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ONE-SIDED APPROXIMATION BY TRIGONOMETRIC POLYNOMIALS IN L_p-NORM, $0 < p < 1$

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Let T_n be the space of trigonometric polynomials of degree at most n, $n = 1, 2, 3, \ldots$ We consider 2π-periodic functions.

DEFINITION 1. The *best one-sided approximation* of a fixed function $f \in L_p(0, 2\pi)$ in $L_p(0, 2\pi)$ norm by T_n is given by

$$(1)\qquad \tilde{E}_n^T(f)_p = \inf \{ (\int_0^{2\pi} |t_1(x) - t_2(x)|^p dx)^{1/p};\ t_1(x) \geqslant f(x) \geqslant t_2(x);\ t_1, t_2 \in T_n \}.$$

DEFINITION 2. Let f be a 2π-periodic bounded measurable function. The *p-th mean modulus of smoothness of order k* (k a positive integer) with parameter $\delta > 0$ for f is the expression

$$(2)\qquad \tau_k(f, \delta)_p = (\int_0^{2\pi} |\omega_k(f, x, \delta)|^p dx)^{1/p}$$

where

$$(3)\qquad \omega_k(f, x, \delta) = \sup(|\Delta_h^k f(t)|;\ t, t+kh \in [x - k\delta/2, x + k\delta/2]).$$

For the properties of this modulus in the case $p \geqslant 1$ see [6]. For this modulus the following results connected with $\tilde{E}_n^T(f)_p$ have been obtained in the case $p \geqslant 1$:

(a) Jackson type direct theorem – see [7],
(b) Stechkin type direct theorem – see [4],
(c) Salem–Stechkin type inverse theorem – see [5].

The purpose of this paper is to obtain direct and inverse theorems for $\tilde{E}_n^T(f)_p$ in the case $0 < p < 1$, using the modulus $\tau_k(f, \delta)_p$.

First we prove a lemma concerning the one-sided approximation of the

2π-periodic function

$$B(x) = (\pi - x)/2, \quad x \in [0, 2\pi).$$

B appears in the integral representation of a given function together with its derivatives. In this paper $c(p)$, $c_1(p)$, $c_1(k, p)$, $c'(k, p)$, ... are constants depending on p and k.

LEMMA 1. *The following estimate holds*:

$$\tilde{E}_n^T(B)_p \leqslant c(p)\tau_2(B, n^{-1})_p = c_1(p)n^{-1/p}.$$

Proof. We consider the function

$$\bar{B}(x) = (\pi - x)/2, \quad x \in [0, 2\pi] \tag{4}$$

and the following interpolation problems:

(a) Find a trigonometric polynomial t_1 with the following interpolation conditions ($x_k = 2k\pi/n$, $k = 0, 1, 2, \ldots, n-1$):

$$t_1(x_0) = \bar{B}(x_0),$$

$$t_1(x_1) = \bar{B}(x_1), t_1'(x_1) = \bar{B}'(x_1), \ldots, t_1^{(2\{1/p\}-1)}(x_1) = \bar{B}^{(2\{1/p\}-1)}(x_1),$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$t_1(x_{n-2}) = \bar{B}(x_{n-2}), t_1'(x_{n-2}) = \bar{B}'(x_{n-2}), \ldots, t_1^{(2\{1/p\}-1)}(x_{n-2}) = \bar{B}^{(2\{1/p\}-1)}(x_{n-2}),$$

$$t_1(x_{n-1}) = \bar{B}(x_{n-1}), t_1'(x_{n-1}) = \bar{B}'(x_{n-1}), \ldots, t_1^{(2\{1/p\}-1)}(x_{n-1}) = \bar{B}^{(2\{1/p\}-1)}(x_{n-1}).$$

Here we have $2\{1/p\}(n-1)+1$ conditions and t_1 is of degree $\{1/p\}(n-1)$ where $\{1/p\} = \min$ (z an integer, $z \geqslant 1/p$).

