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Lower and upper bounds for the number of solutions of $p+h = P_r$

by

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1. Introduction and the main results. Let x be a sufficiently large positive number, h ($\neq 0$) a fixed even number, p a prime and P_r an almost prime with at most r prime factors counted with multiplicity. Set

$$c_h = \prod_{p>2} (1-(p-1)^{-2}) \prod_{2<p|h} (p-1)(p-2)^{-1}.$$

The work to determine the exact order of magnitude for

$$\# := |\{P_r : p+h = P_r,\ p \leqslant x\}|$$

is closely connected with the well-known Prime Twins Conjecture. In all papers published up to date on the lower bounds of $\#$, only the P_r's with no prime factor less than $x^{1/w}$ ($w > 0$, fixed) are counted. This leads to an order of $c_h x \log^{-2} x$ for all r, much smaller than the presumably correct order, i.e. $c_h x \log^{-2} x (\log\log x)^{r-1}$. On the other hand, the upper bounds of $\#$ seem to be ignored for all $r \geqslant 2$. The purpose of this paper is to improve on these situations.

We get the following main results.

THEOREM 1. $|\{P_r : p+h = P_r,\ p \leqslant x\}| \ll c_h x \log^{-2} x (\log\log x)^{r-1}, \quad r \geqslant 1.$

THEOREM 2. *Let δ be a fixed number with $0 < \delta < 1$. For any $r \geqslant 3$,*

$$|\{p : p \leqslant x,\ p+h = p_1 \cdot \ldots \cdot p_{r-1} \text{ or } p_1 \cdot \ldots \cdot p_r,$$
$$p_r > p_{r-1} > \ldots > p_1 \geqslant \exp(\log^{\delta} x)\}|$$
$$> 0.965\big((1-\delta)^{r-2}/(r-2)!\big) c_h x \log^{-2} x (\log\log x)^{r-2}.$$

2. Lemmas. Let $\mathscr{A}$ denote a finite set of integers, $|\mathscr{A}|$ the number of elements in $\mathscr{A}$, and $\mathscr{P}$ a set of primes. Suppose that $|\mathscr{A}| \sim X$, and for square-free d,

$$(\mathrm{A}_1) \qquad |\mathscr{A}_d| = \frac{\omega(d)}{d} X + r_d, \qquad \mathscr{A}_d = \{a : a \in \mathscr{A},\ d \mid a\},$$

$$\omega(d) \text{ is multiplicative, } 0 \leqslant \omega(p) < p.$$

For $z \geq 2$, let

$$P(z) = \prod_{p<z, p\in\mathscr{P}} p,$$

$$S(\mathscr{A}; \mathscr{P}, z) = |\{a: a\in\mathscr{A}, (a, P(z)) = 1\}|.$$

LEMMA 1. $|\{p': p+h = p', p \leq x\}| \ll c_h x \log^{-2} x.$

Cf. e.g. [4], p. 177, (7.1).

LEMMA 2. *Suppose* (A_1) *and*

$$(A_2) \qquad \sum_{z_1\leq p<z_2} \omega(p)/p = \log(\log z_2/\log z_1) + O(\log^{-1} z_1), \qquad z_2 > z_1 \geq 2.$$

Then

$$(1) \qquad S(\mathscr{A}; \mathscr{P}, z) \leq XV(z)\{F(s) + O(\log^{-1/3} D)\} + R_D$$

and

$$(2) \qquad S(\mathscr{A}; \mathscr{P}, z) \geq XV(z)\{f(s) + O(\log^{-1/3} D)\} - R_D$$

where $s = \log D/\log z$, $R_D = \sum_{d<D, d|P(z)} |r_d|$, *and*

$$(3) \qquad V(z) = \prod_{p|P(z)} (1-\omega(p)/p) = c(\omega) e^{-\gamma} \log^{-1} z(1+O(\log^{-1} z)),$$

γ *is the Euler constant,* $c(\omega) = \prod_p (1-\omega(p)/p)(1-1/p)^{-1}$. *The functions* F, f *are defined by the following differential-difference equations*:

$$(4) \qquad \begin{aligned} &F(s) = 2e^{\gamma}/s, \quad f(s) = 0 \quad \text{if } 0 < s \leq 2, \\ &(sF(s))' = f(s-1), \quad (sf(s))' = F(s-1) \quad \text{if } s \geq 2. \end{aligned}$$

For this lemma, cf. [5], (6), (7), (8), (9) with $\varkappa = 1$, $\beta = 2$, and [4], p. 28, (4.12), (4.16), p. 145, (2.5) with $\varkappa = 1$. Note that the W-function in [4] is just the V-function in [5] (and here), and that $\Omega(1, L)$ of [4], (A_2) on p. 205 of [5] are both implied in (A_2) here.

Hereafter, we always take

$$\mathscr{A} = \{p+h: p \leq x\}, \quad \mathscr{P} = \{p: p \nmid h\}, \quad \omega(p) = p/(p-1), \quad p \nmid h.$$

It is easy to see that both (A_1) and (A_2) are satisfied.

LEMMA 3. *For any given* $A > 0$ *and any small* $\varepsilon > 0$,

$$\sum_{d\leq x^{1/2-\varepsilon}} \max_{y\leq x} \max_{(l,d)=1} |\pi(y; d, l) - \mathrm{li}\, y/\varphi(d)| \ll x \log^{-A} x$$

where

$$\pi(y; d, l) = \sum_{p\leq y, p\equiv l(d)} 1, \quad \mathrm{li}\, y = \int_2^y \frac{dt}{\log t},$$

$\varphi(d)$ *is the Euler function.*

This is a consequence of the well-known Bombieri–Vinogradov Theorem.

