COLLOQUIUM MATHEMATICUM

VOL. LVIII 1989 FASC. 1

PRIMES AND POWER-PRIMES

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Let us recall first three definitions of the Möbius function μ, which we shall use below. $\mu(1) = 1$ and

$$\mu(n) = \begin{cases} 0 & \text{if } n \text{ is divisible by a prime power,} \\ (-1)^k & \text{if } n \text{ is a product of } k \text{ distinct primes.} \end{cases} \tag{1}$$

$$\sum_{k \leqslant y} \mu(k)[y/k] = 1 \quad \text{for all } y \geqslant 1, \tag{2}$$

where k runs over positive integers and $[u]$ is the largest integer $\leqslant u$.

$$\sum_{k=1}^{\infty} \frac{\mu(k)}{k^s} = \frac{1}{\zeta(s)} \quad \text{for all } s > 1, \tag{3}$$

where

$$\zeta(s) = \sum_{k=1}^{\infty} k^{-s}$$

is the Riemann ζ function.

Riemann in 1859 (see [2], p. 305) and Ramanujan around 1910 (still before he became acquainted with the European studies in number theory, see [10], p. 349) invented the function

$$R(x) = \sum_{k=1}^{\infty} \frac{\mu(k)}{k} \operatorname{li}(x^{1/k}), \tag{4}$$

where

$$\operatorname{li} x = \lim_{\delta \downarrow 0} \left(\int_0^{1-\delta} \frac{du}{\log u} + \int_{1+\delta}^{x} \frac{du}{\log u} \right).$$

They did not have any proof of convergence of the series in (4), but they suspected that it does and probably they thought that $\pi(x)$, the number of primes $\leqslant x$, satisfies, for every $\varepsilon > 0$,

$$\pi(x) = R(x) + O(x^{\varepsilon}). \tag{5}$$

Today we know that (4) converges, but that (5) is far from truth. Littlewood discovered that $\pi(x) > \mathrm{li}x$ for infinitely many integers x (see [1] and [9] for references and subsequent improvements). Since

$$R(x) = \mathrm{li}x - \tfrac{1}{2}\mathrm{li}x^{1/2} + O(\mathrm{li}x^{1/3}),$$

it follows that (5) fails unless $\varepsilon \geqslant 1/2$, and it is known that the relation

$$\pi(x) = \mathrm{li}x + O(x^{1/2+\varepsilon})$$

for every $\varepsilon > 0$ is equivalent to the Riemann ζ conjecture. Still, R could have the following significance for the distribution of primes. We can introduce an averaging linear operator

$$f^{\mathrm{M}}(x) = \frac{1}{x-1}\int_1^x f(u)du \tag{6}$$

and conjecture that, for each $\varepsilon > 0$,

$$\pi^{\mathrm{M}}(x) = R^{\mathrm{M}}(x) + O(x^{\varepsilon}).$$

But this is still false [H. Montgomery, 1989 letter to the author]. Perhaps applying the operation (6) several times to the equation (5) yields a truth. (See the Addendum at the end of this note concerning possible motivations of Riemann.)

In this state of affairs, should one forget the function $R(x)$? It is the purpose of this note to show that R has a demonstrable significance in the theory of power-primes. *Power-primes* are those positive integers like 2, 3, 5, 6, 7, 10, 11, 12, ... which are not of the form a^b, where a and b are positive integers and $b > 1$. We will prove two theorems which show a tight relationship between this sequence and the function $R(x)$. Those theorems were partly proved in [3] and partly announced without proofs in [6]. For other papers on power-primes see [7] and [8].

At one point of our proof we will have to use the non-elementary formulas

$$\sum_{k=1}^{\infty} \frac{\mu(k)}{k} = 0 \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{\mu(k)\log k}{k} = -1, \tag{7}$$

which were proved by von Mangoldt (see [2], p. 92, or [4] and [5]).

Let $P(x)$ denote the number of power-primes $\leqslant x$, and put

$$H(x) = \sum_{n=2}^{\infty} \frac{(\log x)^n}{n!\zeta(n)}. \tag{8}$$

THEOREM 1 (Kössler [3]).

$$P(x) = \sum_{k \leqslant \log_2 x} \mu(k)[x^{1/k} - 1] = H(x) + O(\log x),$$

and $H(x)$ *is related to* $R(x)$ *by the equations*

$$R(x) = 1 + \int_1^x \frac{dH(t)}{\log t} \quad \textit{and} \quad H(x) = \int_1^x \log t\, dR(t).$$

Proof. (Unlike Kössler we shall not use (7) in the proof of the first part of this theorem.) The first equation follows immediately from the definition of $P(x)$ by an inclusion-exclusion argument using the definition (1) of $\mu(k)$.

