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ON SHAPE PRESERVING SPLINE INTERPOLATION: EXISTENCE THEOREMS AND DETERMINATION OF OPTIMAL SPLINES

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1. Introduction

In the practical use of interpolation it is often required to preserve such properties of the data as monotonicity or convexity. As is well known, polynomial and even spline interpolants, in general, do not have this property. There are data sets in convex position for which, e.g., the quadratic or cubic interpolating splines are not convex (see Passow and Roulier [8]). In the monotone case the situation is analogous; only for cubic C^1-splines the result is always positive (see Fritsch and Carlson [3]).

In this lecture, for some kinds of spline interpolation necessary and sufficient conditions are given under which the shape is preserved. Further, since convex or monotone spline interpolants, if existing at all, are not in general uniquely determined, strategies for selecting one of these splines are proposed. This is done for quadratic, cubic and related splines by using earlier results of Schmidt and Heß [10], Burmeister, Heß and Schmidt [1], Dietze and Schmidt [2], Heß and Schmidt [5] and Schmidt and Heß [11]. Now, a unified presentation is given. Also some new aspects are considered; see especially Sections 3.2, 3.4, 4.2 and 5.5.

2. Weakly coupled systems of inequalities

Let α_i, β_i, τ_i be given constants with

$$\alpha_i > 0, \quad \beta_i > 0, \quad \alpha_i+\beta_i = 1 \quad (i = 1, \ldots, n), \tag{2.1}$$

$$\tau_1 \leqslant \tau_2 \leqslant \ldots \leqslant \tau_n. \tag{2.2}$$

PROBLEM 1. *Are there numbers* $m_0, m_1, \ldots, m_{n-1}$ *such that*

$$\alpha_i m_{i-1}+\beta_i m_i = \tau_i \quad (i = 1, \ldots, n-1), \qquad m_{i-1} \geqslant \tau_i \quad (i = 1, \ldots, n)?$$

ALGORITHM 1 [11]. $c_0 = 1$, $d_1 = \tau_1$,

$$c_i = \beta_i c_{i-1}/\alpha_i, \quad d_{i+1} = d_i + (-1)^i c_i (\tau_{i+1} - \tau_i) \quad (i \geqslant 1);$$

$$d_* = \max\{d_i: i \text{ even}\}, \quad d^* = \min\{d_i: i \text{ odd}\}.$$

PROPOSITION 1 [11]. *Under the assumptions* (2.1) *and* (2.2), *Problem* 1 *is solvable if and only if*

$$d_* \leqslant d^*. \tag{2.3}$$

Further, one gets

$$\tau_i - m_{i-1} = (-1)^i (m_0 - d_i)/c_{i-1} \quad (i \geqslant 1). \tag{2.4}$$

Next, let the constants α_i, β_i, τ_i satisfy (2.1) and

$$\tau_1 \geqslant 0, \quad \tau_2 \geqslant 0, \quad \ldots, \quad \tau_n \geqslant 0. \tag{2.5}$$

PROBLEM 2. *Are there numbers $m_0, m_1, \ldots, m_n$ such that*

$$\alpha_i m_{i-1} + \beta_i m_i = \tau_i \quad (i = 1, \ldots, n), \quad m_i \geqslant 0 \quad (i = 0, \ldots, n)?$$

ALGORITHM 2 [11]. $e_0 = 0$, $c_0 = 1$,

$$e_i = e_{i-1} + (-1)^{i-1} c_{i-1} \tau_i/\alpha_i, \quad c_i = \beta_i c_{i-1}/\alpha_i \quad (i \geqslant 1);$$

$$e_* = \max\{e_i: i \text{ even}\}, \quad e^* = \min\{e_i: i \text{ odd}\}.$$

PROPOSITION 2 [11]. *Under the assumptions* (2.1), (2.5), *Problem* 2 *is solvable if and only if*

$$e_* \leqslant e^*. \tag{2.6}$$

Further, one gets

$$m_i = (-1)^i (m_0 - e_i)/c_i \quad (i \geqslant 0). \tag{2.7}$$

Now, suppose that the constants α_i, β_i, γ_i, δ_i, τ_i satisfy (2.2) and

$$0 < \gamma_i < \alpha_i, \quad 0 < \beta_i < \delta_i, \quad \alpha_i + \beta_i = 1, \quad \gamma_i + \delta_i = 1 \quad (i = 1, \ldots, n). \tag{2.8}$$