LEMMA 4. *Let* α *be a fixed number with* $0 < \alpha \leq 1$,

$$\pi(y; a, d, l) = \sum_{ap\leq y, ap\equiv l(d)} 1,$$

$f(a)$ *a real function,* $f(a) \ll 1$.

For any given $A > 0$ *and any small* $\varepsilon > 0$,

$$\sum_{d\leq x^{1/2-\varepsilon}} \max_{y\leq x} \max_{(l,d)=1} \Big|\sum_{a\leq x^{1-\alpha}, (a,d)=1} f(a)\big(\pi(y; a, d, l) - \mathrm{li}(y/a)/\varphi(d)\big)\Big| \ll x \log^{-A} x.$$

This is a consequence of the mean value theorem of Ding and Pan, cf. [6].

LEMMA 5. *For* $z_1 \geq 2$,

$$\sum_{z_1\leq p\in\mathscr{P}} S(\mathscr{A}_p; \mathscr{P}, p) \leq S(\mathscr{A}; \mathscr{P}, z_1),$$

$$\sum_{z_1\leq p\in\mathscr{P}} S(\mathscr{A}_{pq}; \mathscr{P}, p) \leq S(\mathscr{A}_q; \mathscr{P}, z_1).$$

These follow from the meaning of the sifting function S, or from the Buchstab identity, cf. e.g. [4], p. 39 (1.10), p. 204 (1.1).

Moreover, we need two other deep lemmas, i.e. [2], p. 199 (1.3) and [1], Theorem 10 (or [3], Lemma 7). But they are too long (with some new concepts which should be defined previously) to be restated here. The reader may consult the original papers.

3. Preliminary results for the lower bounds.

PROPOSITION 1. *Let* δ *be a fixed number with* $0 < \delta < 1$. *For any* $r \geq 3$,

$$\begin{aligned} |\{p: p \leq x,\ p+h = p_1\cdot\ldots\cdot p_{r-2} \text{ or } p_1\cdot\ldots\cdot p_{r-1} \text{ or } p_1\cdot\ldots\cdot p_r,& \\ p_r > p_{r-1} > \ldots > p_1 \geq \exp(\log^{\delta} x)\}|& \\ > 0.965((1-\delta)^{r-2}/(r-2)!)\, c_h x \log^{-2} x (\log\log x)^{r-2}.& \end{aligned}$$

Proof. We divide the proof into five parts.

1. *Weighted sieve.* Let $v = (\log x)^{1-\delta}$, $u = \log\log x$. We have, for $r \geq 3$, $h > 0$ (if $h < 0$, then cancel the second expression in the sequel),

$$\begin{aligned} (5) \quad &|\{p+h: p \leq x,\ p+h = p_1\cdot\ldots\cdot p_{r-2} \text{ or } p_1\cdot\ldots\cdot p_{r-1} \text{ or } p_1\cdot\ldots\cdot p_r, \\ &\qquad p_r > p_{r-1} > \ldots > p_1 \geq x^{1/v}\}| \\ &\geq |\{p+h: p \leq x-h,\ p+h = p_1\cdot\ldots\cdot p_{r-2} \text{ or } p_1\cdot\ldots\cdot p_{r-1} \text{ or } p_1\cdot\ldots\cdot p_r, \\ &\qquad p_r > p_{r-1} > \ldots > p_1 \geq x^{1/v}\}| \\ &\geq S - S_1 - S_2 - O(x \log^{-3} x) \end{aligned}$$

where

$$S=\sum_{x^{1/v}\leqslant p_1<\ldots<p_{r-2}<x^{1/u}} S\bigl(\mathscr{A}_{p_1\cdot\ldots\cdot p_{r-2}};\ \mathscr{P}_{p_1\cdot\ldots\cdot p_{r-2}},\ (x/(p_1\cdot\ldots\cdot p_{r-2}))^{1/5}\bigr)$$

(recall $S(\mathscr{A}_d;\ \mathscr{P}_q, z)=\left|\{a:\ a\in\mathscr{A}_d,\ (a,\ P_q(z))=1\}\right|$, $\mathscr{A}_d=\{a:\ a\in\mathscr{A},\ d|a\}$, $\mathscr{P}_q=\{p:\ p\in\mathscr{P},\ p\nmid q\}$, $P_q(z)=\prod_{p<z,\,p\in\mathscr{P}_q}p$),

$$S_1=\sum_{x^{1/v}\leqslant p_1<\ldots<p_{r-2}<x^{1/u}}\ \sum_{(x/(p_1\cdot\ldots\cdot p_{r-2}))^{1/5}\leqslant p_{r-1}<(x/(p_1\cdot\ldots\cdot p_{r-2}))^{1/4}}\ \sum_{p_{r-1}<p_r<(x/(p_1\cdot\ldots\cdot p_{r-1}))^{1/3}}\ \sum_{p_r<p_{r+1}<(x/(p_1\cdot\ldots\cdot p_r))^{1/2}}\ \sum_{p=p_1\cdot\ldots\cdot p_{r+2}-h,\ p_{r+1}<p_{r+2}<x/(p_1\cdot\ldots\cdot p_{r+1})} 1,$$

$p_i\nmid h$, $i=1, 2, \ldots, r+2$, and

$$S_2=\sum_{x^{1/v}\leqslant p_1<\ldots<p_{r-2}<x^{1/u}}\ \sum_{(x/(p_1\cdot\ldots\cdot p_{r-2}))^{1/5}\leqslant p_{r-1}<(x/(p_1\cdot\ldots\cdot p_{r-2}))^{1/3}}\ \sum_{p_{r-1}<p_r<(x/(p_1\cdot\ldots\cdot p_{r-1}))^{1/2}}\ \sum_{p=p_1\cdot\ldots\cdot p_{r+1}-h,\ p_r<p_{r+1}<x/(p_1\cdot\ldots\cdot p_r)} 1,$$

$p_i\nmid h$, $i=1, 2, \ldots, r+1$.

