Conspectus materiae tomi LIII, fasciculi 5

La revue est consacrée à la Théorie des Nombres
The journal publishes papers on the Theory of Numbers
Die Zeitschrift veröffentlicht Arbeiten aus der Zahlentheorie
Журнал посвящен теории чисел

L'adresse de la Rédaction — Address of the Editorial Board — Die Adresse der Schriftleitung — Адрес редакции

ACTA ARITHMETICA
ul. Śniadeckich 8, skr. poczt. 137, 00-950 Warszawa, telex PL 816112

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Published by PWN–Polish Scientific Publishers

ISBN 83-01-09787-6 ISSN 0065-1036

PRINTED IN POLAND

WROCŁAWSKA DRUKARNIA NAUKOWA

A limit theorem for the Riemann zeta-function in the complex space

by

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In memory of V. G. Sprindžuk

Let $s = \sigma + it$ be a complex variable and let $\zeta(s)$ denote, as usual, the Riemann zeta-function. It is known by [3], [7] that the function $\zeta(s)$ has a limiting distribution in the half-plane $\sigma > 1/2$. In modern terminology it is formulated as follows. Let $\boldsymbol{C}$ be the complex space and let $\mathfrak{B}(\boldsymbol{C})$ denote the class of Borel sets of the space $\boldsymbol{C}$. For $T > T_0$ let

$$\nu_T(\ldots) = \frac{1}{T}\operatorname{mes}\{t \in [0, T], \ldots\}$$

where instead of dots we will write the condition which is satisfied by t and $\operatorname{mes}\{A\}$ denotes the Lebesgue measure of the set A. We define the probability measure

$$P_T(A) = \nu_T\big(\zeta(\sigma + it) \in A\big), \qquad A \in \mathfrak{B}(\boldsymbol{C}),\ \sigma > 1/2.$$

The function $\zeta(s)$ has a limiting distribution if on the space $(\boldsymbol{C}, \mathfrak{B}(\boldsymbol{C}))$ there exists a probability measure P such that P_T weakly converges to P as $T \to \infty$.

More general results are obtained in [1] where it was proved that the function $\zeta(s)$ has a limiting distribution in the space of functions meromorphic in the half-plane $\sigma > 1/2$.

The aim of this paper is to prove the limit theorem for the Riemann zeta-function in the complex space, when σ depends on T and tends to $1/2$ as $T \to \infty$.

In [13] the theorem of this kind has been obtained for the modulus of the function $\zeta(s)$. It turns out that in this case some power norming is necessary. It has been proved there that the distribution function

$$\nu_T\big(|\zeta(\tilde{\sigma}_T + it)|^{(2^{-1}\ln\ln T)^{-1/2}} < x\big)$$

converges as $T \to \infty$ to the lognormal distribution function, i.e. to the distribution function

$$G(x)=\begin{cases}\Phi(\ln x), & x>0,\\ 0 & x\leqslant 0,\end{cases}$$

$$\Phi(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-u^2/2}\,du.$$

Here

$$\tilde{\sigma}_T=\frac{1}{2}+\frac{\psi_T\sqrt{\ln\ln T}}{\ln T}$$

where $\psi_T>0$, $\psi_T\to\infty$ and $\ln\psi_T=o(\ln\ln T)$ as $T\to\infty$.

Let us put

$$l_T=\ln\ln T,\qquad \sigma_T=\frac{1}{2}+\frac{\varphi_T\sqrt{\ln\ln\ln T}}{\ln\ln T}$$

where $\varphi_T>0$, $\varphi_T\to\infty$ and $\ln\varphi_T=o(\ln l_T)$ as $T\to\infty$. As was noted in [10], the distribution function

$$\nu_T\big(|\zeta(\sigma_T+it)|^{\tilde{\varkappa}_T}<x\big) \tag{1}$$

also converges to $G(x)$ as $T\to\infty$. Here $\tilde{\varkappa}_T=(2^{-1}\ln\ln\ln T)^{-1/2}$. In particular, it follows that

$$\nu_T\big(\zeta(\sigma_T+it)=0\big)=o(1). \tag{2}$$

If $\zeta(\sigma_T+it)\neq 0$, then $\zeta^{\tilde{\varkappa}_T}(\sigma_T+it)$ will be understood as

$$\exp\{\tilde{\varkappa}_T\ln\zeta(\sigma_T+it)\}$$

where $\arg\zeta(\sigma_T+it)$ is defined by continuous displacement from the point $s=2$ along the path joining the points 2, $2+it$ and σ_T+it. Taking into account (2) we set for simplicity $\zeta^{\tilde{\varkappa}_T}(\sigma_T+it)=0$ if $\zeta(\sigma_T+it)=0$. For sets $A\in\mathfrak{B}(C)$ we define the probability measure as follows:

$$\mu_T(A)\overset{\text{def}}{=}\nu_T\big(\zeta^{\tilde{\varkappa}_T}(\sigma_T+it)\in A\big).$$

THEOREM. *On $(C,\mathfrak{B}(C))$ there exists a non-degenerate probability measure μ such that the measure μ_T weakly converges to μ as $T\to\infty$.*

To prove this theorem we shall make use of the method of characteristic transforms of probability measures given on $(C,\mathfrak{B}(C))$ [10], but first we will replace the study of the measure μ_T by a study of the measure defined by means of simpler functions than $\zeta^{\tilde{\varkappa}_T}(\sigma_T+it)$.