PROBLEM 3. *Are there numbers $m_0, m_1, \ldots, m_n$ such that*

$$\alpha_i m_{i-1} + \beta_i m_i \leqslant \tau_i, \quad \gamma_i m_{i-1} + \delta_i m_i \geqslant \tau_i \quad (i = 1, \ldots, n)?$$

ALGORITHM 3 [10]. $a_0 = -\infty$, $b_0 = +\infty$,

$$a_i = \max\{\tau_i, (\tau_i - \gamma_i b_{i-1})/\delta_i\}, \quad b_i = (\tau_i - \alpha_i a_{i-1})/\beta_i \quad (i \geqslant 1).$$

PROPOSITION 3 [10]. *Under the assumptions* (2.2), (2.8), *Problem* 3 *is solvable if and only if*

$$a_{i-1} \leqslant \tau_i \quad (i = 1, \ldots, n). \tag{2.9}$$

For the proofs of these propositions we refer to the cited papers.

3. Shape preserving quadratic and related spline interpolants

Let Δ be a grid on the interval [0, 1],

$$\Delta: x_0 = 0 < x_1 < \ldots < x_n = 1. \tag{3.1}$$

A set of points (x_0, y_0), (x_1, y_1), ..., (x_n, y_n) is said to be *in convex or monotone position* if, for the slopes

$$\tau_i = (y_i - y_{i-1})/h_i, \quad h_i = x_i - x_{i-1}, \tag{3.2}$$

(2.2) or (2.5) is valid, respectively.

3.1. Quadratic splines

Let $Sp_2(1)$ be the set of quadratic C^1-splines on Δ. The quadratic spline s given by

$$s(x) = y_{i-1} + m_{i-1} h_i t + (\tau_i - m_{i-1}) h_i t^2, \quad 0 \leqslant t \leqslant 1, \tag{3.3}$$

with $x = x_{i-1} + h_i t$ $(i = 1, \ldots, n)$ satisfies the interpolation condition

$$s(x_i) = y_i \quad (i = 0, \ldots, n). \tag{3.4}$$

Further, s is continuously differentiable if and only if

$$m_{i-1} + m_i = 2\tau_i \quad (i = 1, \ldots, n-1) \tag{3.5}$$

where

$$m_i = s'(x_i) \quad (i = 0, \ldots, n). \tag{3.6}$$

The derivatives $m_0, m_1, \ldots, m_n$ are used here to control the shape. Obviously, $s \in Sp_2(1)$ is convex on [0, 1] if and only if

$$m_{i-1} \leqslant \tau_i \quad (i = 1, \ldots, n), \tag{3.7}$$

and monotone on [0, 1] if and only if

$$m_i \geqslant 0 \quad (i = 0, \ldots, n). \tag{3.8}$$

Thus one is led to Problems 1 and 2 with $\alpha_i = \beta_i = 1/2$. Propositions 1 and 2 yield necessary and sufficient conditions for the existence of convex and of monotone $Sp_2(1)$-splines, respectively.

The corresponding tests (2.3) and (2.6) may fail for $n \geqslant 4$. E.g. in the convex case it follows that $d_2 \leqslant d_1$, $d_2 \leqslant d_3$, $d_4 \leqslant d_3$, but there are slopes τ_1, τ_2, τ_3, τ_4 with $d_2 \leqslant d_1 < d_4 \leqslant d_3$. This implies $d_* > d^*$.

Next, three types of splines will be named by which the shape can be preserved.

