DISTRIBUTIONS AT INFINITY FOR RIEMANN SURFACES

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It is possible to view Riemann surfaces (with negative Euler characteristic) from several different viewpoints. Firstly, there is a classical algebraic approach viewing them as complex algebraic curves. Alternatively, it is possible to study them geometrically through, for example, the length spectra of closed geodesics or the spectrum of the Laplace–Beltrami operator. A third viewpoint is to extrapolate to infinity (more specifically the Furstenberg boundary of the Poincaré disc – namely the unit circle) and to study the dynamical behaviour there.

Our principal aim in this note is to clarify the relationship between the geometric and dynamical aspects of the Riemann surface. Corresponding to the spectrum of the Laplacian are certain distributions on the unit circle which are distinguished by the way they transform under the extended action of the covering group. Their prototype is the well-known Patterson measure. (This would correspond to the zero eigenvalue of the Laplacian.)

Central to the construction of these distributions is a Perron–Frobenius-type operator acting on a special class of functions on the unit circle. This establishes a close relationship between the spectra of the Laplacian and of the Perron–Frobenius operators. This correspondence was already suggested by the position of poles for the zeta function (using either the Lie-group orientated trace formula approach or the dynamical approach initiated by Ruelle).

The rest of this paper is dedicated to an investigation of this correspondence.

This note has benefited from conversations with Caroline Series and Terry Lyons.

§ 1. The Patterson measure

In this section we recall the definitions and basic facts for compact Fuchsian groups and the associated measure on the unit circle.

Let $D = \{z \in \mathbb{C} \mid |z| < 1\}$ be the open unit disc then this becomes a standard model for hyperbolic space when equipped with the Poincaré metric $ds^2 = \frac{dx^2 + dy^2}{(1 - (x^2 + y^2))^2}$. For this model the isometries of D have a simple form: a map $g: D \rightarrow D$ is an isometry for the Poincaré metric if and only if we can express g as $g(z) = (az + b)/\bar{b}z + \bar{a}$ where $|a|^2 - |b|^2 = 1$, $a, b \in \mathbb{C}$. (Here we are only referring to orientation preserving isometries.)

In this model the unit circle S^1 plays an important role. The ratio of hyperbolic to euclidean distance is given by $(1 - (x^2 + y^2))^{-2}$ which blows up to infinity when $(x, y) \in D$ approaches S^1 (in the euclidean sense). Thus S^1 lies "at infinity" in the hyperbolic metric. However, the isometries described above extend naturally to S^1 to give an action $g: S^1 \rightarrow S^1$. (In the more general theory of semi-simple Lie groups S^1 would correspond to the Furstenberg-Satake compactification.)

Let Γ be a finitely generated discrete subgroup of isometries of D . The group Γ is a *Fuchsian* group. We shall also assume that the quotient D/Γ is compact.

S. J. Patterson showed that there is a natural probability measure μ on S^1 which is equivalent to Lebesgue measure and characterised by its transformation under the action of Γ on S^1 .

$$g^* \mu = |g'| \mu, \quad \text{for all } g \in \Gamma. \quad (\text{cf. [5], [12]}).$$

(We interpret this as $\int h \circ g d\mu = \int h |g'| d\mu$ for all $h \in C(S^1)$.)

There is an explicit construction of the Patterson measure which captures some of its geometric flavour. For any point $x \in D$ we have associated to Γ a function of a single complex variable

$$g_x(s) = \sum_{g \in \Gamma} \exp -sd(x, gx) \quad (s \in \mathbb{C})$$

called the *Poincaré series*. As a Dirichlet series $g_x(s)$ can be shown to have an abscissa of convergence of $\text{Re}(s) = 1$. For any $1 < s$ we can define a probability measure μ_s^x on the closed unit disc $\bar{D}$ by

$$\int h d\mu_s^x = \frac{\sum_{g \in \Gamma} \exp -sd(x, gx) \cdot h(x)}{\sum_{g \in \Gamma} \exp -sd(x, gx)}.$$

On taking the limit as $s \rightarrow 1$ the measures μ_s^x converge in the weak* topology to the Patterson measure μ [5] (notice that the limit is independent

of the choice of $x \in D$ and has support on S^1). Thus we can interpret that the density of μ on S^1 (compared with Lebesgue measure) reflects the accumulation of images of the point x under Γ .

