APPROXIMATION AND FUNCTION SPACES
BANACH CENTER PUBLICATIONS, VOLUME 22
PWN–POLISH SCIENTIFIC PUBLISHERS
WARSAW 1989

SOME REMARKS ON RELATIVE DIAMETERS

V. M. TIKHOMIROV

Department of Mathematics and Mechanics, Moscow State University
Moscow, U.S.S.R.

Recently V. N. Konovalov [1] has observed a new interesting phenomenon. It is well known that the nth Kolmogorov diameter of the Sobolev space $W_\infty^r(T^1)$ in the sup norm has order n^{-r}. Surprisingly enough, it turns out that if we require the approximating functions to belong to the class $W_\infty^r(T^1)$ itself, then, starting from $r \geqslant 2$, the accuracy of approximation (in the sense of weak asymptotics) stabilizes at the function $n \to n^{-2}$. The original proof was rather complicated, which has hindered a little the analysis of a number of related problems. In the present paper we give a simple proof of a slight generalization of V. N. Konovalov's remarkable theorem and then consider some related questions (however, the general problem that arises here is far from being solved).

1. Formulation of the problem, statement of the results and preliminaries

Let X be a normed space and $W \subset X$ a central-symmetric subset. The *n-th Konovalov diameter* or *n-th relative diameter* of W in X is defined by

$$kd_n(W, X) = \inf \sup_{x \in W} \inf_{y \in W \cap L_n} \|x-y\|_X,$$

where the first infimum is taken over all n-dimensional subspaces L_n of X (in the case of the Kolmogorov diameter $d_n(W, X)$, the second infimum is just over $y \in L_n$).

Let $\alpha \in \boldsymbol{R}_+$. Define $W_p^\alpha(T^1)$ (T^1 being the circle realized as $[-\pi, \pi]$ with $\pm\pi$ identified) to be the collection of 2π-periodic functions $x(\cdot)$ representable as a convolution $x(t) = (\varphi_\alpha * u)(t)$, where $u(\cdot) \in L_p(T^1)$, $\|u(\cdot)\|_{L_p(T^1)} \leqslant 1$ and $\varphi_\alpha(t) = \sum_{k \in N} (ik)^{-\alpha} \exp(ikt)$.

THEOREM. *Let* $\alpha > 0$. *Then*

$$kd_n(W^\alpha_\infty(T^1), L_\infty(T^1)) \asymp n^{-\min(\alpha,2)}.$$

For $\alpha = 1, 2, \ldots$ this result was obtained in [1].

In the proof of the theorem we use the following four facts.

1) Ismagilov's theorem. Let $y(\cdot) = \sum_{k\in Z} y_k \exp(ik\cdot)$ be an $L_2(T^1)$ function and let $K_{y(\cdot)}$ be the curve in $L_2(T^1)$ formed by the translates of $y(\cdot)$: $K_{y(\cdot)} = \{T_\tau y(\cdot)\}_{\tau\in T^1}$, $T_\tau y(t) = y(t+\tau)$. Then

$$d_n(K_{y(\cdot)}, L_2(T^1)) = \Big(\sum_{s\geqslant n} |y_{i_s}|^2\Big)^{1/2},$$

where $(i_1, i_2, \ldots)$ is a permutation of Z such that $|y_{i_1}| \geqslant |y_{i_2}| \geqslant \ldots$

2) The following Fourier expansion holds:

$$\xi(t) := \operatorname{sign}\sin t = 4\pi^{-1} \sum_{k\in N} (2k-1)^{-1} \sin(2k-1)t.$$

3) The Neyman–Pearson lemma. Let $x(\cdot)\in C(\Delta)$, $x(t) \geqslant 0$ $\forall t\in\Delta = [t_0, t_1]$, $Y = \{y(\cdot)\in L_1(\Delta) \mid 0 \leqslant y(t) \leqslant A \text{ a.e.}, \int_\Delta y(t)\,dt \geqslant B\}$. Then

$$\int_\Delta x(t)\,y(t)\,dt \geqslant \int_{D(A,B)} x(t)\,dt, \qquad \forall y(\cdot)\in Y$$

where $D(A, B)$ is the set of t satisfying $0 \leqslant x(t) \leqslant C(A, B)$ with $C(A, B)$ chosen so as to have $A\int_{D(A,B)} dt = B$.