3.2. Lacunary splines

Let $k_i \geqslant 2$ $(i = 1, \ldots, n)$ be given integers, and $k = (k_1, \ldots, k_n)$. Set

$$s(x) = y_{i-1} + m_{i-1} h_i t + (\tau_i - m_{i-1}) h_i t^{k_i}, \quad 0 \leqslant t \leqslant 1, \tag{3.9}$$

with $x = x_{i-1} + h_i t$ $(i = 1, \ldots, n)$. This interpolating spline is from $C^1[0, 1]$ if and only if

$$(k_i - 1) m_{i-1} + m_i = k_i \tau_i \quad (i = 1, \ldots, n-1); \tag{3.10}$$

the corresponding set of splines will be denoted by $LSp_2(1; k)$. Notice that $LSp_2(1; 2) = Sp_2(1)$. Further, $s \in LSp_2(1; k)$ is convex if (3.7) and monotone if (3.8) holds. Thus, Problems 1 and 2 with $\alpha_i = (k_i - 1)/k_i$, $\beta_i = 1/k_i$ arise.

Now, for sufficiently large $k_1, \ldots, k_n$ the necessary and sufficient conditions (2.3) and (2.6) are met if (2.2) and (2.5) are sharpened to

$$\tau_1 < \tau_2 < \ldots < \tau_n \tag{3.11}$$

and

$$\tau_1 > 0, \quad \tau_2 > 0, \quad \ldots, \quad \tau_n > 0, \tag{3.12}$$

respectively. Indeed, let e.g. for even i the parameters $k_1, \ldots, k_{i-1}$ be already chosen such that

$$d_2 < d_4 < \ldots < d_i < d_{i-1} < \ldots < d_3 < d_1.$$

Then, in view of $\alpha_i \to 1$, $\beta_i \to 0$ as $k_i \to +\infty$, the inequality

$$d_i < d_{i+1} < d_{i-1}$$

follows for sufficiently large k_i. Hence, by induction one gets $d_{2i} < d_{2i-1}$ $(i \geqslant 1)$ ensuring $d_* < d^*$, and thus the existence of convex spline interpolants $s \in LSp_2(1; k)$ follows. Analogously, the condition $e_* < e^*$ which guarantees the existence of monotone splines $s \in LSp_2(1; k)$ can be verified for large k.

3.3. Exponential splines

Let $\lambda_i > 0$ $(i = 1, \ldots, n)$ be given parameters, and $\lambda = (\lambda_1, \ldots, \lambda_n)$. Introduce a spline s by

$$s(x) = y_{i-1} + m_{i-1} h_i t + (\tau_i - m_{i-1}) h_i \frac{\cosh \mu_i t - 1}{\cosh \mu_i - 1}, \quad 0 \leqslant t \leqslant 1, \tag{3.13}$$

with $x = x_{i-1} + h_i t$, $\mu_i = \lambda_i h_i$ $(i = 1, \ldots, n)$. This spline interpolates the data set. Further, $s \in C^1[0, 1]$ is equivalent to

$$\alpha_i m_{i-1} + \beta_i m_i = \tau_i \quad (i = 1, \ldots, n-1) \tag{3.14}$$

where α_i, β_i are defined by

$$\alpha_i = 1-\beta_i, \quad \beta_i = \frac{1}{\mu_i}\tanh\frac{\mu_i}{2} \in (0, 1). \tag{3.15}$$

In [11] this set of splines is called $CSp(1; \lambda)$, and the results just described for $LSp_2(1; k)$ there are derived for $CSp(1; \lambda)$. This is possible because of $\alpha_i \to 1$, $\beta_i \to 0$ as $\lambda_i \to +\infty$.

3.4. Rational splines

Let $r_i > 0$ $(i = 1, \ldots, n)$ be given constants, and let $r = (r_1, \ldots, r_n)$. Set

$$s(x) = y_{i-1}+\tau_i h_i t+(m_{i-1}-\tau_i) h_i \frac{t(1-t)}{1+r_i t}, \quad 0 \leqslant t \leqslant 1, \tag{3.16}$$

with $x = x_{i-1}+h_i t$ $(i = 1, \ldots, n)$. This is an interpolating spline which belongs to $C^1[0, 1]$ if and only if

$$m_{i-1}+(1+r_i) m_i = (2+r_i)\tau_i \quad (i = 1, \ldots, n-1). \tag{3.17}$$

The corresponding set of splines will be denoted by $RSp_2(1; r)$. Because of

$$s'(x) = \tau_i+(m_{i-1}-\tau_i)\left(\frac{1+r_i}{r_i(1+r_i t)^2}-\frac{1}{r_i}\right),$$

$$s''(x) = -(m_{i-1}-\tau_i)\frac{2(1+r_i)}{h_i(1+r_i t)^3}$$

the convexity and monotonicity of $s \in RSp_2(1; r)$ are again controlled by (3.7) and (3.8), respectively. Hence, Problems 1 and 2 with $\alpha_i = 1/(2+r_i)$, $\beta_i = (1+r_i)/(2+r_i)$ are to be solved.