(b) Find a trigonometric polynomial t_2 with the following interpolation conditions ($x_k = 2k\pi/n$, $k = 1, 2, \ldots, n$):

$$t_2(x_1) = \bar{B}(x_1), t_2'(x_1) = \bar{B}'(x_1), \ldots, t_2^{(2\{1/p\}-1)}(x_1) = \bar{B}^{(2\{1/p\}-1)}(x_1),$$

$$t_2(x_2) = \bar{B}(x_2), t_2'(x_2) = \bar{B}'(x_2), \ldots, t_2^{(2\{1/p\}-1)}(x_2) = \bar{B}^{(2\{1/p\}-1)}(x_2),$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$t_2(x_{n-1}) = \bar{B}(x_{n-1}), t_2'(x_{n-1}) = \bar{B}'(x_{n-1}), \ldots, t_2^{(2\{1/p\}-1)}(x_{n-1}) = \bar{B}^{(2\{1/p\}-1)}(x_{n-1}),$$

$$t_2(x_n) = \bar{B}(x_n).$$

Analogously with $2\{1/p\}(n-1)+1$ conditions t_2 is of degree $\{1/p\}(n-1)$.

If we assume that t_1 crosses the graph of B then $t_1 - B$ will have $2\{1/p\}(n-1)+2$ zeros (counting multiplicities) in the interval $[0, 2\pi)$. Then the trigonometric polynomial

$$[t_1(x)-B(x)]' = t_1'(x)-\tfrac{1}{2} \tag{5}$$

is of degree $\{1/p\}(n-1)$ but has $2\{1/p\}(n-1)+1$ zeros in the interval $(0, 2\pi)$. This is a contradiction. It follows that $t_1(x) \geqslant B(x)$.

Analogously, $t_2(x) \leqslant B(x)$.

Let now

$$T(x) = t_1(x)-t_2(x). \tag{6}$$

This trigonometric polynomial is uniquely determined by the following interpolation conditions ($x_k = 2k\pi/n$, $k = 0, 1, 2, \ldots, n$):

$$T(x_0) = \pi,$$

$$T(x_1) = 0,\ T'(x_1) = 0, \ldots, T^{(2\{1/p\}-1)}(x_1) = 0,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$T(x_{n-1}) = 0,\ T'(x_{n-1}) = 0, \ldots, T^{(2\{1/p\}-1)}(x_{n-1}) = 0.$$

We can write it explicitly:

$$T(x) = \pi\left(\frac{\sin(nx/2)}{n\sin(x/2)}\right)^{2\{1/p\}} \tag{7}$$

On the other hand,

$$\begin{aligned}\int_0^{2\pi} |T(x)|^p\,dx &= \pi^p \int_0^{2\pi}\left(\frac{\sin(nx/2)}{n\sin(x/2)}\right)^{2\{1/p\}p} dx \\ &= 2\pi^p\int_0^{\pi}\left(\frac{\sin(nx/2)}{n\sin(x/2)}\right)^{2p\{1/p\}} dx \\ &\leqslant 2\pi^p\left(\int_0^{\pi/n} dx + \int_{\pi/n}^{\pi}\frac{1}{(nx/\pi)^{2p\{1/p\}}}\,dx\right) \\ &\leqslant \frac{4\pi^{p+1}}{n}\end{aligned} \tag{8}$$

and this ends the proof.

THEOREM 1. *Let $f^{(k-1)}$ be absolutely continuous and $f^{(k)}$ be bounded and measurable. Then the following estimate holds:*

$$\tilde{E}_n^T(f)_p \leqslant [c(p)]^k \frac{\tilde{E}_n^T(f^{(k)})_p}{n^k}.$$

(If $f^{(k)}$ is unbounded then $\tilde{E}_n^T(f^{(k)})_p = \infty$.)