4) For $\alpha \geqslant 2$ and $c_1 \geqslant 2$ we have the inequality

$$\int_{D(2,c_1/n)} (\varphi_\alpha)_+(t)\,dt \geqslant c_2 n^{-2}, \qquad c_2 > 0$$

$(u_+ = \max(u, 0))$.

For the proof of Ismagilov's theorem, see e.g. [2]; the well-known expansion of $t \to \operatorname{sign}\sin t$ can also be found there. The Neyman–Pearson lemma is stated in many places; an easy proof can be obtained via the theory of extremal problems (a discrete analogue of the Neyman–Pearson lemma was used in [2] to prove Ismagilov's theorem). The proof of 4) is elementary.

2. Proof of the theorem

The upper estimate is trivial: the Jackson operator approximates the class $W^\alpha_\infty(T^1)$ with accuracy $n^{-\alpha}$ for $0 < \alpha \leqslant 2$ and n^{-2} for $\alpha \geqslant 2$. Let L'_n be an n-dimensional space of $L_\infty(T^1)$ functions with mean zero, and let L_n be the subspace of αth integrals of functions from L'_n. By Ismagilov's theorem and 2), there is τ_0 such that the distance $d(T_{\tau_0}\xi(\cdot), L'_n, L_2(T^1))$ in the $L_2(T^1)$

norm from $T_{\tau_0}\xi(\cdot)$ to the subspace L'_n satisfies the inequality

$$d(T_{\tau_0}\xi(\cdot), L'_n, L_2(T^1)) \geqslant d_n(K_{\xi(\cdot)}, L_2(T^1)) = \frac{4}{\pi}\Big(\sum_{k\geqslant n}\frac{1}{(2k-1)^2}\Big)^{1/2} \geqslant \frac{c}{\sqrt{n}}. \tag{1}$$

Without loss of generality we can assume that $\tau_0 = 0$. Denote by $\eta(\cdot)$ an element realizing the best approximation of $(\varphi_\alpha * \xi)(\cdot)$ in $L_\infty(T^1)$ by elements of $L_n \cap W^\alpha_\infty(T^1)$, and let $\zeta(\cdot) = \xi(\cdot) - \eta^{(\alpha)}(\cdot)$ (which means that $(\varphi_\alpha * \zeta)(\cdot) = (\varphi_\alpha * \xi)(\cdot) - \eta(\cdot)$). Thus

$$d((\varphi_\alpha * \xi)(\cdot), L_n \cap W^\alpha_\infty(T^1), L_\infty(T^1)) = \|(\varphi_\alpha * \zeta)(\cdot)\|_{L_\infty(T^1)}. \tag{2}$$

By (1), $\|\zeta(\cdot)\|_{L_2(T^1)} \geqslant d(\xi(\cdot), L'_n, L_2(T^1)) \geqslant c/\sqrt{n}$. Let Δ_α be the interval where $\varphi_\alpha(t) \geqslant 0$, $|\Delta_\alpha| \asymp \pi$. Without loss of generality we assume that $\Delta_\alpha = [0, \pi]$ and $\|\zeta(\cdot)\|_{L_2([0,\pi])} \geqslant c/(2\sqrt{n})$. By our choice, $|\eta^{(\alpha)}(t)| \leqslant 1$, which yields the obvious inequalities $0 \leqslant \zeta(t) \leqslant 2$, $t \in [0, \pi]$, and $\zeta(t) \leqslant 0$, $t \in [-\pi, 0]$ (since $\xi(t) \equiv 1$, $t \in [0, \pi]$). Then

$$\frac{1}{\sqrt{n}} \leqslant \frac{2}{c}\Big(\int_0^\pi \zeta^2(t)\,dt\Big)^{1/2} \leqslant \frac{2}{c}\Big(\max_{t\in[0,\pi]}\zeta(t)\int_0^\pi \zeta(\tau)\,d\tau\Big)^{1/2} \leqslant \frac{2\sqrt{2}}{c}\Big(\int_0^\pi \zeta(t)\,dt\Big)^{1/2}. \tag{3}$$