Now, because of $\alpha_i \to 0$, $\beta_i \to 1$ as $r_i \to +\infty$, under (3.11) and (3.12)

$$d_{2i} < \ldots < d_4 < d_2 < d_1 < d_3 < \ldots < d_{2i-1}$$

and an analogous relation for $e_1, \ldots, e_n$ can be attained if $r_1, \ldots, r_n$ are sufficiently large. Hence, $d_* < d^*$ and $e_* < e^*$ follow ensuring the existence of convex or monotone interpolants $s \in RSp_2(1; r)$, respectively.

4. Minimal quadratic and related splines

For $d_* < d^*$ or $e_* < e^*$ there exist an infinite number of spline interpolants. For selecting one of them a choice function is introduced. In view of Holladay's theorem (for cubic splines) it suggests itself to minimize

$$\int_0^1 s''(x)^2\, dx.$$

But it seems more appropriate to use the geometric curvature

$$\int_0^1 \frac{s''(x)^2}{(1+s'(x)^2)^3}\,dx.$$

Thus, with the approximation $s'(x) \approx \tau_i$ for $x_{i-1} \leqslant x \leqslant x_i$, one is led to the objective function

$$(4.1) \qquad f_2(s) = \sum_{i=1}^{n} w_i \int_{x_{i-1}}^{x_i} s''(x)^2\,dx \to \min!$$

with $w_i = 1$ or $w_i = 1/(1+\tau_i^2)^3$ (see [1], [11], [15]). There are also other acceptable choice functions, e.g.

$$(4.2) \qquad f_\infty(s) = \max_{i=1,\ldots,n} \sqrt{w_i} \max_{x_{i-1} \leqslant x \leqslant x_i} |s''(x)| \to \min!$$

In the spaces $CSp(1;\lambda)$ and $RSp_2(1;r)$ these functions should be somewhat modified in order to avoid piecewise linear splines which occur for large parameter vectors λ or r (see e.g. [11]).

4.1. The objective function f_2

In all cases treated up to now the function (4.1) reads

$$f_2(s) = \sum_{i=1}^{n} W_i(\tau_i - m_{i-1})^2$$

where the W_i's depend on the chosen spline space. E.g. in $LSp_2(1;k)$ one gets

$$W_i = k_i^2(k_i-1)^2/(h_i(2k_i-3)).$$

Upon using relations (2.4) and (2.7) the following one-dimensional optimization problems appear [11]: In the convex case

$$(4.3) \qquad \sum_{i=1}^{n} W_i(m_0-d_i)^2/c_{i-1}^2 \to \min! \text{ s.t. } d_* \leqslant m_0 \leqslant d^*,$$

and in the monotone case

$$(4.4) \quad \sum_{i=1}^{n} W_i\left(m_0 - e_{i-1} + (-1)^i c_{i-1}\tau_i\right)^2/c_{i-1}^2 \to \min! \text{ s.t. } e_* \leqslant m_0 \leqslant e^*.$$

Both (4.3) and (4.4) are easily solved by comparing the unconstrained minimizers with the boundary values.

4.2. The objective function f_∞

In the treated cases the function (4.2) is of the form

$$f_\infty(s) = \max_{i=1,\ldots,n} V_i|\tau_i - m_{i-1}|$$

where the V_i's, e.g. in $LSp_2(1; k)$, are

$$V_i = \sqrt{w_i}\, k_i(k_i-1)/h_i.$$

Thus in the convex case one has to solve

$$\max_{i=1,\dots,n} V_i|m_0-d_i|/c_{i-1} \to \min! \text{ s.t. } d_* \leqslant m_0 \leqslant d^*, \tag{4.5}$$

and in the monotone case

$$\max_{i=1,\dots,n} V_i|m_0-e_{i-1}+(-1)^i c_{i-1}\tau_i|/c_{i-1} \to \min! \text{ s.t. } e_* \leqslant m_0 \leqslant e^*. \tag{4.6}$$