Proof. If $k \geqslant 1$ then the following integral representation holds:

$$f(x) = \frac{1}{2\pi}\int_0^{2\pi} f(t)\,dt + \frac{1}{\pi}\int_0^{2\pi} B(x-t) f'(t)\,dt.$$

Let $Q_n(t) \geqslant f'(t) \geqslant q_n(t)$, $Q_n \in T_n$, $q_n \in T_n$,

$$\tilde{E}_n^T(f')_p = \Big(\int_0^{2\pi} |Q_n(t)-q_n(t)|^p\,dt\Big)^{1/p}.$$

On the other hand, $t_1(t) \geqslant B(t) \geqslant t_2(t)$, $t_1 \in T_n$, $t_2 \in T_n$,

$$\tilde{E}_n^T(B) \leqslant \Big(\int_0^{2\pi} |t_1(t)-t_2(t)|^p\,dt\Big)^{1/p} \leqslant c_1(p)\,n^{-1/p} \quad \text{(see Lemma 1).}$$

First we prove the estimate for $k = 1$; then the estimate for all positive integers k follows easily.

The following inequality is true (see [3] for $r_n \in T_n$, $0 < p_2 \leqslant p_1 \leqslant \infty$):

$$\Big(\int_0^{2\pi} |r_n(t)|^{p_1}\,dt\Big)^{1/p_1} \leqslant c(p_1, p_2)\,n^{1/p_2 - 1/p_1}\Big(\int_0^{2\pi} |r_n(t)|^{p_2}\,dt\Big)^{1/p_2}. \tag{10}$$

Let us define

$$\begin{aligned} P_n^-(x) &= \frac{1}{\pi}\int_0^{2\pi} B(x-t)\,q_n(t)\,dt,\\ t_n^-(x) &= \frac{1}{\pi}\int_0^{2\pi} t_2(x-t)\,[f'(t)-q_n(t)]\,dt + \frac{1}{2\pi}\int_0^{2\pi} f(t)\,dt;\\ &\qquad\qquad P_n^- \in T_n,\ t_n^- \in T_n,\\ P_n^+(x) &= \frac{1}{\pi}\int_0^{2\pi} B(x-t)\,Q_n(t)\,dt,\\ t_n^+(x) &= -\frac{1}{\pi}\int_0^{2\pi} t_2(x-t)\,[Q_n(t)-f'(t)]\,dt + \frac{1}{2\pi}\int_0^{2\pi} f(t)\,dt;\\ &\qquad\qquad P_n^+ \in T_n,\ t_n^+ \in T_n. \end{aligned} \tag{11}$$

Obviously $P_n^- \in T_n$, $t_n^- \in T_n$, $P_n^+ \in T_n$ and $t_n^+ \in T_n$. We have (see [1])

$$f(x) - [P_n^-(x) + t_n^-(x)] = \frac{1}{\pi}\int_0^{2\pi} [B(x-t)-t_2(x-t)]\,[f'(t)-q_n(t)]\,dt \geqslant 0,$$

$$[P_n^+(x)+t_n^+(x)] - f(x) = \frac{1}{\pi}\int_0^{2\pi} [B(x-t)-t_2(x-t)]\,[Q_n(t)-f'(t)]\,dt \geqslant 0. \tag{12}$$

From (10) and (12) it follows that

$$(13)\ \tilde{E}_n^T(f)_p \leqslant \Big(\int_0^{2\pi} [(P_n^+ + t_n^+)(t) - (P_n^- + t_n^-)(t)]^p\, dt\Big)^{1/p}$$
$$= \frac{1}{\pi}\Big(\int_0^{2\pi} \Big|\int_0^{2\pi} [B(x-t) - t_2(x-t)]\,[Q_n(t) - q_n(t)]\, dt\Big|^p dx\Big)^{1/p}$$
$$\leqslant \frac{1}{\pi}\Big(\int_0^{2\pi} \Big|\int_0^{2\pi} [t_1(x-t) - t_2(x-t)]\,[Q_n(t) - q_n(t)]\, dt\Big|^p dx\Big)^{1/p}$$
$$\leqslant c_1(p)\, n^{1/p-1}\Big(\int_0^{2\pi}\int_0^{2\pi} [t_1(x-t) - t_2(x-t)]\,[Q_n(t) - q_n(t)]^p\, dt\, dx\Big)^{1/p}$$
$$\leqslant c(p)\frac{n^{1/p-1}}{n^{1/p}}\tilde{E}_n^T(f')_p = \frac{c(p)}{n}\tilde{E}_n^T(f')_p.$$

By induction,

$$(14)\qquad \tilde{E}_n^T(f)_p \leqslant \frac{c(p)}{n}\tilde{E}_n^T(f')_p \leqslant \frac{[c(p)]^2}{n^2}\tilde{E}_n^T(f'')_p \leqslant \ldots \leqslant \frac{[c(p)]^k}{n^k}\tilde{E}_n^T(f^{(k)})_p,$$

which ends the proof.