The reason is as follows. First of all, we may disregard those a's ($a=p+h$) for which $(a, h)>1$; for then necessarily $(a, h)=p$, so that the number of such elements a is at most $\nu(h)$ (ν denotes the number of distinct prime factors) $=O(\log x)$, and can be absorbed into the error term. Next, since

$$\sum_{p\geqslant x^{1/v}}|\mathscr{A}_{p^2}|\ll\sum_{p\geqslant x^{1/v}}x/p^2\ll x^{1-1/v}\ll x\log^{-3}x,$$

we need only consider those squarefree a's ($a=p+h$) which are divisible by $p_1\cdot\ldots\cdot p_{r-2}$ with $x^{1/v}\leqslant p_1<\ldots<p_{r-2}<x^{1/u}$.

If $\Omega(a)\geqslant r+3$ (Ω denotes the total number of prime factors), a must contain a prime factor less than $(x/(p_1\cdot\ldots\cdot p_{r-2}))^{1/5}$ other than $p_1, \ldots, p_{r-2}$. Therefore by the definition of $S\bigl(\mathscr{A}_{p_1\cdot\ldots\cdot p_{r-2}};\ \mathscr{P}_{p_1\cdot\ldots\cdot p_{r-2}},\ (x/(p_1\cdot\ldots\cdot p_{r-2}))^{1/5}\bigr)$, such an a is sieved.

If $\Omega(a)=r+2$ or $r+1$, clearly such an a will be numbered in S_1 or S_2 and then subtracted in either case. Hence the remaining a's are those with $r-2\leqslant\Omega(a)\leqslant r$, $\mu(a)\neq 0$ ($\mu(a)$ denotes the Möbius function), and $q(a)$ (the least prime factor of a) $\geqslant x^{1/v}$. So we get (5).

2. *Lower bound of S*. To estimate S from below, we apply mainly the above-cited two deep lemmas from [2] and [1] (or [3]).

By [2], p. 199 (1.3), [4], p. 28 (4.16) and p. 145 (2.5), with

$$X=\frac{\omega(p_1\cdot\ldots\cdot p_{r-2})}{p_1\cdot\ldots\cdot p_{r-2}}\operatorname{li}x,\qquad D=x^{4/7-\varepsilon}/(p_1\cdot\ldots\cdot p_{r-2}),\qquad z=(x/(p_1\cdot\ldots\cdot p_{r-2}))^{1/5},$$

$$V(z)=\prod_{p|P_{p_1\cdot\ldots\cdot p_{r-2}}(z)}(1-\omega(p)/p)\geqslant\prod_{p|P(z)}(1-\omega(p)/p)$$
$$=c(\omega)e^{-\gamma}\log^{-1}z\bigl(1+O(\log^{-1}z)\bigr),$$

since $c(\omega)=2c_h$ (this may be easily deduced from $\omega(p)=p/(p-1)$, $p\nmid h$), $p_1\cdot\ldots\cdot p_{r-2}<x^{(r-2)/u}$, and $f(s)$ is continuous, we get

$$S\geqslant 2c_h x\log^{-2}x\,(1+o(1))e^{-\gamma}5f\left(5\times\frac{4}{7}\right)\times\sum_{x^{1/v}\leqslant p_1<\ldots<p_{r-2}<x^{1/u}}\frac{1}{p_1\cdot\ldots\cdot p_{r-2}}-|R| \tag{6}$$

where

$$R=\sum_{x^{1/v}\leqslant p_1<\ldots<p_{r-2}<x^{1/u}}\ \sum_{m<x^{4/7-\varepsilon}/(p_1\cdot\ldots\cdot p_{r-2}),\ m|P_{p_1\cdot\ldots\cdot p_{r-2}}((x/(p_1\cdot\ldots\cdot p_{r-2}))^{1/5})}\bigl(\pi(x;\ p_1\cdot\ldots\cdot p_{r-2}m,\ -h)-\operatorname{li}x/\varphi(p_1\cdot\ldots\cdot p_{r-2}m)\bigr).$$

To estimate R, we should note that, among its multiple sum, a fixed $L=p_1\cdot\ldots\cdot p_{r-2}m$ may be counted more than once. This is because, among all the prime factors of L, we may take $r-2$ of them to be $p_1, \ldots, p_{r-2}$ while $L/(p_1\cdot\ldots\cdot p_{r-2})$ to be m; and there may be more than one way for the suitable choice (i.e. satisfying all the summing conditions — $x^{1/v}\leqslant p_1<\ldots<p_{r-2}<x^{1/u}$, $m<x^{4/7-\varepsilon}/(p_1\cdot\ldots\cdot p_{r-2})$, $m\,|\,P_{p_1\cdot\ldots\cdot p_{r-2}}\bigl((x/(p_1\cdot\ldots\cdot p_{r-2}))^{1/5}\bigr)$ — in the multiple sum). But the number of ways for such a choice is at most

$$\binom{\Omega(L)}{r-2}\ll\binom{O(\log x)}{r-2}\ll\log^{r-2}x$$

because of $L<x^{4/7-\varepsilon}$. Hence L may be counted at most $O(\log^{r-2}x)$ times.