Let $\varkappa=[\sqrt{2^{-1}\ln l_T}]^{-1}$ where $[u]$ denotes the integral part of the number u. By $d_\varkappa(m)$, $m=1,2,3,\ldots$, we will denote the coefficients of the Dirichlet series expansion of the function $\zeta^\varkappa(s)$ in the half-plane $\sigma>1$ (see [5]). For $u>0$ and

$N=T^{3/4}$ we put

$$S_u(s)=\sum_{m\leqslant u}\frac{d_\varkappa(m)}{m^s},\qquad g(s)=\zeta(s)-S_N^{1/\varkappa}(s).$$

$c_1,c_2,\ldots$ are suitably chosen positive constants, B denotes a number (not always the same) which is bounded by a constant. Further on it is assumed that $T\to\infty$.

We will need the mean value theorem of Montgomery–Vaughan for Dirichlet polynomials:

Let $a_1,\ldots,a_n$ be arbitrary complex numbers. Then

$$\int_0^T\left|\sum_{m\leqslant n}\frac{a_m}{m^{it}}\right|^2 dt=T\sum_{m\leqslant n}|a_m|^2+B\sum_{m\leqslant n}m|a_m|^2.$$

This theorem is the special case of the results of [13]. For the proof see [6], p. 130–134.

LEMMA 1. *We have*

$$\int_0^T|g(\sigma_T+it)|^{2\varkappa}dt=BT\exp\left\{-c_1\frac{\ln T}{l_T}\right\}.$$

Proof. Following [5], [8] we will define the functions

$$w(t)=\int_{\ln^2 T}^{T}\exp\{-2\varkappa(t-2\tau)^2\}\,d\tau,\qquad L(\sigma)=\int_{-\infty}^{\infty}|S_N(\sigma+it)|^2w(t)\,dt,$$

$$J(\sigma)=\int_{-\infty}^{\infty}|\zeta(\sigma+it)|^{2\varkappa}w(t)\,dt,\qquad K(\sigma)=\int_{-\infty}^{\infty}|g(\sigma+it)|^{2\varkappa}w(t)\,dt.$$

In [8] it was shown that

$$J(\tfrac{1}{2})=B\sqrt{\varkappa^{-1}}\,T(\ln T)^{\varkappa^2}. \tag{3}$$

It follows from the definition of $w(t)$ that

$$w(t)=\frac{1}{4\sqrt{\varkappa}}\int_{2\sqrt{\varkappa}(2\ln^2T-t)}^{2\sqrt{\varkappa}(2T-t)}e^{-u^2/2}\,du.$$

Consequently,

$$w(t)=\begin{cases}B\sqrt{\varkappa^{-1}}\exp\{-c_9\varkappa(2\ln^2T-t)^2\} & \text{for } t\leqslant 0,\\ B\sqrt{\varkappa^{-1}}\exp\{-c_{10}\varkappa(2T-t)^2\} & \text{for } t\geqslant 2T+\ln^2 T,\\ B\sqrt{\varkappa^{-1}} & \text{for } 0\leqslant t\leqslant 2T+\ln^2T.\end{cases}$$

Since $S_N(\frac{1}{2}+it)=B\sqrt{N}$, from these properties of $w(t)$ we obtain

$$L(\tfrac{1}{2})=B\sqrt{\varkappa^{-1}}\int_0^{2T}|S(\tfrac{1}{2}+it)|^2dt+BN\varkappa^{-1}\ln^4 T. \tag{4}$$

By use of the estimation [8]

$$\sum_{m\leqslant N} d_\varkappa^2(m)/m = B(\ln T)^{\varkappa^2}$$

and applying the Montgomery–Vaughan theorem we have

$$\int_0^{2T} |S_N(\tfrac{1}{2}+it)|^2 dt = 2T\sum_{m\leqslant N} d_\varkappa^2(m)/m + B\sum_{m\leqslant N} d_\varkappa^2(m)$$
$$= BT\sum_{m\leqslant N} d_\varkappa^2(m)/m = BT(\ln T)^{\varkappa^2}.$$