The one-dimensional problems

$$\max_{i=1,\dots,n} U_i|m-u_i| \to \min! \text{ s.t. } u_* \leqslant m \leqslant u^* \tag{4.7}$$

can be solved in a simple way. Assume that $u_1 \leqslant u_2 \leqslant \dots \leqslant u_n$ and $U_1 > 0, \dots, U_n > 0$. Let (x_{jk}, y_{jk}) be the intersection point of the two lines

$$y = U_j(x-u_j), \quad y = -U_k(x-u_k),$$

i.e.

$$x_{jk} = \frac{u_j U_j + u_k U_k}{U_j + U_k}, \quad y_{jk} = \frac{(u_k-u_j) U_j U_k}{U_j+U_k}. \tag{4.8}$$

Then the unconstrained minimizer $\tilde{m}$ of (4.7) is given by

$$\tilde{m} = x_{\mu\nu} \quad \text{where } y_{\mu\nu} = \max_{j<k} y_{jk} \tag{4.9}$$

and the constrained minimizer $\bar{m}$ of (4.7) is

$$\bar{m} = \begin{cases} \tilde{m} & \text{for } u_* \leqslant \tilde{m} \leqslant u^*, \\ u^* & \text{for } \tilde{m} > u^*, \\ u_* & \text{for } \tilde{m} < u_*. \end{cases} \tag{4.10}$$

However, it is possible to compute $\tilde{m}$ more effectively: At the beginning, set

$$j_0 = p_0 = 1, \quad k_0 = q_0 = n, \quad z_0 = y_{1n}, \quad A_0 = \{j_0\}, \quad B_0 = \{k_0\}$$

and then for $l = 1, 2, \dots$ do:

if $J_l = \{j\colon U_j > U_{j_{l-1}},\ u_j < x_{p_{l-1}q_{l-1}}\} = \emptyset$ and
$K_l = \{j\colon U_j > U_{k_{l-1}},\ u_j > x_{p_{l-1}q_{l-1}}\} = \emptyset$ then
set $\tilde{m} = x_{p_{l-1}q_{l-1}}$, stop;
if $J_l \neq \emptyset$ set $j_l = \min J_l$, $A_l = A_{l-1} \cup \{j_l\}$ else $j_l = j_{l-1}$, $A_l = A_{l-1}$;
if $K_l \neq \emptyset$ set $k_l = \max K_l$, $B_l = B_{l-1} \cup \{k_l\}$ else $k_l = k_{l-1}$, $B_l = B_{l-1}$;
set $z_l = \max\{z_{l-1}, y_{j_l\nu}\ (\nu \in B_l,\ \nu \leqslant q_{l-1}),\ y_{\mu k_l}\ (\mu \in A_l,\ \mu \geqslant p_{l-1})\}$;
if $z_l = z_{l-1}$ set $p_l = p_{l-1}$, $q_l = q_{l-1}$;
if $z_l = y_{\mu\nu}$ set $p_l = \mu$, $q_l = \nu$.

This procedure for computing $\bar{m}$ terminates after at most $n-1$ steps. Moreover, some refinements are possible if the boundary values u_* and u^* are taken into account. E.g., in the case $u^* \leqslant u_1$ the procedure can be stopped with $\bar{m} = u^*$, and so on.

5. Shape preserving cubic and related splines

5.1. Cubic splines

Let $Sp_3(l)$ be the set of cubic C^l-splines on the grid Δ. The spline s defined by

$$(5.1) \qquad s(x) = y_{i-1} + m_{i-1} h_i t + (3\tau_i - 2m_{i-1} - m_i) h_i t^2 + (m_{i-1} + m_i - 2\tau_i) h_i t^3, \quad 0 \leqslant t \leqslant 1,$$

with $x = x_{i-1} + h_i t$ $(i = 1, \ldots, n)$ is from $Sp_3(1)$ and satisfies the interpolation condition (3.4). Moreover, $m_i = s'(x_i)$ $(i = 0, \ldots, n)$. Now, the spline (5.1) is convex on [0, 1] if and only if

$$(5.2) \qquad 2m_{i-1} + m_i \leqslant 3\tau_i, \quad m_{i-1} + 2m_i \geqslant 3\tau_i \quad (i = 1, \ldots, n)$$