To prove the second equation we need two elementary facts. For all $y \geqslant 1$

$$\left|\sum_{k \leqslant y} \frac{\mu(k)}{k}\right| \leqslant 1$$

(this follows easily from (2)), and

$$\sum_{y \leqslant k < \infty} (e^{y/k} - 1 - y/k) = O(y),$$

which is an easy exercise.

Putting $y = \log x$ in the above relations, and using (3) and (8) we get

$$\begin{aligned}
\sum_{k \leqslant \log_2 x} \mu(k)[x^{1/k} - 1] &= \sum_{k=1}^{\infty} \mu(k)\left(x^{1/k} - 1 - \frac{y}{k}\right) + O(y) \\
&= \sum_{k=1}^{\infty} \mu(k) \sum_{n=2}^{\infty} \frac{y^n}{n!\,k^n} + O(y) \\
&= \sum_{n=2}^{\infty} \frac{y^n}{n!} \sum_{k=1}^{\infty} \frac{\mu(k)}{k^n} + O(y) \\
&= \sum_{n=2}^{\infty} \frac{y^n}{n!\,\zeta(n)} + O(y) = H(x) + O(\log x),
\end{aligned}$$

which is the second equation of Theorem 1. (The interchange of the summations $\sum_{k=1}^{\infty}$ and $\sum_{n=2}^{\infty}$ was legal since the double series converges absolutely.)

The third and fourth equations are essentially due to J. P. Gram and can be shown as follows:

$$\begin{aligned}
\int_1^x \frac{dH(t)}{\log t} &= \int_1^x \sum_{n=2}^{\infty} \frac{(\log t)^{n-2}}{(n-1)!\,\zeta(n)\,t}\,dt = \sum_{m=1}^{\infty} \frac{(\log x)^m}{m!\,m\,\zeta(m+1)} \\
&= \sum_{m=1}^{\infty} \frac{(\log x)^m}{m!\,m} \sum_{k=1}^{\infty} \frac{\mu(k)}{k^{m+1}} \qquad \text{(by (3))} \\
&= \sum_{k=1}^{\infty} \frac{\mu(k)}{k} \sum_{m=1}^{\infty} \frac{(\log x^{1/k})^m}{m!\,m} \\
&= \sum_{k=1}^{\infty} \frac{\mu(k)}{k} \int_0^{\log x^{1/k}} \sum_{m=1}^{\infty} \frac{t^{m-1}}{m!}\,dt = \sum_{k=1}^{\infty} \frac{\mu(k)}{k} \int_0^{\log x^{1/k}} \frac{e^t - 1}{t}\,dt \\
&= \sum_{k=1}^{\infty} \frac{\mu(k)}{k} \int_2^{\log x^{1/k}} \frac{e^t - 1}{t}\,dt \qquad \text{(by (7))}
\end{aligned}$$

$$= \sum_{k=1}^{\infty} \frac{\mu(k)}{k} \operatorname{li} x^{1/k} - \sum_{k=1}^{\infty} \frac{\mu(k)}{k} \log\log x^{1/k} \qquad \text{(by (7))}$$

$$= R(x) + \sum_{k=1}^{\infty} \frac{\mu(k)\log k}{k} \qquad \text{(by (7))}$$

$$= R(x) - 1 \qquad \text{(by (7))}.$$

So the third equation of Theorem 1 is proved.

The fourth follows from the third by differentiation, multiplication by $\log x$ and integration.

Now we will prove a very accurate asymptotic formula involving directly Riemann's function R.

THEOREM 2.

$$\sum_{P \leqslant x} \frac{1}{\log P} = R(x) + O(\log\log x),$$

where P runs over power-primes.

Proof. By Theorem 1, and since $\log x$ is a monotonic function, we have

$$\sum_{P \leqslant x} \frac{1}{\log P} = \int_{1.5}^{x} \frac{dP(u)}{\log u} = \frac{P(x)}{\log x} - \int_{1.5}^{x} P(u)\, d\left(\frac{1}{\log u}\right)$$

$$= \frac{H(x)}{\log x} + O(1) - \int_{1.5}^{x} (H(u) + O(\log u))\, d\left(\frac{1}{\log u}\right)$$

$$= \frac{H(x)}{\log x} - \int_{1.5}^{x} H(u)\, d\left(\frac{1}{\log u}\right) + O\left(\int_{1.5}^{x} \log u\, d\left(\frac{1}{\log u}\right)\right)$$

$$= \int_{1.5}^{x} \frac{dH(u)}{\log u} + O\left(\int_{1.5}^{x} \frac{du}{u \log u}\right) = R(x) + O(\log\log x).$$

(Several other theorems about the sequence of power-primes are given in [3] and [6]–[8].)