Therefore, by [1], Theorem 10 (or [3], Lemma 7) with $a=-h$, $\lambda(q)=1$ if $q=p_1\cdot\ldots\cdot p_{r-2}m$ and 0 otherwise, and $A\geqslant r+1$, we get

$$R\ll\log^{r-2}x\sum_{q\leqslant x^{4/7-\varepsilon},\,(q,\,-h)=1}\lambda(q)\bigl(\pi(x;\ q,\ -h)-\operatorname{li}x/\varphi(q)\bigr)\ll x\log^{-3}x. \tag{7}$$

As for the main term, by [4], p. 227, (2.9), $5f\left(5\times\frac{4}{7}\right) = 3.5e^{\gamma}\log(13/7)$; while by an elementary combination and the Prime Number Theorem,

$$(8)\quad \sum_{x^{1/v}\leqslant p_1<\ldots<p_{r-2}<x^{1/u}} \frac{1}{p_1\cdot\ldots\cdot p_{r-2}}$$
$$\geqslant \frac{1}{(r-2)!}\Big(\sum_{x^{1/v}\leqslant p<x^{1/u}}\frac{1}{p}\Big)^{r-2} - O\Big(\sum_{x^{1/v}\leqslant p<x^{1/u}}\frac{1}{p^2}\Big)$$
$$= (1+o(1))\big(\log(v/u)\big)^{r-2}/(r-2)! - O(1)$$
$$= (1+o(1))\big(\log(v/u)\big)^{r-2}/(r-2)!.$$

Hence it follows that

$$(9)\quad S \geqslant (1+o(1))\,7\log(13/7)\,c_h x\log^{-2}x\big(\log(v/u)\big)^{r-2}/(r-2)! - O(x\log^{-3}x)$$
$$\geqslant (1+o(1))\,4.3332\,c_h x\log^{-2}x\big(\log(v/u)\big)^{r-2}/(r-2)! - O(x\log^{-3}x).$$

3. *Estimate of* S_1. Consider the sets

$$\mathscr{E} = \{e:\ e = p_1\cdot\ldots\cdot p_{r+1},\ x^{1/v}\leqslant p_1<\ldots<p_{r-2}<x^{1/u},$$
$$\big(x/(p_1\cdot\ldots\cdot p_{r-2})\big)^{1/5}\leqslant p_{r-1}<\big(x/(p_1\cdot\ldots\cdot p_{r-2})\big)^{1/4},$$
$$p_{r-1}<p_r<\big(x/(p_1\cdot\ldots\cdot p_{r-1})\big)^{1/3},\ p_r<p_{r+1}<\big(x/(p_1\cdot\ldots\cdot p_r)\big)^{1/2}\},$$
$$\mathscr{L} = \{l:\ l = ep-h,\ ep\leqslant x,\ e\in\mathscr{E}\}.$$

Clearly $|\mathscr{E}| \ll x^{3/4+(r-2)/u}$ and $e > x^{3/5}$, $e\in\mathscr{E}$. Moreover,

$$|\{l:\ l\in\mathscr{L},\ l\leqslant x^{3/5}\}| \ll x^{3/4+(r-2)/u},$$

and $S_1 \leqslant$ the number of primes in $\mathscr{L}$. It follows that

$$(10)\quad S_1 \leqslant S(\mathscr{L};\mathscr{P},x^{3/5}) + O(x^{3/4+(r-2)/u}).$$

To estimate $S(\mathscr{L};\mathscr{P},x^{3/5})$, we apply Lemma 2 with

$$X = \sum_{e\in\mathscr{E}}\mathrm{li}(x/e),\quad \omega(d) = d/\varphi(d),\quad \mu(d)\neq 0,\quad (d,h)=1,$$
$$D = x^{1/2-\varepsilon},\quad z = x^{3/5}.$$

Since F is continuous, from (4),

$$F(s) = (1+o(1))F(5/6) = (1+o(1))\,2e^{\gamma}(5/6)^{-1}.$$

Hence from (1), (3) with $c(\omega) = 2c_h$, we have

$$(11)\quad S(\mathscr{L};\mathscr{P},x^{3/5}) \leqslant (1+o(1))\,8c_h X\log^{-1}x + R_1 + R_2$$

where

$$X = \sum_{e\in\mathscr{E}}\mathrm{li}(x/e),$$
$$R_1 = \sum_{d\leqslant D,(d,h)=1}\Big|\sum_{e\in\mathscr{E},(e,d)=1}\Big(\sum_{ep\leqslant x,\,ep\equiv h(d)}1 - \mathrm{li}(x/e)/\varphi(d)\Big)\Big|,$$
$$R_2 = \sum_{d\leqslant D,(d,h)=1}\frac{1}{\varphi(d)}\sum_{e\in\mathscr{E},(e,d)>1}\mathrm{li}(x/e).$$

Since $x^{3/5} < e < x^{(r+1)/(r+2)}$, $e\in\mathscr{E}$, it follows that

$$R_1 = \sum_{d\leqslant D,(d,h)=1}\Big|\sum_{\substack{x^{3/5}<a<x^{(r+1)/(r+2)}\\(a,d)=1}} f(a)\Big(\sum_{ap\leqslant x,\,ap\equiv h(d)}1 - \mathrm{li}(x/a)/\varphi(d)\Big)\Big|$$

where $f(a) = \sum_{e=a,e\in\mathscr{E}}1 \ll 1$. Hence by Lemma 4 with $A = 3$, $R_1 \ll x\log^{-3}x$.