Let us formulate a lemma which is used in the proof of a Jackson type theorem.

LEMMA 2. *Let g be a piecewise linear function with points of breaking linearity $2\pi k/n$, $k = 0, 1, 2, \ldots, n$. If g is absolutely continuous then*

$$\tilde{E}_n^T(g)_p \leqslant c(p)\,\tau_1(g, n^{-1})_p.$$

Proof. The proof is based on an integral representation for 2π-periodic functions (see [1]) and on the inequality

$$(15)\qquad \Big(\sum x_i^{p_1}\Big)^{1/p_1} \leqslant \Big(\sum x_i^{p_2}\Big)^{1/p_2}, \quad 0 < p_2 \leqslant p_1,\ x_i \geqslant 0.$$

If $g'(x) = c_k$, $x \in (2\pi k/n, 2\pi(k+1)/n)$, $k = 0, 1, \ldots, n-1$, then

$$(16)\qquad g(x) = \frac{1}{2\pi}\int_0^{2\pi} g(t)\,dt + \sum_{k=0}^{n-1}\frac{c_k}{\pi}\int_{2\pi k/n}^{2\pi(k+1)/n} B(x-t)\,dt.$$

On the other hand, we construct the polynomials (for the definition of t_1 and t_2 see Lemma 1)

$$
\begin{aligned}
(17)\qquad T_g(x) &= \frac{1}{2\pi}\int_0^{2\pi} g(t)\,dt \\
&\quad + \sum_{k=0}^{n-1} \frac{c_k}{2\pi} \int_{2\pi k/n}^{2\pi(k+1)/n} [(\operatorname{sign}(c_k)+1)\,t_1(x-t) \\
&\quad -(\operatorname{sign}(c_k)-1)\,t_2(x-t)]\,dt \\
&\geqslant g(x) \geqslant t_g(x) \\
&= \frac{1}{2\pi}\int_0^{2\pi} g(t)\,dt \\
&\quad + \sum_{k=0}^{n-1} \frac{c_k}{2\pi} \int_{2k\pi/n}^{2(k+1)\pi/n} [(\operatorname{sign}(c_k)+1)\,t_2(x-t) \\
&\quad -(\operatorname{sign}(c_k)-1)\,t_1(x-t)]\,dt.
\end{aligned}
$$

Using (15), (16), (17) one obtains

$$
\begin{aligned}
(18)\qquad \tilde{E}_n^T(g)_p &\leqslant \Big(\int_0^{2\pi} |T_g(x)-t_g(x)|^p\,dx\Big)^{1/p} \\
&= \int_0^{2\pi} \left| \sum_{k=0}^{n-1} \frac{|c_k|}{\pi} \int_{2k\pi/n}^{2(k+1)\pi/n} \big(t_1(x-t)-t_2(x-t)\big)\,dt \right|^p dx \\
&\leqslant \left(\sum_{k=0}^{n-1} \frac{|c_k|^p}{\pi^p} \int_0^{2\pi} \Big| \int_{2k\pi/n}^{2(k+1)\pi/n} \big(t_1(x-t)-t_2(x-t)\big)\,dt\Big|^p dx \right)^{1/p}
\end{aligned}
$$

