots)$ is its unique (up to a multiplicative constant) invariant measure. Hence, our chain is m'-ergodic.

Let X be the space of (one-sided) trajectories of our Markov chain with the product σ-field and the natural Markov measure m'' with initial distribution m'. Let S be the shift on the space X. Then m'' is S-invariant and S-ergodic [1].

The dynamical system (I, m, f) is in the natural way isomorphic to the dynamical system (X, m'', S). Hence the measure m is f-ergodic.

Since m is f-ergodic and the support of m is the whole interval I, f has no other a.c.i.m. In particular, f admits no finite a.c.i.m.

The transformations f constructed in these examples satisfy Rényi's condition

$$\sup_{x, y \in I \setminus A} \left|f''(x)/(f'(y))^2\right| < \infty,$$

so they are simple counterexamples to Bowen's theorem [2, p.1]. Another counterexample was given by Bugiel [3].

After completing this work the author was informed that another example of countably piecewise expanding transformation without an a.c.i.m. has been constructed by Bernard Schmitt [6]. In this example the p-variations ($p > 1$) of the reciprocal of derivative are infinite.

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