As for R_2, note that for squarefree q, $d(q) = 2^{\nu(q)}$ ($d(q)$ denotes the number of divisors of q, $\nu(q)$ denotes the number of different prime factors of q), $\varphi(q) > q/d(q)$. Hence

$$R_2 \ll \sum_{q\leqslant D} d(q)/q \sum_{e\in\mathscr{E},(e,q)>1} x/(e\log(x/e))$$
$$\ll x\log^{-1}x\sum_{q\leqslant D}d(q)/q\sum_{a<x^{(r+1)/(r+2)},(a,q)\geqslant x^{1/v}}1/a$$
$$= x\log^{-1}x\sum_{q\leqslant D}d(q)/q\sum_{m|q,m\geqslant x^{1/v}}1/m\sum_{b<x^{(r+1)/(r+2)}/m,(b,q)=1}1/b$$
$$\ll x\sum_{q\leqslant D}d(q)/q\sum_{m|q,m\geqslant x^{1/v}}1/m$$
$$\ll x^{1-1/v}\sum_{q\leqslant D}d^2(q)/q \ll x^{1-1/v}(\log D)^{2^2} \ll x^{1-1/v}(\log x)^4 \ll x\log^{-3}x.$$

(Here we have used the inequality $\sum_{q\leqslant x}d^n(q)/q \ll (\log x)^{2^n}$, which can be proved by induction.)

It remains to calculate X. By the Prime Number Theorem and Stieltjes' integration,

$$X = (1+o(1))\,x\log^{-1}x\int_{1/v}^{1/u}\int_{t_1}^{1/u}\ldots\int_{t_{r-3}}^{1/u}\int_{(1-t_1-\ldots-t_{r-2})/5}^{(1-t_1-\ldots-t_{r-2})/4}$$
$$\int_{t_{r-1}}^{(1-t_1-\ldots-t_{r-1})/3}\int_{t_r}^{(1-t_1-\ldots-t_r)/2}\frac{dt_{r+1}\cdot\ldots\cdot dt_1}{t_1\cdot\ldots\cdot t_{r+1}(1-t_1-\ldots-t_{r+1})}$$
$$= (1+o(1))\big(x\log^{-1}x\big(\log(v/u)\big)^{r-2}/(r-2)!\big)$$
$$\times\int_{1/5}^{1/4}\int_a^{(1-a)/3}\int_b^{(1-a-b)/2}\frac{dc\,db\,da}{abc(1-a-b-c)}.$$

Numerical calculation by computer shows the last triple integral is < 0.0149.

Combining all these estimates, by (10), (11) we have

(12) $S_1 \leqslant (1+o(1))\,8\cdot 0.0149 c_h x\log^{-2}x\log^{r-2}(v/u)/(r-2)!+O(x\log^{-3}x)$.

4. *Estimate of* S_2. This is similar to 3. Consider the sets

$$\mathscr{E}' = \{e\colon e = p_1\cdot\ldots\cdot p_r,\ x^{1/v}\leqslant p_1<\ldots<p_{r-2}<x^{1/u},$$
$$\left(x/(p_1\cdot\ldots\cdot p_{r-2})\right)^{1/5}\leqslant p_{r-1}<\left(x/(p_1\cdot\ldots\cdot p_{r-2})\right)^{1/3},$$
$$p_{r-1}<p_r<\left(x/(p_1\cdot\ldots\cdot p_{r-1})\right)^{1/2}\},$$
$$\mathscr{L}' = \{l\colon l = ep-h,\ ep\leqslant x,\ e\in\mathscr{E}'\}.$$

Clearly $|\mathscr{E}'|\ll x^{2/3+(r-2)/u}$ and $e>x^{2/5}$, $e\in\mathscr{E}'$. Moreover, $|\{l\colon l\in\mathscr{L}', l\leqslant x^{2/5}\}|\ll x^{2/3+(r-2)/u}$, and $S_2\leqslant$ the number of primes in $\mathscr{L}'$. It follows that

$$S_2\leqslant S(\mathscr{L}';\mathscr{P},x^{2/5})+O(x^{2/3+(r-2)/u}). \tag{13}$$

By the method of 3, we get

$$S(\mathscr{L}';\mathscr{P},x^{2/5})\leqslant(1+o(1))\,8c_h Y\log^{-1}x+R_1'+R_2' \tag{14}$$

where

$$R_1',\ R_2'\ll x\log^{-3}x$$

and

$$Y = (1+o(1))\,x\log^{-1}x\int_{1/v}^{1/u}\int_{t_1}^{1/u}\ldots\int_{t_{r-3}}^{1/u}\int_{(1-t_1-\ldots-t_{r-2})/5}^{(1-t_1-\ldots-t_{r-2})/3}\int_{t_{r-1}}^{(1-t_1-\ldots-t_{r-1})/2}\frac{dt_r\cdot\ldots\cdot dt_1}{t_1\cdot\ldots\cdot t_r(1-t_1-\ldots-t_r)}$$
$$= (1+o(1))\left(x\log^{-1}x\log^{r-2}(v/u)/(r-2)!\right)\int_{1/5}^{1/3}\int_{a}^{(1-a)/2}\frac{db\,da}{ab(1-a-b)}$$
$$\leqslant (1+o(1))\,0.4061\,x\log^{-1}x\log^{r-2}(v/u)/(r-2)!.$$

Hence we have

(15) $S_2\leqslant(1+o(1))\,8\cdot 0.4061 c_h x\log^{-2}x\log^{r-2}(v/u)/(r-2)!+O(x\log^{-3}x)$.