On the other hand, using Taylor's expansion and Bernstein's inequality one gets

$$
\begin{aligned}
(19)\qquad &\int_{2k\pi/n}^{2(k+1)\pi/n} |t_1(x-t)-t_2(x-t)|\,dt \\
&= \int_0^{2\pi/n} \left| t_1\left(x-t-\frac{2k\pi}{n}\right) - t_2\left(x-t-\frac{2k\pi}{n}\right)\right| dt \\
&= \int_0^{2\pi/n} \sum_{l=0}^{\infty} (t_1-t_2)^{(l)}\left(x-\frac{2k\pi}{n}\right)\frac{(-t)^l}{l!}\,dt \\
&\leqslant \sum_{l=0}^{\infty} \left(\frac{2\pi}{n}\right)^{l+1} \frac{1}{(l+1)!} \left|(t_1-t_2)^{(l)}\left(x-\frac{2k\pi}{n}\right)\right|,
\end{aligned}
$$

whence (see (15))

$$\int_0^{2\pi} \Big| \int_{2\pi k/n}^{2\pi(k\pm 1)/n} \big(t_1(x-t)-t_2(x-t)\big)dt\Big|^p dx \tag{20}$$
$$\leqslant \sum_{l=0}^{\infty}\left(\frac{2\pi}{n}\right)^{(l+1)p}\frac{1}{[(l+1)!]^p}\int_0^{2\pi}|(t_1-t_2)^{(l)}(x)|^p dx$$
$$\leqslant \left(\sum_{l=0}^{\infty}\frac{(2\pi)^{(l+1)p}}{n^{(l+1)p}}n^{lp}\frac{1}{[(l+1)!]^p}\right)c_1(p)\int_0^{2\pi}|(t_1-t_2)(x)|^p dx$$
$$=\frac{c'(p)}{n^p}\int_0^{2\pi}|(t_1-t_2)(x)|^p dx.$$

Finally, by (18), (20) and Lemma 1, we get

$$\tilde{E}_n^T(g)_p \leqslant c''(p)\Big(\int_0^{2\pi}(t_1(x)-t_2(x))^p dx\Big)^{1/p}\left(\sum_{k=0}^{n-1}|c_k|^p\frac{1}{n^p}\right)^{1/p} \tag{21}$$
$$\leqslant c'''(p)\left(\sum_{k=0}^{n-1}|c_k|^p\frac{1}{n^p}\frac{1}{n}\right)^{1/p} \leqslant c(p)\tau_1(g, n^{-1})_p$$

(for Bernstein's inequality in the case $0 < p < 1$, see [2]), and this ends the proof.

Let us note that L_p, $0 < p < 1$, is a metric space with the metric

$$\varrho(f, g) = \int_0^{2\pi}|f(x)-g(x)|^p dx.$$

Let us formulate a direct theorem of Jackson's type for $\tilde{E}_n^T(f)_p$:

THEOREM 2. *If $f \in L_p[0, 2\pi]$ then the following estimate holds:*

$$\tilde{E}_n^T(f)_p \leqslant c(p)\tau_1(f, n^{-1})_p.$$

Proof is based on Lemma 2 and a method developed in [7]. One can construct functions $g_n^+(x)$ and $g_n^-(x)$ (analogously to [7]) such that

$$\begin{gathered} g_n^+(x) \geqslant f(x) \geqslant g_n^-(x),\\ |f(x)-g_n^+(x)| \leqslant \omega_1(f, x, 8\pi/n),\\ |f(x)-g_n^-(x)| \leqslant \omega_1(f, x, 8\pi/n),\\ \omega_1(g_n^+, x, 2\pi/n) \leqslant \omega_1(f, x, 12\pi/n),\\ \omega_1(g_n^-, x, 2\pi/n) \leqslant \omega_1(f, x, 12\pi/n), \end{gathered} \tag{22}$$

and g_n^+, g_n^- are absolutely continuous piecewise linear functions with points of breaking linearity $2k\pi/n$, $k = 0, 1, 2, \ldots, n$ (see [7]).