Remark. There is an interesting connection between the cohomology of the Riemann surface and distributions at infinity. By Hodge theory any element of the cohomology group $H^1(S, \mathbb{C})$ has a unique realisation as a harmonic 1-form, which in turn corresponds to a smooth 1-current w on the unit circle. We can then define a distribution $\nu \in C^\infty(S^1)^*$ with $\nu(\mathbb{C}) = 0$ by $\langle \nu, f \rangle = -\langle w, df \rangle$, for $f \in C^\infty(S^1)$. Furthermore, the action of Γ commutes with the correspondences above to ensure that ν is Γ -invariant. (These observations were made by Haefliger and Banghe.)

§ 2. Distributions at infinity

In this section we shall show that the Patterson measure is only one particular member of a family of distributions on the unit circle characterised by similar transformation properties. These distributions at infinity are intimately related to the analytic structure of the surface through the spectrum of the Laplacian.

In the Poincaré metric the Laplacian takes the form:

$$\Delta = (1 - (x^2 + y^2))^2 (\partial^2/\partial x^2 + \partial^2/\partial y^2).$$

We now want to describe the eigenfunctions for Δ acting on $C^\infty(D)$. We first introduce a map $\langle, \rangle: D \times S^1 \rightarrow \mathbb{R}$ giving the (signed) distance $\langle x, b \rangle$ from 0 to the unique horocycle tangent to S^1 at $b \in S^1$ and passing through $x \in D$. (In this model a horocycle is simply a euclidean circle in $\bar{D}$ tangent to S^1 at a single point.)

Let $\mathcal{A}$ be the space of analytic functions on S^1 (defined on some open neighbourhood of S^1). The space $\mathcal{A}$ becomes a Frechet space when equipped with the induced topology of uniform convergence on annuli containing S^1 .

We define an *analytic distribution* v to be a continuous linear functional on $\mathcal{A}$. We denote the dual space of all such distributions by $\mathcal{A}^*$.

The close connection between the Laplacian and these analytic distributions is described by the following result of Helgason ([1], p. 60):

PROPOSITION 1 (Helgason). *There is a bijection between*

- (a) *eigenfunctions h for $\Delta: C^\infty(D) \rightarrow C^\infty(D)$, and*
- (b) *analytic distributions $v \in \mathcal{A}^*$*

where

$$h(z) = \int_{S^1} e^{(i\lambda+1)\langle z, b \rangle} dv(b)$$

satisfies $\Delta h = -(\lambda^2 + 1)h$, all provided $i\lambda \neq -1, -3, -5, \dots$

(Here $\int_{S^1} g dv$ denotes $v(g)$.)

We really want to study the Laplacian for the compact Riemann surface $S = D/\Gamma$ associated to Γ . Here D can be interpreted as the covering space for S and Γ is then the covering group.

Consider the Laplacian $\Delta_S: C^\infty(S) \rightarrow C^\infty(S)$ for the surface S . If $h \in C^\infty(S)$ is an eigenfunction for Δ_S then it corresponds to an automorphic function on D (which we also denote by h) i.e. $h(gx) = h(x)$ for all $g \in \Gamma$. Furthermore, this function $h \in C(D)$ is also an eigenfunction for $\Delta_D: C^\infty(D) \rightarrow C^\infty(D)$.

As a consequence we have that for any $g \in \Gamma$:

$$\begin{aligned} \int_{S^1} e^{(i\lambda+1)\langle z, b \rangle} dv(b) &= h(z) = h(gz) \\ &= \int_{S^1} e^{(i\lambda+1)\langle gz, b \rangle} dv(b) = \int e^{(i\lambda+1)\langle gz, gb \rangle} dv(gb) \\ &= \int_{S^1} e^{(i\lambda+1)\langle z, b \rangle} \cdot [e^{(i\lambda+1)\langle g0, gb \rangle} dv(gb)] \end{aligned}$$

(where we have used the simple geometric identity $\langle gz, gb \rangle = \langle z, b \rangle + \langle g0, gb \rangle$ cf. [1] (p. 83)).