5. *Completion of the proof of Proposition* 1. By (5), (9), (12), (15) (recall $v=(\log x)^{1-\delta}$, $0<\delta<1$, $u=\log\log x$), for $r\geqslant 3$, we get

$$|\{p\colon p\leqslant x,\ p+h = p_1\cdot\ldots\cdot p_{r-2}\ \text{or}\ p_1\cdot\ldots\cdot p_{r-1}\ \text{or}\ p_1\cdot\ldots\cdot p_r, \tag{16}$$
$$p_r>p_{r-1}>\ldots>p_1\geqslant\exp(\log^{\delta}x)\}|$$
$$\geqslant(1+o(1))\,0.9652 c_h x\log^{-2}x\left(\log(v/u)\right)^{r-2}/(r-2)!-O(x\log^{-3}x)$$
$$>0.965\left((1-\delta)^{r-2}/(r-2)!\right)c_h x\log^{-2}x(\log\log x)^{r-2}.$$

COROLLARY 1. *Let δ be a fixed number with $0<\delta<1$, and let $q(a)$ denote the least prime factor of a. Then for any $r\geqslant 3$,*

$$|\{P_r\colon p+h=P_r,\ p\leqslant x,\ q(P_r)\geqslant\exp(\log^{\delta}x)\}|$$
$$>0.965\left((1-\delta)^{r-2}/(r-2)!\right)c_h x\log^{-2}x(\log\log x)^{r-2}.$$

COROLLARY 2. *For any $r\geqslant 3$,*

$$|\{P_r\colon p+h=P_r,\ p\leqslant x\}|\geqslant(0.965/(r-2)!)\,c_h x\log^{-2}x(\log\log x)^{r-2}.$$

4. Results for the upper bounds.

PROPOSITION 2. *Let $\nu(a)$ be the number of different prime factors of a. For $r\geqslant 1$, $1\leqslant i\leqslant r$,*

$$|\{P_r\colon p+h=P_r,\ p\leqslant x,\ \nu(P_r)=i\}|\ll c_h x\log^{-2}x(\log\log x)^{i-1}.$$

Proof. By Lemma 1, we need only consider the case of $r\geqslant 2$.

Let $p_1,\ldots,p_i$ denote the i different prime factors of P_r in Proposition 2, $p_1<\ldots<p_i$, and let δ be a fixed number with $0<\delta<1$. Set

$$|\{P_r\colon p+h=P_r,\ p\leqslant x,\ \nu(P_r)=i\}| \tag{17}$$
$$= |\{P_r\colon p+h=P_r,\ p\leqslant x,\ \nu(P_r)=i,\ P_r<x^{\delta}\}|$$
$$+|\{P_r\colon p+h=P_r,\ p\leqslant x,\ \nu(P_r)=i,\ P_r\geqslant x^{\delta}\}|\overset{\text{say}}{=}\#_1+\#_2.$$

Clearly

$$\#_1<x^{\delta}\ll x\log^{-3}x. \tag{18}$$

By the sieve method we have

$$\#_2 = |\{p+h\colon p\leqslant x,\ p+h=P_r,\ \nu(P_r)=i,\ P_r\geqslant x^{\delta}\}| \tag{19}$$
$$\leqslant\sum_{p_1<\ldots<p_i<(x+h)/(p_1\cdot\ldots\cdot p_{i-1}),\,p_i\geqslant x^{\delta/r}} S(\mathscr{A}_{p_1\cdot\ldots\cdot p_i};\mathscr{P}_{p_1\cdot\ldots\cdot p_i},x+h)\overset{\text{say}}{=}\sum{}^{(i)}.$$

This is because, for $a\in\mathscr{A}$ (recall $\mathscr{A}=\{p+h\colon p\leqslant x\}$) with $a=p_1\cdot\ldots\cdot p_i m$, $\nu(a)=i$, and $p_1<\ldots<p_i$, we have

$$p_i<(x+h)/(p_1\cdot\ldots\cdot p_{i-1})\quad\text{and}\quad(a,P_{p_1\cdot\ldots\cdot p_i}(x+h))=1.$$

Moreover,

$$p_i^r\geqslant P_r\geqslant x^{\delta}\Rightarrow p_i\geqslant x^{\delta/r}.$$

Hence such an a must be numbered in $\sum^{(i)}$, and (19) follows.

For $1\leqslant j\leqslant i$, hereafter let $(p_1,\ldots,p_j)$ denote

$$p_1<\ldots<p_j<\left((x+h)/(p_1\cdot\ldots\cdot p_{j-1})\right)^{1/(i-j+1)}$$

(this last inequality follows from $p_1\cdot\ldots\cdot p_{j-1}p_j^{i-(j-1)}\leqslant p_1\cdot\ldots\cdot p_i<x+h$).

By the meaning of the sifting function S and Lemma 5,

$$(20)\qquad \sum^{(i)} \leqslant \sum_{(p_1,\dots,p_i),\, p_i \geqslant x^{\delta/r}} S(\mathscr{A}_{p_1\cdot\dots\cdot p_i};\mathscr{P}_{p_1\cdot\dots\cdot p_{i-1}}, p_i)$$

(if $i=1$, see (29) below)

$$\leqslant \sum_{(p_1,\dots,p_{i-1}),\, p_1\cdot\dots\cdot p_{i-1} \leqslant x^{1/2-\varepsilon'}} S(\mathscr{A}_{p_1\cdot\dots\cdot p_{i-1}};\mathscr{P}_{p_1\cdot\dots\cdot p_{i-1}}, x^{\delta/r})$$

$$+ \sum_{(p_1,\dots,p_{i-1}),\, p_1\cdot\dots\cdot p_{i-1} > x^{1/2-\varepsilon'}} S(\mathscr{A}_{p_1\cdot\dots\cdot p_{i-1}};\mathscr{P}_{p_1\cdot\dots\cdot p_{i-2}}, p_{i-1}) \overset{\text{say}}{=} \sum\nolimits_1 + \sum\nolimits_1'$$

where ε' is a fixed number with $\varepsilon < \varepsilon' < 1/2$.