According to Lemma 2 we have trigonometric polynomials

$$T_{g_n^+}(x) \geq g_n^+(x) \geq f(x) \geq g_n^-(x) \geq t_{g_n^-}(x)$$

such that

$$\tilde{E}_n^T(f)_p^p \leq \int_0^{2\pi} (T_{g_n^+}(x) - t_{g_n^-}(x))^p\, dx \tag{23}$$
$$\leq \int_0^{2\pi} (T_{g_n^+}(x) - g_n^+(x))^p\, dx + \int_0^{2\pi} (g_n^+(x) - f(x))^p\, dx$$
$$+ \int_0^{2\pi} (f(x) - g_n^-(x))^p\, dx + \int_0^{2\pi} (g_n^-(x) - t_{g_n^-}(x))^p\, dx$$
$$\leq c_1(p)[\tau_1^p(g_n^+, n^{-1})_p + \tau_1^p(f, n^{-1})_p + \tau_1^p(g_n^-, n^{-1})_p]$$
$$\leq c(p)\tau_1^p(f, n^{-1})_p,$$

where we have used the trivial inequality $\tau_1(f, \lambda\delta)_p \leq c(\lambda, p)\tau_1(f, \delta)_p$ for $\lambda > 0$, $\delta > 0$. This ends the proof.

We now prove the following theorem of Stechkin type:

THEOREM 3. *If* $f \in L_p[0, 2\pi]$, *then*

$$\tilde{E}_n^T(f)_p \leq c(k, p)\tau_k(f, n^{-1})_p.$$

Proof. Consider the modified Steklov functions

$$f_{k,h}(x) = h^{-k}\int_0^h \dots \int_0^h [\Delta^k_{(t_1+\dots+t_k)/k} f(x) + f(x)]\, dt_1\, dt_2 \dots dt_k. \tag{24}$$

The following properties hold:

(a) $|f_{k,h}(x) - f(x)| \leq \omega_k(f, x, h)$,

(b) $\tilde{E}_n^T(f_{k,h})_p \leq c(k, p)\dfrac{\tilde{E}_n^T(f_{k,h}^{(k)})_p}{n^k}$

$$\leq c_1(k, p)(hn)^{-k} \sum_{i=1}^{k} \tau_1(\Delta^k_{ih/k} f, n^{-1})_p \tag{25}$$
$$\leq c_2(k, p)\tau_k(f, n^{-1})_p \text{ if } h = n^{-1},$$

(c) $[\tilde{E}_n^T(f)_p]^p \leq [\tilde{E}_n^T(\omega_k(f, \bullet, h))_p]^p$
$$+ 2[\tilde{E}_n^T(f_{k,h})_p]^p + 2[\tau_k(f, h)_p]^p.$$

(a) trivial.

(b) follows from Theorem 1 and the fact that

$$(26)\quad \begin{aligned}&\omega_1(\Delta^k_{ih/k}f, x, 1/n)\\ &= \sup_{t_1,t_2\in[x-1/2n,\, x+1/2n]} |\Delta^k_{ih/k}f(t_1)-\Delta^k_{ih/k}f(t_2)|\\ &\leqslant 2 \sup_{z,z+ky\in[x-1/2n-ih,\, x+1/2n+ih]} |\Delta^k_y f(z)|\\ &\leqslant 2\omega_k(f, x, 1/kn+2ih/k).\end{aligned}$$

On the other hand,

$$(27)\qquad \tau_k(f, lh)_p \leqslant c(k, l, p)\tau_k(f, h)_p, \qquad l>0,\ h>0,$$

and the proof is analogous to that in the case $p \geqslant 1$ (see [6]).

(c) follows as in the case $p \geqslant 1$, using the fact that $L_p[0, 2\pi]$ is a metric space.

Substituting $h = n^{-1}$, we now obtain from (25)–(27)

$$(28)\quad \begin{aligned}&[\tilde{E}^T_n(f)_p]^p \leqslant c_1^p(k, p)\left[\tau_1\left(\omega_k(f, \bullet, 1/n), 1/n\right)_p\right]^p + [\tau_k(f, 1/n)_p]^p,\\ &\tilde{E}^T_n(f)_p \leqslant c(k, p)\left[\tau_1\left(\omega_k(f, \bullet, 1/n), 1/n\right)_p + \tau_k(f, 1/n)_p\right].\end{aligned}$$

But we have

$$(29)\qquad \tau_1\left(\omega_k(f, \bullet, 1/n), 1/n\right)_p \leqslant c(k, p)\tau_k(f, 1/n)_p$$

as in the case $p \geqslant 1$ (see [4]), and this ends the proof.