From the uniqueness part of the above proposition we have that $g^* v = e^{-(i\lambda+1)\langle g0, gb \rangle} v$. However, it is simple to show that $|g'(b)| = e^{-2\langle g0, gb \rangle}$ (cf. [1] p. 58) and so we have that $g^* v = |g'|^s v$ where $s = (i\lambda + 1)/2$. We summarise this below:

PROPOSITION 2. *There is a bijection between*

- (a) *eigenvalues $-(\lambda^2 + 1)$ for $\Delta_S: C^\infty(S) \rightarrow C^\infty(S)$, and*
- (b) *analytic distributions $v \in \mathcal{A}^*$ such that $g^* v = |g'|^s v$, $\forall g \in \Gamma$ where $\lambda = (2s - 1)/i$ (provided $i\lambda \neq -1, -3, -5, \dots$).*

Since Δ_s is an unbounded normal operator with discrete spectrum, we have that $\{s \mid \exists v \text{ s.t. } g^* v = |g'|^s v\}$ is a countable subset of $[0, 1] \cup \{\frac{1}{2} + i\mathbf{R}\}$ with no points of accumulation.

The Patterson measure is an analytic distribution corresponding to the zeroth eigenvalue of $\Delta_S: C^\infty(S) \rightarrow C^\infty(S)$. (The corresponding eigenfunction $h \in C^\infty(S)$ with $\Delta_S h = 0$ is simply a constant function.)

Remark. An alternative viewpoint is to study harmonic functions on Γ , whereupon S^1 becomes the Martin boundary. In this situation harmonic functions on Γ are decomposed into minimal harmonic functions according to a measure supported on S^1 . (For more details, cf. [10].)

§ 3. Perron–Frobenius type operators

The action of the Fuchsian group Γ on S^1 leads to a situation where the orbits have very complicated dynamical behaviour. Since the action is hyperfinite we can find a single transformation on S^1 (essentially) the same orbit structure as the action of Γ . Caroline Series has carried out a careful study of this situation and constructed an explicit map $f: S^1 \rightarrow S^1$ from the action of Γ . We begin by considering the appropriate class of maps.

Let $\{J_1, \dots, J_n\}$ be a partition of S^1 into disjoint arcs and let $f: S^1 \rightarrow S^1$ be a map satisfying the following conditions:

- (i) *Markov*: If $f(\text{int } J_e) \cap (\text{int } J_k) \neq \emptyset$ then $f(\text{int } J_e) \supset (\text{int } J_k)$;
- (ii) *Expanding*: $\exists \alpha > 1$ with $|f'(z)| \geq \alpha$ for all $z \in S^1$;
- (iii) *Strong Renyi*: $\exists K > 0$ with $|f''(z)/f'(z)| \leq K$;
- (iv) *Analytic*: $\exists$ neighbourhoods $J_i \subset U_i \subset \mathbf{C}$ such that $f|_{J_i}$ extends to an analytic function $f: U_i \rightarrow \mathbf{C}$.

(We can assume that if $f(\text{int } J_e) \cap (\text{int } J_k) \neq \emptyset$ then $f^{-1} U_k \subset U_e$.) These hypotheses are not completely independent.

Choose neighbourhoods $J_i \subset V_i \subset \bar{V}_i \subset U_i$; then we can define a Banach space B_i of analytic functions on each U_i with the (supremum) norm: $\|g\|_i = \sup \{|g(z)| \mid z \in \bar{V}_i\}$. We then consider the Banach space $B = \bigoplus_i (B_i, \|\cdot\|_i)$.

The space $(B, \|\cdot\|)$ is also a nuclear space.

We define an operator $L_s: B \rightarrow B$ ($s \in \mathbf{C}$) by

$$(L_s h)(z) = \sum_i \frac{h \circ f_i^{-1}(z)}{|f'_i \circ f_i^{-1}(z)|^s}$$

(where the summation is over those i 's for which $f(\text{int } J_i) \cap (\text{int } J_k) \neq \emptyset$ when $z \in U_k$). These operators are of a general type called Perron–Frobenius type operators.