Addendum. One may ask how Riemann invented the function $R(x)$ (this is not explained in his paper, see [2], p. 305). One possible "derivation" of $R(x)$ is the following. Let us recall Chebyshev's functions

$$\theta(x) = \sum_{p \leqslant x} \log p \quad \text{and} \quad \psi(x) = \sum_{p^n \leqslant x} \log p.$$

Then $\psi(x) = \sum_{k=1}^{\infty} \theta(x^{1/k})$ and, by Möbius inversion,

$$\theta(x) = \sum_{m=1}^{\infty} \mu(m)\psi(x^{1/m}).$$

Hence we have the identities

$$\pi(x) = \int_0^x \frac{d\theta(t)}{\log t} = \sum_{m=1}^{\infty} \frac{\mu(m)}{m} \int_0^{x^{1/m}} \frac{d\psi(t)}{\log t}.$$

Now, if we substitute t for $\theta(t)$ or for $\psi(t)$, we get the approximations li$x$ and $R(x)$, respectively. Such substitutions were natural after the work of Chebyshev (which was known to Riemann).

The claim that $R(x)$, or at least $R(x)-1$, is a natural approximation to $\pi(x)$ can be also seen from the following identity:

$$\int_1^{\infty} (\pi(x)-R(x)+1)\frac{dx}{x(x-1)} = 0. \tag{9}$$

The convergence at 1 and at ∞ will be shown below. We will show also

$$\int_1^{\infty} (\pi(x)-R(x)+1)\frac{dx}{x(x^{1+\varepsilon}-1)} = (\gamma-1)\varepsilon+O(\varepsilon^2) \tag{10}$$

as $\varepsilon \downarrow 0$, where γ is Euler's constant.

Proof of (10) and (9). Euler's identity tells

$$\zeta(s) = \prod_p \sum_{k=0}^{\infty} p^{-ks} = \prod_p (1-p^{-s})^{-1} \quad \text{for } s > 1.$$

Hence

$$\log\zeta(s) = -\int_1^{\infty} \log(1-x^{-s})d\pi(x) = s\int_1^{\infty} \frac{\pi(x)dx}{x(x^s-1)}.$$

We have also

$$\begin{aligned} s\int_1^{\infty} \frac{(R(x)-1)dx}{x(x^s-1)} &= -\int_1^{\infty} \log(1-x^{-s})dR(x) \\ &= \int_1^{\infty} \sum_{k=1}^{\infty} \frac{1}{kx^{ks}} \sum_{n=1}^{\infty} \frac{(\log x)^{n-1}}{n!\zeta(n+1)x}dx \\ &= \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n!\zeta(n+1)k} \int_1^{\infty} \frac{(\log x)^{n-1}}{x^{ks+1}}dx \\ &= \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{ns^n\zeta(n+1)k^{n+1}} = \sum_{n=1}^{\infty} \frac{1}{ns^n} = \log\frac{s}{s-1}. \end{aligned}$$

Putting $\varepsilon = s-1$ we have

$$\zeta(s) = 1/\varepsilon+\gamma+O(\varepsilon) \quad \text{as } \varepsilon\downarrow 0.$$

Then subtracting the previous equations we get

$$\int_1^{\infty} (\pi(x)-R(x)+1)\frac{dx}{x(x^s-1)} = \frac{1}{s}\log\frac{(s-1)\zeta(s)}{s} = (\gamma-1)\varepsilon+O(\varepsilon^2),$$

which yields (10).

Finally, (9) follows easily from (10) and the theorem of de la Vallée Poussin which implies that the integrals (9) and (10) converge absolutely. (The convergence of (9) at the point $x = 1$ requires a little calculation. Namely, computing as above we get

$$\int_1^2 \frac{(R(x)-1)dx}{x(x-1)} = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n!\zeta(n+1)k} \int_1^2 \frac{(\log x)^{n-1}}{x^{k+1}} dx + C$$
$$\leqslant \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{(\log 2)^{n-1}}{n!\zeta(n+1)k^2} + C < \infty.)$$

Remark. We do not know if $P(x) - H(x)$ or $\sum_{P \leqslant x} 1/\log P - R(x)$ change sign for arbitrarily large x. **(P 1375)**

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Reçu par la Rédaction le 2.11.1988