Now we shall consider the inverse theorem. It is based on Bernstein's inequality (see [2]) and the following lemma:

LEMMA 3. *Let t_n be a trigonometric polynomial of degree n and suppose $0 < n\delta \leqslant 1$. Then the following estimate holds*:

$$\tau_k(t_n, \delta)_p \leqslant c(p)\,\delta^k\,\|t_n^{(k)}\|_p.$$

Proof. We have

$$(30)\qquad \Delta^k_h t_n(t) = \int_0^h\int_0^h\cdots\int_0^h t_n^{(k)}(t+v_1+\ldots+v_k)\,dv_1\ldots dv_k,$$

whence

$$(31)\quad \begin{aligned}\omega_k(t_n, x, \delta) &\leqslant \overbrace{\int_0^\delta\int_0^\delta\cdots\int_0^\delta}^{k-1}\int_{-k\delta/2}^{k\delta/2} |t_n^{(k)}(x+v_1+v_2+\ldots+v_k)|\,dv_1\,dv_2\ldots dv_k\\ &\leqslant \int_0^\delta\cdots\int_0^\delta\int_{-k\delta/2}^{k\delta/2}\sum_{l_1=0}^{\infty}\sum_{l_2=0}^{\infty}\cdots\sum_{l_k=0}^{\infty}|t_n^{(k+l_1+\ldots+l_k)}(x)|\\ &\quad\times\frac{v_1^{l_1}v_2^{l_2}\ldots v_{k-1}^{l_{k-1}}|v_k|^{l_k}}{l_1!\,l_2!\ldots l_{k-1}!\,l_k!}\,dv_1\ldots dv_k\end{aligned}$$

$$\leqslant \sum_{l_1=0}^{\infty} \cdots \sum_{l_k=0}^{\infty} |t_n^{(k+l_1+\dots+l_k)}(x)| \left(\frac{k}{2}\right)^{l_k+1}$$
$$\times \frac{\delta^{l_1+1}\delta^{l_2+1}\dots\delta^{l_k+1}}{(l_1+1)!(l_2+1)!\dots(l_k+1)!}.$$

Taking the L_p norm in (31) and (15) and using Bernstein's inequality (see [2]) in the case $0<p<1$ we get

$$\text{(32)} \qquad [\tau_k(t_n, \delta)_p]^p \leqslant \sum_{l_1=0}^{\infty} \cdots \sum_{l_k=0}^{\infty} \int_0^{2\pi} |t_n^{(k+l_1+\dots+l_k)}(x)|^p dx$$
$$\times \left(\frac{k}{2}\right)^{(l_k+1)p} \frac{\delta^{(l_1+1)p}\delta^{(l_2+1)p}\dots\delta^{(l_k+1)p}}{[(l_1+1)!(l_2+1)!\dots(l_k+1)!]^p}$$
$$\leqslant c^p(k, p)\delta^{kp}\int_0^{2\pi} |t_n^{(k)}(x)|^p dx.$$

Using a method developed by V. Popov we obtain the following inverse theorem:

THEOREM 4. *For $f \in L_p[0, 2\pi]$ we have the following estimate*:

$$\tau_k(f, n^{-1})_p \leqslant \frac{c(k, p)}{n^k}\Big(\sum_{s=0}^{n} (s+1)^{kp-1}[\tilde{E}_s^T(f)_p]^p\Big)^{1/p}.$$

Proof. Just as in the case $p \geqslant 1$ we get for $n = 2^{s_0}$ $(\delta > 0)$

$$\text{(33)} \qquad \tau_k(f, \delta)_p^p \leqslant 4^p\delta^{kp}\sum_{i=1}^{s_0} 2^{ikp}[\tilde{E}_{2^i-1}^T(f)_p]^p$$
$$+[E_0^T(f)_p]^p + c(k, p)[(\delta n)^p+1][E_n^T(f)_p]^p.$$

On the other hand, there exists a constant $c(k, p)$ such that

$$\text{(34)} \qquad 2^{ikp} \leqslant c(k, p)\sum_{s=2^{i-1}+1}^{2^i} (s+1)^{kp-1}, \quad i = 1, 2, \dots, s_0.$$

Now as in the case $p \geqslant 1$ (see [5]), substituting $\delta = n^{-1}$ in (33) and using (34) we get the assertion of Theorem 4.