Since $(B, \|\cdot\|)$ is nuclear and L_s is a bounded operator it follows that L_s is a nuclear operator [4]. One consequence of this is the following:

PROPOSITION 3. *For any $S \in \mathbb{C}$ the spectrum of $L_s: B \rightarrow B$ consists of isolated eigenvalues (with finite dimensional eigenspaces) except for 0 (which is a point of accumulation for the spectrum).*

Assume that for a particular value of $S \in \mathbb{C}$ the operator $L_s: B \rightarrow B$ has 1 as an eigenvalue. By duality of spectra we know that $L_s^*: B^* \rightarrow B^*$ (given by $(L_s^* \varrho)(g) = \varrho(L_s g)$) has precisely the same spectrum as L_s . In particular, we can choose $v \in B^*$ for which $L_s^* v = v$.

Let $h_i \in B_i \subset B$ be an analytic function with support on U_i then $L_s h_i$ takes the form:

$$L_s h_i = (h_i \circ f_i^{-1}) \cdot (|f'| \circ f_i^{-1})^{-s}.$$

Thus $L_s h_i$ describes an analytic function whose support contains $\bigcup_{U_i \supset U_j} U_j$.

Since $L_s^* v = v$ we have that

$$v(h_i) = v(L_s h_i) = v(h_i \circ f_i^{-1} \cdot |f'|^{-s} \circ f_i^{-1}).$$

More generally, choose a sequence with

$$f_{i_n}^{-1} \circ f_{i_{n-1}}^{-1} \circ \dots \circ f_{i_1}^{-1} \cdot U_{i_0} \neq \emptyset$$

(i.e. $\text{domain}(f_{i_k}) \subset \text{range}(f_{i_{k+1}})$) and let h be an analytic function with support on this neighbourhood. We choose to define $v(h)$ by the expression

$$v(h) = v([|(f_i \circ \dots \circ f_{i_n})'|^{-s} \cdot h] \circ (f_{i_1} \circ \dots \circ f_{i_n})^{-1}).$$

Thus v extends from B^* to analytic functions supported on appropriate neighbourhoods of smaller intervals of the form $\bigcap_{k=0}^n f^{-k} J_{i_k}$, in a well-defined way.

We now return to the setting of Fuchsian groups. The map $f: S^1 \rightarrow S^1$ which Series constructed had all the above properties and took the form $f(z) = g_i(z)$ when $z \in J_i$, where $g_1, \dots, g_n$ are a finite number of generators (and their inverses) for Γ . A particularly useful feature of her construction is the following [11]:

PROPOSITION 4 (Series). *Assume $gx = y$ where $g \in \Gamma$ and $x, y \in S^1$ have f -expansions $\{x\} = \bigcap_{n=0}^{+\infty} f^{-n} J_{x_n}$, $\{y\} = \bigcap_{n=0}^{+\infty} f^{-n} J_{y_n}$. There exists $m, N > 0$ such that $x_n = y_{n+m}$ for $n \geq M$ and $g = g_{y_0}^{-1} \circ \dots \circ g_{y_{N+m}}^{-1} \circ g_{x_N} \circ \dots \circ g_{x_0}$.*

In particular, if h is an analytic function with support on $\bigcap_{i=0}^N f^{-i} U_{x_i} \ni x$ then we have shown above how to extend $v \in B^*$ to define $v(h)$. Similarly, if k is an analytic function with domain $\bigcap_{i=0}^{N+m} f^{-i} U_{y_i} \ni y$ we can define $v(k)$. If we assume that $h \circ f^N = k \circ f^{N+m}$ then the above proposition gives that

$k = h \circ g$. In particular,

$$v(|f^N|^s h) = v(h \circ f^N) = v(k \circ f^{N+m}) = v(|f^{N+m}|^s k).$$

We can conclude that since $k = h \circ g$ then $v(h \circ g) = v(|g'|^s h)$.

If h is analytic on a neighbourhood of S^1 it gives rise to an element of B , which we can further 'decompose' into analytic functions on neighbourhoods of the form $\bigcap_{n=0}^N f^{-n} U_{i_n}$. Thus using the above comments $v \in B^*$ with $f^* v = |f'|^s v$ uniquely defines $v \in \mathcal{A}^*$ (we use the same notation) with $g^* v = |g'|^s v$. The converse can be proved similarly.