To estimate $\sum_1$, let

$$X = \frac{\omega(p_1\cdot\dots\cdot p_{i-1})}{p_1\cdot\dots\cdot p_{i-1}} \operatorname{li} x, \qquad D = x^{1/2-\varepsilon}/(p_1\cdot\dots\cdot p_{i-1}),$$

$$z = x^{\delta/r}, \qquad V(z) = \prod_{p \mid P_{p_1\cdot\dots\cdot p_{i-1}}(z)} (1-\omega(p)/p)$$

in (1).

Note that $D \geqslant x^{\varepsilon'-\varepsilon}$ provided $p_1\cdot\dots\cdot p_{i-1} \leqslant x^{1/2-\varepsilon'}$, hence $\log^{-1/3} D \ll \log^{-1/3} x$, $s = \log D/\log z \geqslant (\varepsilon'-\varepsilon) r/\delta$, $F(s) \ll 1$ (since F is decreasing). In addition, from (3) and $\omega(p) = p/(p-1)$, here we have

$$V(z) = \prod_{p \mid P(z)} (1-\omega(p)/p) \prod_{t=1}^{i-1} \left(1-(p_t-1)^{-1}\right)^{-1}$$

$$\ll c(\omega) e^{-\gamma} \log^{-1} z \left(1+O(\log^{-1} z)\right) \ll c_h \log^{-1} x.$$

Therefore by Lemma 2,

$$(21)\qquad \sum\nolimits_1 \ll c_h x \log^{-2} x \sum\nolimits^{+} + R$$

where

$$\sum\nolimits^{+} = \sum_{(p_1,\dots,p_{i-1}),\, p_1\cdot\dots\cdot p_{i-1} \leqslant x^{1/2-\varepsilon'}} \frac{\omega(p_1\cdot\dots\cdot p_{i-1})}{p_1\cdot\dots\cdot p_{i-1}},$$

$$R = \sum_{\substack{(p_1,\dots,p_{i-1})\\ p_1\cdot\dots\cdot p_{i-1} \leqslant x^{1/2-\varepsilon'}}} \ \sum_{\substack{m < x^{1/2-\varepsilon}/(p_1\cdot\dots\cdot p_{i-1})\\ m \mid P_{p_1\cdot\dots\cdot p_{i-1}}(x^{\delta/r})}} |\pi(x; p_1\cdot\dots\cdot p_{i-1} m, -h) - \operatorname{li} x/\varphi(p_1\cdot\dots\cdot p_{i-1} m)|.$$

By an elementary argument and the Prime Number Theorem,

$$(22)\qquad \sum\nolimits^{+} \leqslant \left(\sum_{p \leqslant x^{1/2-\varepsilon'}} \frac{\omega(p)}{p}\right)^{i-1} \Big/ (i-1)! \leqslant \left(\sum_{p<x} \frac{1}{p-1}\right)^{i-1} \Big/ (i-1)!$$

$$\ll (\log\log x)^{i-1}.$$

Similarly to the argument for (7) (cf. the explanation before (7)), by Lemma 3 with $A = i+1$,

$$(23)\qquad R \ll \log^{i-1} x \cdot R_D \ll \log^{i-1} x \cdot x \log^{-(i+1)} x \ll x \log^{-2} x.$$

From (21), (22), (23),

$$(24)\qquad \sum\nolimits_1 \ll c_h x \log^{-2} x (\log\log x)^{i-1}.$$

Now we turn to the estimation for $\sum_1'$. By the meaning of the sifting function S and Lemma 5 again, similarly to (20), we may generally have

$$(25)\qquad \sum\nolimits_j' \leqslant \sum\nolimits_{j+1} + \sum\nolimits_{j+1}', \qquad 1 \leqslant j \leqslant i-2$$

where

$$\sum\nolimits_j = \sum_{(p_1,\dots,p_{i-j}),\, p_1\cdot\dots\cdot p_{i-j} \leqslant x^{1/2-\varepsilon'}} S(\mathscr{A}_{p_1\cdot\dots\cdot p_{i-j}};\mathscr{P}_{p_1\cdot\dots\cdot p_{i-j}}, x^{(1/2-\varepsilon')/(i-j+1)}),$$

$$2 \leqslant j \leqslant i-1,$$

$$\sum\nolimits_j' = \sum_{(p_1,\dots,p_{i-j}),\, p_1\cdot\dots\cdot p_{i-j} > x^{1/2-\varepsilon'}} S(\mathscr{A}_{p_1\cdot\dots\cdot p_{i-j}};\mathscr{P}_{p_1\cdot\dots\cdot p_{i-j-1}}, p_{i-j}),$$

$$1 \leqslant j \leqslant i-1.$$

Hence

$$(26)\qquad \sum\nolimits_1' \leqslant \sum\nolimits_2 + \sum\nolimits_2' \leqslant \dots \leqslant \sum_{j=2}^{i-1} \sum\nolimits_j + \sum\nolimits_{i-1}'.$$