From Theorem 3 and Theorem 4 the following characterization of $\tilde{E}_n^T(f)_p$ by the average modulus of smoothness $\tau_k(f, \delta)_p$ is obtained:

THEOREM 5. *Let $f \in L_p[0, 2\pi]$. Then*

$$\tilde{E}_n^T(f)_p = O(n^{-\alpha}) \Leftrightarrow \tau_k(f, \delta)_p = O(\delta^\alpha), \quad k > \alpha.$$

Proof. Let $\tilde{E}_n^T(f)_p = O(n^{-\alpha})$. Then from Theorem 4 it follows that

$$\tau_k(f, n^{-1})_p \leqslant \frac{c(k, p)}{n^k}\Big(\sum_{s=1}^{n} (s+1)^{kp-1} s^{-\alpha p}\Big)^{1/p} + O(n^{-k}) \tag{35}$$

$$\leqslant \frac{c_1(k, p)}{n^k}\Big(\int_0^{n+1} v^{(k-\alpha)p-1}\, dv\Big)^{1/p} + O(n^{-k})$$

$$= \frac{c_2(k, p)}{n^k} n^{k-\alpha} + O(n^{-k}) = O(n^{-\alpha}).$$

Now let $\tau_k(f, \delta)_p = O(\delta^\alpha)$. Then from Theorem 3 it follows that

$$\tilde{E}_n^T(f)_p = O(n^{-\alpha}), \quad n \to \infty. \tag{36}$$

Note. After this paper was written the author received a preprint of R. Taberski's paper [9], where independently of the present paper and using another method Theorem 4 in the case $k = 1$ and Theorem 2 are proved, but Taberski's method is inapplicable in the case $k > 1$.

For example, from Theorem 5 one may deduce that the optimal order of one-sided approximation of B is $O(n^{-1/p})$.

Now we will only sketch how the one-sided approximation can be applied in the classical case. We give another proof of the following theorem:

Theorem (see [1]). *If $f \in L_p$ then the following estimate holds:*

$$E_n^T(f)_p \leqslant c(p)\,\omega_1(f, n^{-1})_p.$$

Proof. Using Whitney's theorem we can find a piecewise constant function with points of breaking polynomiality $2k\pi/n$, $k = 0, 1, \ldots, n-1$, such that

$$\Big(\int_0^{2\pi} |f(x) - S_n(x)|^p dx\Big)^{1/p} \leqslant c(p)\,\omega_1(f, n^{-1})_p.$$

By Theorem 2 we can find $S_n^+(x) \geqslant S_n(x)$, $S_n^+ \in T_n$ and $S_n^-(x) \leqslant S_n(x)$, $S_n^- \in T_n$ such that

$$\Big(\int_0^{2\pi} (S_n^+(x) - S_n(x))^p dx\Big)^{1/p} \leqslant c_1(p)\,\tau_1(S_n, n^{-1})_p$$

$$\leqslant c(p)\,\omega_1(S_n, n^{-1})_p;$$

$$\Big(\int_0^{2\pi} (S_n(x) - S_n^-(x))^p dx\Big)^{1/p} \ll \omega_1(S_n, n^{-1})_p.$$

On the other hand,

$$\omega_1^p(S_n, n^{-1})_p \leqslant \omega_1^p(f, n^{-1})_p + 2\int_0^{2\pi} |S_n(x) - f(x)|^p dx$$
$$\leqslant c(p)\,\omega_1^p(f, n^{-1})_p,$$
$$\int_0^{2\pi} |f(x) - S_n^+(x)|^p dx \leqslant \int_0^{2\pi} |f(x) - S_n(x)|^p dx + \int_0^{2\pi} |S_n(x) - S_n^+(x)|^p dx$$
$$\leqslant c^p(p)\,\omega_1^p(f, n^{-1})_p.$$

It is obvious that this method can be used for obtaining the classical Stechkin type theorem for approximation by trigonometric polynomials in the case $0 < p < 1$. Another proof of this theorem can be found in [8].

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