Using Proposition 2, the spectra of $\Delta_s: C^\infty(S) \rightarrow C^\infty(S)$ and $L_s: B \rightarrow B$ ($s \in \mathbb{C}$) are related as follows:

THEOREM 1. *There is a bijection between:*

- (a) *eigenvalues $-(1+\lambda^2)$ for $\Delta_s: C^\infty(S) \rightarrow C^\infty(S)$, and*
- (b) *parameters $s \in \mathbb{C}$ for which 1 is an eigenvalue for $L_s: B \rightarrow B$ where $s = (i\lambda + 1)/2$ (provided $i\lambda \neq -1, -3, -5, \dots$).*

§ 4. Eigenvalue estimates

In the previous section we established the correspondence between the spectra of the Laplacian and of Perron–Frobenius-type operators. There is an established theory for estimating the eigenvalues of the Laplacian which consequently has a bearing on this other class of operators, and vice versa.

There exists a simple geometric bound for the smallest non-zero eigenvalue of the Laplacian, due to Schoen, Wolpert and Yau [9], which we state as follows:

PROPOSITION 5 (Laplacian spectra). *Let l be the length of the shortest closed geodesic on S and let α be the smallest non-zero eigenvalue of $-\Delta_s$. Then $l \asymp \alpha$ (i.e. α/l is bounded above and below by constants depending only on the genus and not on the Riemann metric).*

Notice that $\dot{g} \in \Gamma$ corresponds to a closed geodesic of length $\log |g'(x)|$ where $gx = x$. Using Theorem 1 we can reformulate this proposition as:

PROPOSITION 6 (distribution spectra). *Choose $g \in \Gamma$ and $gx_0 = x_0$ such that $|g'(x_0)| > 1$ is minimised then*

- (i) *the largest value $\sigma = \sigma_0$ satisfying $0 < \sigma < 1$ and $g^* v = |g'|^\sigma v$ having a solution satisfies $\sigma_0 \asymp |g'(x_0)|$;*
- (ii) *the largest value of $\sigma = \sigma_0$ with $0 < \sigma < 1$ such that L_σ has unity as an eigenvalue satisfies $\sigma_0 \asymp |g'(x_0)|$.*

In view of this correspondence between spectra we can make the following conjecture:

CONJECTURE. It is possible to find an independent proof of the Schoen–Wolpert–Yau result by analysing the spectrum of Perron–Frobenius-type operators.

Remark. A claim by the author in this direction (in [6] § 3) appears to be falsely based on a claim of Collet–Epstein–Gallovotti.

§ 5. Closed geodesics

In this section we consider how the two types of spectra influence the geometry of the surface through the lengths of closed geodesics (without knowing which free homotopy classes they are associated) as a subset of the real line then this is insufficient to determine the metric. (This is because a complete knowledge of the lengths gives a complete knowledge of the Laplacian spectrum, but there exists examples of Riemann surfaces which are isospectral but *not* isometric.)

However, it is true that there exists (for a given genus g) a finite number $N = N(g)$, say, of fixed free homotopy classes the lengths of whose associated closed geodesics uniquely determines a Riemann metric (cf. Astérisque volumes 66–67). In the situation we have been studying the geodesic flow corresponds to a suspension flow over $f: S^1 \rightarrow S^1$ of $r(x) = \log |f'(x)|$. Since f (and thus r) are derived from rational functions they are finite-dimensional and so can be specified by a finite number of closed geodesic lengths i.e. values of $|(f^n)'(x)|$ where $f^n x = x$. (This can be thought of as a finite-dimensional Lifšic theorem.)

The appropriate way to study the lengths of closed geodesics (without associating free homotopy classes) is to introduce a *zeta function* defined by

$$\zeta(s) = \prod_{\gamma} (1 - \exp - sl(\gamma))^{-1} \quad (s \in \mathbb{C})$$

where the Euler product runs over all closed geodesics γ of length $l(\gamma)$. (The product converges to a non-zero analytic function for $\operatorname{Re}(s) > 1$.) The following is a standard result (cf. [2]).