From (3) we obtain $\int_0^\pi \zeta(t)\,dt \geqslant c_1/n$. By the Neyman–Pearson lemma,

$$\begin{aligned} d((\varphi_\alpha * \xi)(\cdot), L_n \cap W^\alpha_\infty(T^1), L_\infty(T^1)) &= \|(\varphi_\alpha * \zeta)(\cdot)\|_{L_\infty(T^1)} \quad \text{by (2)} \\ &\geqslant |(\varphi_\alpha * \zeta)(0)| = \Big|\int_{-\pi}^{\pi} \varphi_\alpha(t)\zeta(t)\,dt\Big| \\ &\geqslant \int_0^\pi \varphi_\alpha(t)\zeta(t)\,dt \\ &\geqslant \int_{D(2,c_1/n)} \varphi_\alpha(t)\,dt \quad \text{(by 3)} \\ &\geqslant c_2 n^{-2} \quad \text{(by 4)}, \end{aligned}$$

which completes the proof.

3. Remarks

3.1. The proof presented in Section 2 is an improved exposition of V. N. Konovalov's own proof. The original proof followed the same lines, but its author proved himself some assertions similar to Ismagilov's theorem and the Neyman–Pearson lemma. A discussion on these subjects between the present author and S. V. Konyagin resulted in a search for the driving forces of Konovalov's proof.

3.2. The natural question arises of the behaviour of $kd_n(W_p^\alpha(T^1), L_q(T^1))$ for different p and q. Clearly,

$$kd_n(W_?^\alpha(T^1), L_2(T^1)) \doteq d_n(W_2^\alpha(T^1), L_2(T^1)) \asymp n^{-\alpha}.$$

It can be shown by the methods of Section 2 that

$$kd_n(W_\infty^\alpha(T^1). L_q(T^1)) \asymp n^{-\min(\alpha,2)}, \quad 2 \leqslant q \leqslant \infty.$$

To all appearance, in the other cases, other ideas have to be searched for.

3.3. The use of de la Vallée Poussin operators shows that there is a constant $\hat{c}(\alpha)$ such that there are trigonometric polynomials of degree n for which the $C(T^1)$ norm of the αth derivative does not exceed $\hat{c}(\alpha)$ and which approximate the class $W_\infty^\alpha(T^1)$ with the rate $n^{-\alpha}$. It is interesting to pass to $\hat{c}(\alpha)$ from the case considered above, where we have restricted ourselves to the constant 1. It is easy to show by the methods of Section 2 that "close to 1" the rate of approximation is $n^{-\min(\alpha,2)}$ as above, but for constants greater than some $\tilde{c}$ our method breaks down.

3.4. One of the natural limit cases for the use of our method of proof is the case of the sphere, where $W_\infty^\alpha(S^m)$ is defined to be the collection of functions $x(\cdot)$ satisfying $\|\Delta^{\alpha/2}x(\cdot)\|_{L_\infty(S^m)} \leqslant 1$. Here it also seems possible to obtain the estimate

$$kd_n(W_\infty^\alpha(S^m), L_\infty(S^m)) \geqslant n^{-\gamma(m)} \quad \text{for } \alpha \geqslant \alpha(m).$$

All these problems seem interesting; I wanted to draw attention to them.

3.5. This paper was written during the author's stay at the Banach Center in the spring of 1986, and discussed there in detail with V. N. Konovalov, whom I wish to express my sincere gratitude. I would also like to thank S. V. Konyagin who clarified part of the way to the final result.

References

[1] V. N. Konovalov, *Estimates of Kolmogorov type diameters for classes of periodic differentiable functions*, Mat. Zametki 36 (3) (1984), 154–160 (in Russian).

[2] V. M. Tikhomirov, *Some Problems of Approximation Theory*, Izdat. Moskov. Gos. Univ., Moscow 1976 (in Russian).

Translated from the Russian
by Jerzy Trzeciak

Presented to the Semester
Approximation and Function Spaces
February 27–May 27, 1986