(see [7]), and monotone on [0, 1] if and only if

$$(5.3) \qquad m_{i-1} - \sqrt{m_{i-1} m_i} + m_i \leqslant 3\tau_i, \quad m_{i-1} \geqslant 0, \quad m_i \geqslant 0 \quad (i = 1, \ldots, n);$$

this is an equivalent form, due to W. Burmeister, of the condition given in [3]. Since (5.3) is valid e.g. for $m_0 = m_1 = \ldots = m_n = 0$, the problem of monotone interpolation is always solvable in $Sp_3(1)$. The convex interpolation in $Sp_3(1)$ leads to Problem 3 with $\alpha_i = \delta_i = 2/3$ and $\beta_i = \gamma_i = 1/3$, and Proposition 3 gives the necessary and sufficient existence criterion developed in [10].

For characterizing the shape preserving interpolation in $Sp_3(2)$ the equalities

$$(5.4) \qquad h_{i+1} m_{i-1} + 2(h_i + h_{i+1}) m_i + h_i m_{i+1} = 3(h_i \tau_{i+1} + h_{i+1} \tau_i) \quad (i = 1, \ldots, n-1)$$

ensuring $s''(x_i+0) = s''(x_i-0)$ have to be added to (5.2) or (5.3). The arising problems (5.2), (5.4) and (5.3), (5.4) are not always solvable. Sufficient solvability conditions are given in [6] while conditions which are both necessary and sufficient are derived in paper [18].

5.2. Minimal cubic splines

If there are any at all, then in general there exist an infinite number of convex or monotone cubic spline interpolants. For selecting one of them the objective functions (4.1) and (4.2) are of interest. In $Sp_3(l)$, $l \geqslant 1$, they read

$$f_2(s) = \sum_{i=1}^{n} F_i(m_{i-1}, m_i) \tag{5.5}$$

with

$$F_i(x, y) = \frac{4w_i}{h_i}\{(x-\tau_i)^2+(x-\tau_i)(y-\tau_i)+(y-\tau_i)^2\}, \tag{5.6}$$

and

$$f_\infty(s) = \max_{i=1,\ldots,n} 2\sqrt{w_i}\max\{|3\tau_i-2m_{i-1}-m_i|, |m_{i-1}+2m_i-3\tau_i|\}. \tag{5.7}$$

Among the problems now arising only the problems in $Sp_3(1)$ with regard to the objective function f_2 are treated in details (see [1], [2]). For solving the problem

$$f_2(s) \to \min! \text{ s.t. } (5.2) \tag{5.8}$$

in [1] it is proposed to consider the corresponding dual problem

$$-\sum_{i=1}^{n} H_i^*(p_{i-1}, -p_i) \to \max! \quad \text{with } p_0 = p_n = 0 \tag{5.9}$$

where

$$H_i^*(\xi, \eta) = \begin{cases} \tau_i(\xi+\eta)+\dfrac{h_i}{12w_i}(\xi^2-\xi\eta+\eta^2) & \text{for } \xi \leqslant 0, \eta \geqslant 0, \\ \tau_i(\xi+\eta)+\dfrac{h_i}{12w_i}\left(\dfrac{\xi}{2}-\eta\right)^2 & \text{for } 0 \leqslant \xi \leqslant 2\eta, \\ \tau_i(\xi+\eta)+\dfrac{h_i}{12w_i}\left(\xi-\dfrac{\eta}{2}\right)^2 & \text{for } 2\xi \leqslant \eta \leqslant 0, \\ \tau_i(\xi+\eta) & \text{for } \xi \geqslant 2\eta, 2\xi \geqslant \eta. \end{cases} \tag{5.10}$$

In contrast to (5.8), the problem (5.9) is unconstrained and, moreover, the Hessian of the objective function is tridiagonal. These properties are very convenient for applying e.g. Newton's method. Further, if a solution of the dual problem (5.9) is known then the solution $(m_0, m_1, \ldots, m_n)$ of (5.8) is explicitly given by means of the partial derivatives of H_i^*,

$$m_{i-1} = \partial_1 H_i^*(p_{i-1}, -p_i), \quad m_i = \partial_2 H_i^*(p_{i-1}, -p_i) \quad (i = 1, \ldots, n). \tag{5.11}$$

In [1] these duality results are derived by using the Kuhn–Tucker theory while in [2] this is done by applying Fenchel's duality theory.