Hence and from (4) we obtain

$$L(\tfrac{1}{2}) = B\sqrt{\varkappa^{-1}}\,T(\ln T)^{\varkappa^2}. \tag{5}$$

From the definition of the function $g(s)$ the inequality

$$K(\tfrac{1}{2}) \leqslant J(\tfrac{1}{2}) + L(\tfrac{1}{2})$$

follows. Then by (3) and (5) we conclude that

$$K(\tfrac{1}{2}) = B\sqrt{\varkappa^{-1}}\,T(\ln T)^{\varkappa^2}. \tag{6}$$

In [12] the following modification of Lemma 7 of [5] was obtained. Let $1/2 \leqslant \sigma \leqslant 3/4$, then

$$K(\sigma) \leqslant \left(1 + B(\ln\ln T)^{-1/2}\right)\left(K(\tfrac{1}{2})\right)^{(5-4\sigma)/3}\left(c_{11}\sqrt{\varkappa^{-1}}\,T^{1-9\varkappa/8}\right)^{(4\sigma-2)/3}$$
$$+ B\left(K(\tfrac{1}{2})\right)^{(7-8\sigma)/3}\exp\{-c_{12}\varkappa(2\sigma-1)\ln^4 T + c_{13}(2\sigma-1)\ln T\}.$$

Now, by the use of (6), we find that

$$K(\sigma_T) = B\sqrt{\varkappa^{-1}}\,T(\ln T)^{\varkappa^2}\exp\left\{-c_2\frac{\ln T}{l_T}\right\} = BT\exp\left\{-c_1\frac{\ln T}{l_T}\right\}. \tag{7}$$

And again, by the properties of $w(t)$, we obtain

$$K(\sigma_T) = B\sqrt{\varkappa^{-1}}\int_0^{2T} |g(\sigma_T+it)|^{2\varkappa} dt + BN\varkappa^{-1}\ln^4 T.$$

Hence and from (7) the assertion of Lemma 1 follows easily.

Let

$$\tilde{\mu}_T(A) = \nu_T\left(S_N(\sigma_T+it)\in A\right), \quad A\in\mathfrak{B}(C).$$

We will prove that the study of the measure μ_T can be replaced by that of the measure $\tilde{\mu}_T$.

LEMMA 2. *If for $T\to\infty$ the measure $\tilde{\mu}_T$ weakly converges to some measure, then the measure μ_T also weakly converges to the same measure.*

Proof. Let $\varepsilon_T = (\ln T)^{-1/\sqrt{\ln l_T}}$. Then by Lemma 1 and by the Chebyshev inequality

$$\nu_T\left(|g(\sigma_T+it)| \geqslant \varepsilon_T\right) \leqslant \frac{\varepsilon_T^{-2\varkappa}}{T}\int_0^T |g(\sigma_T+it)|^{2\varkappa} dt = o(1). \tag{8}$$

Then, since the distribution function (1) converges to $G(x)$ as $T\to\infty$, we have

$$\nu_T\left(|\zeta(\sigma_T+it)| < 2\sqrt{\varepsilon_T}\right) = \nu_T\left(|\zeta(\sigma_T+it)|^{\tilde{\varkappa}_T} < (2\sqrt{\varepsilon_T})^{\tilde{\varkappa}_T}\right) \tag{9}$$
$$= G\left((2\sqrt{\varepsilon_T})^{\tilde{\varkappa}_T}\right) + o(1) = o(1).$$

Now from (8) and (9) we deduce

$$\nu_T\left(|S_N^{1/\varkappa}(\sigma_T+it)| < \sqrt{\varepsilon_T}\right) \tag{10}$$
$$\leqslant \nu_T\left(\left||\zeta(\sigma_T+it)| - |\zeta(\sigma_T+it) - S_N^{1/\varkappa}(\sigma_T+it)|\right| < \sqrt{\varepsilon_T}\right)$$
$$\leqslant \nu_T\left(|\zeta(\sigma_T+it)| - |g(\sigma_T+it)| < \sqrt{\varepsilon_T}\right)$$
$$\leqslant \nu_T\left(|\zeta(\sigma_T+it)| < 2\sqrt{\varepsilon_T}\right) + o(1) = o(1).$$

Let $A\in\mathfrak{B}(C)$. Then in virtue of (8) and (10) we have uniformly in A

$$\nu_T\left(\zeta^\varkappa(\sigma_T+it)\in A\right)$$
$$= \nu_T\left(\left(S_N^{1/\varkappa}(\sigma_T+it) + g(\sigma_T+it)\right)^\varkappa \in A\right)$$
$$= \nu_T\left(\left(S_N^{1/\varkappa}(\sigma_T+it) + g(\sigma_T+it)\right)^\varkappa \in A,\ |g(\sigma_T+it)| < \varepsilon_T\right) + o(1)$$
$$= \nu_T\Big(S_N(\sigma_T+it)\left(1 + g(\sigma_T+it)S_N^{-1/\varkappa}(\sigma_T+it)\right)^\varkappa \in A,$$
$$|g(\sigma_T+it)| < \varepsilon_T,\ |S_N^{1/\varkappa}(\sigma_T+it)| \geqslant \sqrt{\varepsilon_T}\Big) + o(1)$$
$$= \nu_T\Big(S_N(\sigma_T+it) + B\varkappa|g(\sigma_T+it)||S_N(\sigma_T+it)|^{-1/\varkappa+1} \in A,$$
$$|g(\sigma_T+it)| < \varepsilon_T,\ |S_N^{1/\varkappa}(\sigma_T+it)| \geqslant \sqrt{\varepsilon_T}\Big) + o(1)$$
$$= \nu_T\left(S_N(\sigma_T+it) + o(1)\in A,\ |g(\sigma_T+it)| < \varepsilon_T,\ |S_N^{1/\varkappa}(\sigma_T+it)| \geqslant \sqrt{\varepsilon_T}\right) + o(1)$$
$$= \nu_T\left(S_N(\sigma_T+it) + o(1)\in A\right) + o(1).$$