PROPOSITION 7. $\zeta(s)$ has a non-zero meromorphic extension to the entire complex plane with (non-trivial) poles at $s_n = \frac{1}{2} + i(\lambda_n - \frac{1}{4})^{\frac{1}{2}}$.

The traditional proof of this result uses the Selberg trace formula. (The trace refers to an integral operator whose kernel is some function $\Phi: \mathbb{R} \rightarrow \mathbb{R}$ of the distance function $d(x, y)$. The trace can be expressed in one of two ways: either as a sum $\sum_n A(\lambda_n)$ where $A: \mathbb{R} \rightarrow \mathbb{R}$ and $\{\lambda_n\}$ are eigenvalues of the Laplacian, or by explicit computation (for appropriate Φ) which decomposes into a sum over Γ each term of which corresponds to a closed geodesic.)

There is an alternative approach using Perron–Frobenius operators. Ruelle showed that using the nuclear property of similar operators (but for a different type of symbolic dynamics) the zeta function $\zeta(s)$ could be expressed as a Grothendieck determinant of Perron–Frobenius operators [8] (cf. also [4]). However, by following Ruelle’s program using the above symbolic formalism and these particular Perron–Frobenius-type operators we are able to relate the poles for $\zeta(s)$ to the spectrum of the Laplacian (as expressed in the proposition) via our relationship between the two types of operator expressed in Theorem 1.

§ 6. Rate of mixing

The most natural dynamical viewpoint associated to the surface itself is to consider the geodesic flow $\varphi_t: T_1 S \rightarrow T_1 S$ on the unit tangent bundle. Let m be the Liouville measure on $T_1 S$ formed from Riemannian measure π on the surface and Lebesgue measure on each fibre. This flow is known to be strongly mixing, and in particular for any $F, G \in C^\infty(T_1 M)$ we have that $\varrho(t) = \int F\varphi_t \cdot G dm - \int F dm \cdot \int G dm \rightarrow 0$ as $t \rightarrow +\infty$. There exists a much sharper result due to Moore and Ratner [7]:

PROPOSITION. *For geodesic flows for compact surfaces of constant negative curvature $|\varrho(t)| \leq C \exp -\lambda_1 t$ (for some $C > 0$) where $\lambda_1 > 0$ is the smallest non-zero eigenvalue.*

In view of the Schoen–Wolpert–Yau result in Section 4 we see that the rate of decay of $\varrho(t)$ is of the order of the length of the shortest closed geodesic. This is consistent with geometric intuition: If a *short* closed geodesic divided the surface into two separate pieces then the rate of mixing between the two should be proportional to the length of the closed geodesic.

Remark. For Brownian motion on S the natural analogue of the proposition is easy to prove: Let $\varrho_t(x, y)$ be the associated heat kernel (that is, the probability density of going from x to y in time t). Given $F, G \in C^\infty(S)$, define $\alpha(t) = \int \int_S F(x) \cdot G(y) \cdot \varrho_t(x, y) d\pi(x) d\pi(y) - \int_S F d\pi \cdot \int_S G d\pi$. Since the heat kernel has an explicit form in terms of eigenfunctions and eigenvalues of the Laplacian. It is trivial to then show that $|\alpha(t)| \leq C \exp -\lambda_1 t$.

This result for Brownian motion remains true for surfaces of *variable* negative curvature also.

In the case of metrics of variable negative curvature the metric is *conformally* equivalent to a metric of constant negative curvature (i.e. it takes the form $ds'^2 = \sigma(x) ds^2$ where $\sigma: S \rightarrow \mathbf{R}^+$). As a consequence the ‘hitting distribution’ of the Brownian motion on S^1 remains the same for ds^2 and ds'^2 . Using the Ahlfors–Bers mapping theorem, we can find $h: D \rightarrow D$ car-

rying one metric to the other which extends to a Hölder continuous map on S^1 . Let $h^*l = \mu$ be the image of Lebesgue measure under the extension $h: S^1 \rightarrow S^1$; then μ is equivalent to the equilibrium state for $\log|f'| \circ h^{-1}: S^1 \rightarrow \mathbb{R}$. (In general, $h: S^1 \rightarrow S^1$ will not be Lipschitz and μ will not be equivalent to Lebesgue measure.)

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