In numerical tests this procedure for solving (5.8), i.e. for determining optimal convex $Sp_3(1)$-interpolants, turns out to be very effective (cf. [1] and also [5], [11]). To the problem

$$(5.12) \qquad f_2(s) \to \min! \quad \text{s.t. } (5.3)$$

occurring in monotone interpolation there also corresponds a dual problem of the form (5.9). For the corresponding functions H_i^* we refer to [2].

As stated before, with cubic C^1-splines monotonicity can always be preserved, but convexity in general not. Therefore it is of interest to have some types of extended cubic splines for which convex interpolation is successful.

5.3. Lacunary splines

For given integers $k_1 \geqslant 3, \ldots, k_n \geqslant 3$ the spline $s \in LSp_3(1; k)$ defined by

$$(5.13) \qquad s(x) = y_{i-1} + m_{i-1} h_i t + (k_i \tau_i - (k_i - 1) m_{i-1} - m_i) h_i t^{k_i - 1} + ((k_i - 2) m_{i-1} + m_i - (k_i - 1) \tau_i) h_i t^{k_i}, \quad 0 \leqslant t \leqslant 1,$$

with $x = x_{i-1} + h_i t$ $(i = 1, \ldots, n)$ is continuously differentiable and satisfies the interpolation condition (3.4). According to [10], s is convex on [0, 1] if and only if

$$(5.14) \qquad (k_i - 1) m_{i-1} + m_i \leqslant k_i \tau_i, \quad (k_i - 2) m_{i-1} + 2m_i \geqslant k_i \tau_i \quad (i = 1, \ldots, n),$$

and under the assumption (3.11) these inequalities are solvable if $k_1, \ldots, k_n$ are sufficiently large.

5.4. Exponential splines

For given parameters $\lambda_1 > 0, \ldots, \lambda_n > 0$ denote by $ESp(l; \lambda)$ the set of exponential C^l-splines s which are defined by

$$(5.15) \qquad s \in \operatorname{span}\{1, x, \exp(\lambda_i x), \exp(-\lambda_i x)\}, \quad x_{i-1} \leqslant x \leqslant x_i \quad (i = 1, \ldots, n).$$

In [5] necessary and sufficient conditions of type (5.2) are derived under which $s \in ESp(1; \lambda)$ is convex, and they are shown to be satisfied for sufficiently large $\lambda_1, \ldots, \lambda_n$ if (3.11) holds. For conditions which ensure the convexity of $s \in ESp(2; \lambda)$ see [12], [9].

5.5. Rational splines

Let the spline $s \in RSp_3(1; r)$ be given by

$$(5.16) \qquad s(x) = y_{i-1} + \tau_i h_i t + h_i t(1-t) \frac{(\tau_i - m_i) t + (m_{i-1} - \tau_i)(1-t)}{1 + r_i t(1-t)}, \quad 0 \leqslant t \leqslant 1,$$

with $x = x_{i-1} + h_i t$ $(i = 1, \ldots, n)$, where $r_1 > 0, \ldots, r_n > 0$ are parameters. Indeed, s is from $C^1[0, 1]$, and the interpolation condition (3.4) is valid. Further, s is convex on $[0, 1]$ if and only if

$$(5.17) \qquad (2+r_i)m_{i-1} + m_i \leqslant (3+r_i)\tau_i, \qquad m_{i-1} + (2+r_i)m_i \geqslant (3+r_i)\tau_i \qquad (i = 1, \ldots, n).$$