Hence we see that from weak convergence of the measure $\tilde{\mu}_T$ the weak convergence of the measure

$$\tilde{\tilde{\mu}}_T(A) \overset{\text{def}}{=} \nu_T\left(\zeta^\varkappa(\sigma_T+it)\in A\right), \quad A\in\mathfrak{B}(C),$$

follows. It remains to pass from the measure $\tilde{\tilde{\mu}}_T$ to the measure μ_T. For that we will make use of the mappings at weak convergence of measures. Let $\tilde{\tilde{\mu}}_T$ converge weakly to the measure μ and let $h_T\colon C\to C$ be defined by $h_T(s) = s^{(\tilde{\varkappa}_T/\varkappa)}$, $s\in C$, $s\neq 0$, $h_T(0) = 0$. Then by Theorem 5.5 from [2] we find that the measure $\tilde{\mu}_T h_T^{-1}$ weakly converges to the measure μh^{-1} where $h(s) = s$, i.e. μ_T weakly converges to μ.

The sum $S_N(\sigma_T + it)$ is rather long and it is difficult to approximate it by a product. Later on this sum will be replaced by a shorter one. For this we will need the asymptotics of some Dirichlet polynomial with multiplicative coefficients. Let $g(m) = g_{\tau,k}(m; T)$ be a multiplicative function, $|g(m)| \leqslant 1$ and $g(p) = c(\tau, k)\varkappa^2 \overset{\text{def}}{=} g$. Here $c(\tau, k)$ is some function of the parameters $\tau \in \boldsymbol{R}$ and $k \in \boldsymbol{Z}$.

LEMMA 3. *Let $T \geqslant n \geqslant \frac{1}{8}\ln T$. Then uniformly for T and τ, k in the domain $|c(\tau, k)| \leqslant c_3$*

$$\sum_{m \leqslant n} \frac{g(m)}{m^{2\sigma_T}} = l_T^g (H(1) + BR_T)$$

where

$$H(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^g \sum_{\alpha=0}^{\infty} \frac{g(p^\alpha)}{p^{\alpha s}}, \qquad \sigma > 1/2,$$

$$R_T = \max\left(\frac{\ln \varphi_T}{\ln l_T}, \frac{1}{\sqrt{\ln l_T}}\right).$$

Proof. Consider the Dirichlet series

$$Z(s) = \sum_{m=1}^{\infty} g(m)/m^s, \qquad \sigma > 1.$$

Since $g(p) = g$, we have by a simple calculation

$$Z(s) = \zeta^g(s) H(s).$$

The function $Z(s)$ satisfies all the conditions of the theorem from [4]. Let

$$M(x) = \sum_{m \leqslant x} g(m)/m.$$

Then in [4] it has been shown that

$$A(x) \overset{\text{def}}{=} \int_1^x M(u)\,du = \frac{xH(1)(\ln x)^g}{\Gamma(g+1)} + r(x) \tag{11}$$

where $r(x) = Bx(\ln x)^{\operatorname{Re} g - 1}$. By this formula it is easy to find the asymptotics of the function $M(x)$. Let $\Delta = x(\ln x)^{-1/2}$. In view of the identity

$$M(x) = \frac{1}{\Delta}\left(A(x+\Delta) - A(x) - \int_x^{x+\Delta} (M(u) - M(x))\,du\right)$$

and of the estimate

$$M(u) - M(x) = B\Delta x^{-1} \qquad \text{for } x < u \leqslant x + \Delta$$

we deduce

$$M(x) = (A(x+\Delta) - A(x))\frac{1}{\Delta} + \frac{B\Delta}{x}.$$

Thus, applying (11) we obtain

$$M(x) = x(\ln x)^g \left(1 + \frac{\Delta}{x}\right)\left(\left(\frac{\ln(x+\Delta)}{\ln x}\right)^g - 1\right)\frac{H(1)}{\Delta\Gamma(g+1)} + \frac{Bx(\ln x)^{\operatorname{Re} g - 1}}{\Delta} + \frac{B\Delta}{x}$$

$$= \frac{H(1)(\ln x)^g}{\Gamma(g+1)} + B(\ln x)^{\operatorname{Re} g - 1} + B(\ln x)^{-1/2}.$$

Hence, summing by parts and taking into account the equalities

$$\int_1^x \frac{(\ln u)^g\,du}{u^{2\sigma_T}} = (2\sigma_T - 1)^{-g-1}\left(\Gamma(g+1) - \Gamma(g+1, (2\sigma_T - 1)\ln x)\right),$$

$$\int_2^x \frac{(\ln u)^{g-1/2}\,du}{u^{2\sigma_T}} = B\sqrt{l_x} + B(2\sigma_T - 1)^{-\operatorname{Re} g - 1}(\ln l_x)^{-1/2},$$

we find

$$\sum_{m \leqslant x} g(m)/m^{2\sigma_T} = H(1)(2\sigma_T - 1)^{-g} + B(2\sigma_T - 1)^{-\operatorname{Re} g}\Gamma(g+1, (2\sigma_T - 1)\ln x)$$

$$+ Bx^{1-2\sigma_T}(\ln x)^{\operatorname{Re} g} + B(2\sigma_T - 1)\sqrt{l_x} + B(2\sigma_T - 1)^{-\operatorname{Re} g}(\ln l_x)^{-1/2}.$$

Putting $x = n$ in view of the obvious estimate

$$\Gamma(g+1, (2\sigma_T - 1)\ln n) = 1 + B\exp\{-c_4 \varphi_T \sqrt{\ln l_T}\}$$

we obtain the assertion of the lemma.

Let $M = \frac{1}{8}\ln T$ and

$$\varrho_T(A) = \nu_T(S_M(\sigma_T + it) \in A), \qquad A \in \mathfrak{B}(\boldsymbol{C}).$$

LEMMA 4. *If the measure ϱ_T weakly converges to some measure as $T \to \infty$, then μ_T also weakly converges to the same measure.*

Proof. By Lemma 2, it is sufficient to prove that the weak convergence of ϱ_T implies the weak convergence of $\tilde{\mu}_T$. By the Montgomery–Vaughan theorem and Lemma 3 we deduce that

$$\frac{1}{T}\int_0^T |S_N(\sigma_T + it) - S_M(\sigma_T + it)|^2\,dt = \sum_{M < m \leqslant N} \frac{d_\varkappa^2(m)}{m^{2\sigma_T}} + o\left(\sum_{M < m \leqslant N} \frac{d_\varkappa^2(m)}{m^{2\sigma_T}}\right) = BR_T.$$

Thus, by the Chebyshev inequality we obtain

$$\nu_T(|S_N(\sigma_T + it) - S_M(\sigma_T + it)| \geqslant \sqrt[4]{R_T}) = B\sqrt{R_T}. \tag{12}$$

Let $A \in \mathfrak{B}(\boldsymbol{C})$. From the estimate (12) it follows that uniformly in A

$$\begin{aligned}
&\nu_T(S_N(\sigma_T+it) \in A) \\
&\quad = \nu_T(S_M(\sigma_T+it)+o(1) \in A,\ |S_N(\sigma_T+it)-S_M(\sigma_T+it)| < \sqrt[4]{R_T}) + o(1) \\
&\quad = \nu_T(S_M(\sigma_T+it)+o(1) \in A) + o(1).
\end{aligned}$$

The latter equality shows that the weak convergence of ϱ_T to some measure implies the weak convergence of $\tilde{\mu}_T$ to the same measure, which proves the lemma.

Now we turn the sum $S_M(\sigma_T+it)$ into a product. It follows from the multiplicativity of $d_\varkappa(m)$ that

$$S_M(\sigma_T+it) = \prod_{p \leqslant M}\left(1+\sum_{\alpha \leqslant \ln M/\ln p}\frac{d_\varkappa(p^\alpha)}{p^{\alpha(\sigma_T+it)}}\right) - \sum_{M<m\leqslant d}{}^{'}\frac{d_\varkappa(m)}{m^{\sigma_T+it}}$$

where the prime indicates that the sum is extended over those m whose all prime divisors are smaller than M. It is easily seen that

$$d \leqslant \prod_{p^\alpha \leqslant M} p^\alpha = BT^{c_5}$$

where $c_5 < 1$. Therefore, by Lemma 3 we have

$$\nu_T\left(\left|\sum_{M<m\leqslant d}\frac{d_\varkappa(m)}{m^{\sigma_T+it}}\right| \geqslant \sqrt[4]{R_T}\right) = o(1).$$

Thus, in a manner similar to that used in the proof of Lemma 4 we can see that if the measure

$$\tilde{\varrho}_T(A) \stackrel{\text{def}}{=} \nu_T\left(\prod_{p \leqslant M}\left(1+\sum_{\alpha \leqslant \ln M/\ln p}\frac{d_\varkappa(p^\alpha)}{p^{\alpha(\sigma_T+it)}}\right) \in A\right), \quad A \in \mathfrak{B}(\boldsymbol{C}),$$

weakly converges to some measure as $T \to \infty$, then μ_T also weakly converges to the same measure.