For the proof notice that for $x_{i-1} \leqslant x \leqslant x_i$

$$(5.18) \qquad h_i(1+r_i t(1-t))^3 s''(x) = \varrho(t)(\tau_i - m_i) + \varrho(1-t)(m_{i-1} - \tau_i)$$

with $\varrho(t) = 2 - 6t - 2r_i t^3$. Now,

$$(5.19) \qquad \begin{aligned} h_i s''(x_{i-1}) &= \varrho(0)(\tau_i - m_i) + \varrho(1)(m_{i-1} - \tau_i) \geqslant 0, \\ h_i s''(x_i) &= \varrho(1)(\tau_i - m_i) + \varrho(0)(m_{i-1} - \tau_i) \geqslant 0 \quad (i = 1, \ldots, n) \end{aligned}$$

is equivalent to (5.17). Further, (5.19) implies $s''(x) \geqslant 0$ for $x_{i-1} \leqslant x \leqslant x_i$ $(i = 1, \ldots, n)$. Indeed, from (5.19) it follows that $\tau_i - m_i \leqslant 0$, $m_{i-1} - \tau_i \leqslant 0$. Using $\varrho(1-t) < 0$ for $0 \leqslant t \leqslant 1/2$ one gets

$$\varrho(t)(\tau_i - m_i) + \varrho(1-t)(m_{i-1} - \tau_i) \geqslant \left\{\varrho(t) - \frac{\varrho(0)}{\varrho(1)}\varrho(1-t)\right\}(\tau_i - m_i) \geqslant 0$$

since $\varrho(1)\varrho(t) - \varrho(0)\varrho(1-t) \geqslant 0$. Thus $s''(x) \geqslant 0$ for $0 \leqslant t \leqslant 1/2$. Analogously this inequality can be proved for $1/2 \leqslant t \leqslant 1$.

Therefore, the convexity of $s \in RSp_3(1; r)$ leads to Problem 3 with $\alpha_i = \delta_i = (2+r_i)/(3+r_i)$ and $\beta_i = \gamma_i = 1/(3+r_i)$. It is easily shown that the solvability test (2.9) is fulfilled for sufficiently large parameters $r_1, \ldots, r_n$ if (3.11) is assumed.

The convexity and monotonicity of splines from $RSp_3(2; r)$ are treated in [4], where sufficient conditions for these properties are given.

5.6. Quintic splines

As seen before, it is essential to have a finite set of conditions formulated in terms of $m_i = s'(x_i)$ $(i = 0, \ldots, n)$ which ensure convexity or monotonicity. For quintic twice differentiable splines $s \in Sp_5(2)$ such conditions, now in terms of $m_i = s'(x_i)$ and $M_i = s''(x_i)$ $(i = 0, \ldots, n)$, become highly nonlinear (see [20]).

6. Choice of the weights

An example is given to point out the significance of the weights $w_1, \ldots, w_n$ in the objective function (4.1). The broken line (see Fig. 1) represents the optimal monotone $Sp_3(1)$-spline interpolant belonging to the weights $w_i = 1$

while the solid line gives the one with respect to the weights $w_i = (1+\tau_i^2)^3$. It is obvious that the latter spline should be preferred (see Fig. 1). In other examples the same situation has been observed.

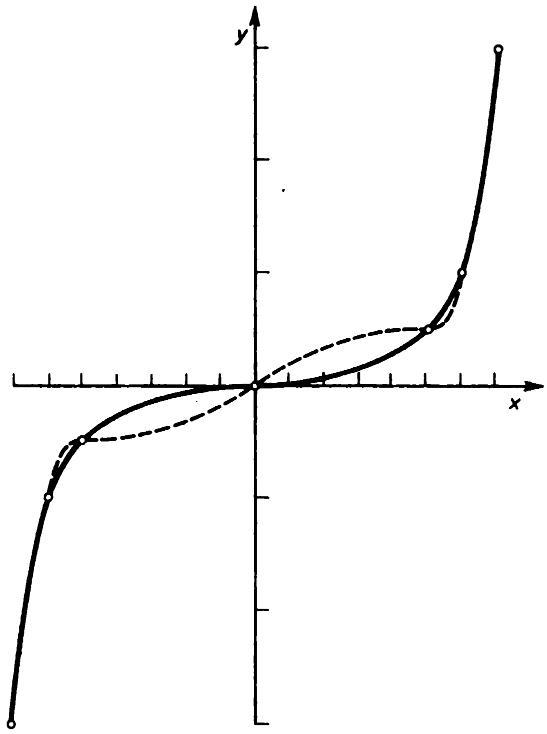

Fig. 1

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