Proof of the Theorem. We will write the product defining $\tilde{\varrho}_T$ in a simpler form. We have

$$\begin{aligned}
(13)\quad \Pi_M(t) &\stackrel{\text{def}}{=} \prod_{p \leqslant M}\left(1+\sum_{\alpha \leqslant \ln M/\ln p}\frac{d_\varkappa(p^\alpha)}{p^{\alpha(\sigma_T+it)}}\right) \\
&= \prod_{p \leqslant \sqrt{M}}\left(1+\frac{d_\varkappa(p)}{p^{\sigma_T+it}}+\frac{d_\varkappa(p^2)}{p^{2(\sigma_T+it)}}+\frac{B}{p^{3/2}\sqrt{\ln l_T}}\right)\prod_{\sqrt{M}<p\leqslant M}\left(1+\frac{d_\varkappa(p)}{p^{\sigma_T+it}}\right) \\
&= (1+o(1))\prod_{p \leqslant \sqrt{M}}\left(1+\frac{d_\varkappa(p)}{p^{\sigma_T+it}}+\frac{d_\varkappa(p^2)}{p^{2(\sigma_T+it)}}\right)\prod_{\sqrt{M}<p\leqslant M}\left(1+\frac{d_\varkappa(p)}{p^{\sigma_T+it}}\right).
\end{aligned}$$

Now we will find the characteristic transform $w_T(\tau, k)$ of $\tilde{\varrho}_T$. We have [11]

$$\begin{aligned}
(14)\quad w_T(\tau, k) &= \int_{\boldsymbol{C}\setminus\{0\}} |s|^{i\tau} e^{ik\arg s}\, d\tilde{\varrho}_T \\
&= \frac{1}{T}\int_0^T |\Pi_M(t)|^{i\tau}\exp\{ik\arg \Pi_M(t)\}\, dt.
\end{aligned}$$

Here $\tau \in \boldsymbol{R}$, $k \in \boldsymbol{Z}$. It can be easily seen that

$$|\Pi_M(t)|^{i\tau} = \Pi_M^{i\tau/2}(t)\bar{\Pi}_M^{i\tau/2}(t)$$

and

$$\exp\{ik\arg \Pi_M(t)\} = \Pi_M^{k/2}(t)\cdot\bar{\Pi}_M^{-k/2}(t).$$

For each prime p we have

$$(15)\quad \begin{aligned}
\left(1+\frac{d_\varkappa(p)}{p^{\sigma_T+it}}+\frac{d_\varkappa(p^2)}{p^{2(\sigma_T+it)}}\right)^{(i\tau+k)/2} &= \sum_{l=0}^\infty c_{\tau,k}(l)\left(\frac{d_\varkappa(p)}{p^{\sigma_T+it}}+\frac{d_\varkappa(p^2)}{p^{2(\sigma_T+it)}}\right)^l, \\
\left(1+\frac{d_\varkappa(p)}{p^{\sigma_T+it}}\right)^{(i\tau+k)/2} &= \sum_{l=0}^\infty c_{\tau,k}(l)\frac{d_\varkappa^l(p)}{p^{l(\sigma_T+it)}}
\end{aligned}$$

where

$$c_{\tau,k}(l) = \frac{i\tau+k}{2}\left(\frac{i\tau+k}{2}-1\right)\dots\left(\frac{i\tau+k}{2}-l+1\right)\frac{1}{l!}.$$

Therefore

$$(16)\quad \left(1+\frac{d_\varkappa(p)}{p^{\sigma_T+it}}+\frac{d_\varkappa(p^2)}{p^{2(\sigma_T+it)}}\right)^{(i\tau+k)/2} = \sum_{l=0}^\infty h_{\tau,k}(p^l)p^{-l(\sigma_T+it)}$$

where for an even l we have

$$h_{\tau,k}(p^l) = c_{\tau,k}(l)d_\varkappa^l(p)+c_{\tau,k}(l-1)C_{l-1}^1 d^{l-2}(p)d_\varkappa(p^2)+\dots+c_{\tau,k}(l/2)d_\varkappa^{l/2}(p^2)$$

and for an odd l

$$\begin{aligned}
h_{\tau,k}(p^l) &= c_{\tau,k}(l)d_\varkappa^l(p)+c_{\tau,k}(l-1)C_{l-1}^1 d_\varkappa^{l-2}(p)d_\varkappa(p^2)+\dots \\
&\quad + c_{\tau,k}([l/2]+1)([l/2]+1)d_\varkappa(p)d_\varkappa^{[l/2]}(p^2).
\end{aligned}$$

Here C_l^r are binomial coefficients. If T is sufficiently large, and τ_T and k_T tend to infinity sufficiently slowly as $T \to \infty$, then for all $|\tau| \leqslant \tau_T$, $|k| \leqslant k_T$ the estimates $h_{\tau,k}(p^l) = o(1)$, $d^l(p)c_{\tau,k}(p^l) = o(1)$ for $l \geqslant 1$ are valid. From (13), (15) and (16) we find that uniformly in t and $|\tau| \leqslant c_6$, $|k| \leqslant c_7$

$$\begin{aligned}
(17)\quad \Pi_M^{(i\tau+k)/2}(t) &= (1+o(1))\prod_{p \leqslant \sqrt{M}}\sum_{l=0}^\infty\frac{h_{\tau,k}(p^l)}{p^{l(\sigma_T+it)}}\prod_{\sqrt{M}<p\leqslant M}\sum_{l=0}^\infty\frac{c_{\tau,k}(l)d_\varkappa^l(p)}{p^{l(\sigma_T+it)}} \\
&= (1+o(1))\prod_{p \leqslant \sqrt{M}}\left(1+\sum_{l \leqslant \ln M/\ln p}\frac{h_{\tau,k}(p^l)}{p^{l(\sigma_T+it)}}+\frac{o(1)}{p^{3/2}}\right) \\
&\quad \times \prod_{\sqrt{M}<p\leqslant M}\left(1+\frac{\tilde{h}_{\tau,k}(p)}{p^{\sigma_T+it}}+\frac{\tilde{h}_{\tau,k}(p^2)}{p^{2(\sigma_T+it)}}+\frac{o(1)}{p^{3/2}}\right)
\end{aligned}$$

$$= (1+o(1)) \prod_{p \leqslant \sqrt{M}} \left(1 + \sum_{l \leqslant \ln M/\ln p} \frac{h_{\tau,k}(p^l)}{p^{l(\sigma_T+it)}}\right)$$

$$\times \prod_{\sqrt{M} < p \leqslant M} \left(1 + \frac{\tilde{h}_{\tau,k}(p)}{p^{\sigma_T+it}} + \frac{\tilde{h}_{\tau,k}(p^2)}{p^{2(\sigma_T+it)}}\right)$$

$$= (1+o(1)) \left(\sum_{m \leqslant M} \frac{g_{\tau,k}(m)}{m^{\sigma_T+it}} + \sum_{M < m \leqslant d}' \frac{g_{\tau,k}(m)}{m^{\sigma_T+it}}\right).$$

Here

$$\tilde{h}_{\tau,k}(p^l) = c_{\tau,k}(l) d_\varkappa^l(p),$$

$$g_{\tau,k}(m) = \prod_{p^l \| m} g_{\tau,k}(p^l),$$

$$g_{\tau,k}(p^l) = \begin{cases} h_{\tau,k}(p^l), & p \leqslant \sqrt{M}, \\ \tilde{h}_{\tau,k}(p^l), & \sqrt{M} < p \leqslant M. \end{cases}$$

We note that in (17) $d \leqslant T^{c_8}$ where $c_8 < 1$. Similarly we find that uniformly for all t and $|\tau| \leqslant c_6$, $|k| \leqslant c_7$

(18) $$\bar{\Pi}_M^{(i\tau-k)/2}(t) = (1+o(1)) \left(\sum_{m \leqslant M} \frac{g_{\tau,-k}(m)}{m^{\sigma_T-it}} + \sum_{M < m \leqslant d}' \frac{g_{\tau,-k}(m)}{m^{\sigma_T-it}}\right).$$

Consequently, from (14), (15) and (18) we deduce that uniformly in $|\tau| \leqslant c_6$, $|k| \leqslant c_7$

(19) $$w_T(\tau, k) = (1+o(1)) \frac{1}{T} \int_0^T \sum_{m \leqslant M} \frac{g_{\tau,k}(m)}{m^{\sigma_T+it}} \sum_{m \leqslant M} \frac{g_{\tau,-k}(m)}{m^{\sigma_T-it}} dt$$

$$+ \frac{B}{T} \int_0^T \left|\sum_{m \leqslant M} \frac{g_{\tau,k}(m)}{m^{\sigma_T+it}}\right| \left|\sum_{M < m \leqslant d}' \frac{g_{\tau,-k}(m)}{m^{\sigma_T-it}}\right| dt$$

$$+ \frac{B}{T} \int_0^T \left|\sum_{m \leqslant M} \frac{g_{\tau,-k}(m)}{m^{\sigma_T-it}}\right| \left|\sum_{M < m \leqslant d}' \frac{g_{\tau,k}(m)}{m^{\sigma_T+it}}\right| dt$$

$$+ \frac{B}{T} \int_0^T \left|\sum_{M < m \leqslant d}' \frac{g_{\tau,k}(m)}{m^{\sigma_T+it}}\right| \left|\sum_{M < m \leqslant d}' \frac{g_{\tau,-k}(m)}{m^{\sigma_T-it}}\right| dt$$

$$\stackrel{\text{def}}{=} I_1 + I_2 + I_3 + I_4.$$

By the Cauchy–Schwarz inequality

$$I_2 \leqslant \left(\frac{1}{T} \int_0^T \left|\sum_{m \leqslant M} \frac{g_{\tau,k}(m)}{m^{\sigma_T+it}}\right|^2 dt\right)^{1/2} \left(\frac{1}{T} \int_0^T \left|\sum_{M < m \leqslant d}' \frac{g_{\tau,-k}(m)}{m^{\sigma_T-it}}\right|^2 dt\right)^{1/2}.$$

Therefore, by the Montgomery–Vaughan theorem and Lemma 3, $I_2 = o(1)$ uniformly in $|\tau| \leqslant c_6$, $|k| \leqslant c_7$. Similarly, $I_3 = o(1)$ and $I_4 = o(1)$. Taking into account the equality

$$\sum_{m \leqslant M} \frac{g_{\tau,k}(m)}{m^{\sigma_T+it}} \sum_{m \leqslant M} \frac{g_{\tau,-k}(m)}{m^{\sigma_T-it}}$$

$$= \sum_{m \leqslant M} \frac{g_{\tau,k}(m) g_{\tau,-k}(m)}{m^{2\sigma_T}} + \sum_{m \leqslant M} \sum_{\substack{n \leqslant M \\ m \neq n}} \frac{g_{\tau,k}(m) g_{\tau,-k}(m)}{m^{\sigma_T} n^{\sigma_T}} \left(\frac{n}{m}\right)^{it}$$

and Lemma 3 we obtain

$$I_1 = \sum_{m \leqslant M} \frac{g_{\tau,k}(m) g_{\tau,-k}(m)}{m^{2\sigma_T}} + \frac{B}{T} \sum_{m \leqslant M} \sum_{\substack{n \leqslant M \\ m \neq n}} \frac{|g_{\tau,k}(m) g_{\tau,-k}(m)|}{m^{\sigma_T} n^{\sigma_T} \left|\ln \frac{n}{m}\right|}$$

$$= \exp\left\{-\left(\frac{\tau^2}{2} + \frac{k^2}{2}\right)\right\} + o(1)$$

uniformly in $|\tau| \leqslant c_6$, $|k| \leqslant c_7$. Hence and from (19) it follows that

$$\breve{w}_T(\tau, k) = \exp\left\{-\left(\frac{\tau^2}{2} + \frac{k^2}{2}\right)\right\} + o(1)$$

uniformly in $|\tau| \leqslant c_6$, $|k| \leqslant c_7$. By the properties of characteristic transforms, we find that the measure $\tilde{\varrho}_T$ weakly converges as $T \to \infty$ to the measure defined by the characteristic transform $\exp\left\{-\left(\frac{\tau^2}{2} + \frac{k^2}{2}\right)\right\}$. It is obvious that the limit measure is non-degenerate. It follows from the lemmas proved above that the measure μ_T also weakly converges to the same measure. The theorem is proved.

Here we have considered only the value $\sigma = \sigma_T$. It is easily seen that a similar result with appropriate changes is also valid for $\sigma > \sigma_T$. The case $\sigma < \sigma_T$ is more complicated.

The author expresses his gratitude to Professor J. Kubilius for constant attention to the present work.

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Received on 18.4.1988
and in revised form on 4.1.1989 (1813)

ACTA ARITHMETICA
LIII (1990)

Обобщенные тэта-функции с характеристиками и представление чисел квадратичными формами

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Светлой памяти Владимира
Геннадиевича Спринджука
посвящается

Пусть

$$f = f(x) = f(x_1, x_2, \ldots, x_s) = \tfrac{1}{2}x'Ax = \tfrac{1}{2}\sum_{j,k=1}^{s} a_{jk}x_jx_k$$

— целочисленная положительная квадратичная форма, где x — вектор-столбец с компонентами $x_1, x_2, \ldots, x_s$, а x' — вектор-строка; Δ — определитель симметрической матрицы $A = (a_{jk})$ с четными диагональными элементами; N — ступень формы f, т.е. наименьшее натуральное число, для которого NA^{-1} — симметрическая целочисленная матрица с четными диагональными элементами. Далее, пусть $r(n; f)$ обозначает число представлений натурального числа n формой f, т.е. число решений в целых числах уравнения

$$n = f(x_1, x_2, \ldots, x_s).$$

Количество работ, посвященных т.н. точным формулам для функции $r(n; f)$, весьма велико. Эта тема привлекала внимание математиков еще в прошлом веке (Гаусс, Эйзенштейн, Лиувилль и др.).

Задача получения формулы для $r(n; f)$, годной для всех n, сводится к задаче получения формулы для тэта-ряда

$$\vartheta(\tau; f) = 1 + \sum_{n=1}^{\infty} r(n; f)e^{2\pi i\tau n} \tag{1}$$

являющегося целой модулярной формой некоторого типа, здесь и всюду в дальнейшем $\tau \in H$ (H — верхняя полуплоскость). Схема метода получения такой формулы заключается в следующем. Тэта-ряд представляется в виде суммы двух слагаемых:

$$\vartheta(\tau; f) = E(\tau; f) + X(\tau),$$