By the method of estimating $\sum_1$, we can get

$$(27)\qquad \sum\nolimits_j \ll c_h x \log^{-2} x (\log\log x)^{i-j}, \qquad 2 \leqslant j \leqslant i-1.$$

As for

$$\sum\nolimits_{i-1}' = \sum_{p_1 < x^{1/2},\, p_1 > x^{1/2-\varepsilon'}} S(\mathscr{A}_{p_1};\mathscr{P}, p_1),$$

by Lemma 5, Lemma 2 (with $X = \operatorname{li} x$, $D = x^{1/2-\varepsilon}$, $z = x^{1/2-\varepsilon'}$, thus $s = (1/2-\varepsilon)/(1/2-\varepsilon')$, $F(s) \ll 1$, $V(z) \ll c_h \log^{-1} x$) and Lemma 3 (with $A = 2$), we have

$$(28)\qquad \sum\nolimits_{i-1}' \leqslant S(\mathscr{A};\mathscr{P}, x^{1/2-\varepsilon'}) \ll c_h x \log^{-2} x + R_D$$

$$\ll c_h x \log^{-2} x + O(x \log^{-2} x) \ll c_h x \log^{-2} x.$$

If $i=1$, the same method will give

$$(29)\qquad \sum\nolimits^{(1)} \leqslant \sum_{x^{\delta/r} \leqslant p_1 < x+h} S(\mathscr{A}_{p_1};\mathscr{P}, p_1) \leqslant S(\mathscr{A};\mathscr{P}, x^{\delta/r}) \ll c_h x \log^{-2} x.$$

By (26), (27), (28),

$$(30)\qquad \sum\nolimits_1' \ll c_h x \log^{-2} x (\log\log x)^{i-2}, \qquad i \geqslant 2.$$

From (20), (24), (30),

$$(31)\qquad \sum\nolimits^{(i)} \ll c_h x \log^{-2} x (\log\log x)^{i-1}, \qquad i \geqslant 2.$$

Finally, combining (17), (18), (19), (29) with (31), Proposition 2 is proved.

THEOREM 1. $|\{P_r\colon p+h = P_r,\ p \leqslant x\}| \ll c_h x \log^{-2} x (\log\log x)^{r-1},\ r \geqslant 1.$

Proof. Since

$$|\{P_r\colon p+h = P_r,\ p \leqslant x\}| = \sum_{i=1}^{r} |\{P_r\colon p+h = P_r,\ p \leqslant x,\ \nu(P_r) = i\}|,$$

by Proposition 2, Theorem 1 follows.

5. More precise results for the lower bounds.

THEOREM 2. *Let δ be a fixed number with $0 < \delta < 1$. Then for any $r \geqslant 3$,*

$$|\{p\colon p \leqslant x,\ p+h = p_1 \cdot \ldots \cdot p_{r-1} \text{ or } p_1 \cdot \ldots \cdot p_r,\ p_r > p_{r-1} > \ldots$$
$$\ldots > p_1 \geqslant \exp(\log^\delta x)\}|$$
$$> 0.965\big((1-\delta)^{r-2}/(r-2)!\big) c_h x \log^{-2} x (\log\log x)^{r-2}.$$

Proof. From Proposition 2 we have

$$|\{p\colon p \leqslant x,\ p+h = p_1 \cdot \ldots \cdot p_{r-2}\}| \ll c_h x \log^{-2} x (\log\log x)^{r-3}, \quad r \geqslant 3. \tag{32}$$

From Proposition 1 and (32), Theorem 2 follows.

COROLLARY 3. *For all $r \geqslant 3$,*

$$|\{p\colon p \leqslant x,\ p+h = p_1 \cdot \ldots \cdot p_{r-1} \text{ or } p_1 \cdot \ldots \cdot p_r,\ p_r > p_{r-1} > \ldots > p_1\}|$$
$$\geqslant 0.965\big((1-\delta)^{r-2}/(r-2)!\big) \geqslant 0.965/(r-2)!.$$

Proof. In Theorem 2 let $\delta \to 0^+$.

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On an estimate for the orders of zeros of Mahler type functions

by

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Nesterenko [6] gives a very good measure of the algebraic independence for the values of functions of Mahler type.

NESTERENKO'S THEOREM [6]. *Let $f_1(z), \ldots, f_m(z)$ be power series in z with coefficients in an algebraic number field K, which converge in some neighborhood U of the point $z = 0$, which satisfy the equalities*

$$f_i(z^d) = a_i(z) f_i(z) + b_i(z), \quad a_i(z), b_i(z) \in K(z), \quad i = 1, \ldots, m,$$

where d is an integer, $d \geqslant 2$, and which are algebraically independent over $C(z)$. Suppose that α is an algebraic number, $\alpha \in U$, $0 < |\alpha| < 1$, and the numbers $\alpha, \alpha^d, \alpha^{d^2}, \ldots$ are distinct from the poles of the functions $a_i(z)$ and $b_i(z)$. Then there exists a function $\varphi(s)$ such that, for any H and $s \geqslant 1$ with $H \geqslant \varphi(s)$ and for any polynomial $R \in \mathbf{Z}[x_1, \ldots, x_m]$ whose degree does not exceed s and whose coefficients are not greater than H in absolute value, the following inequality holds:

$$|R(f_1(\alpha), \ldots, f_m(\alpha))| > H^{-\gamma s^m}, \tag{0}$$

where γ is a positive constant which depends only on α and the functions $f_1, \ldots, f_m$.

The above function $\varphi(s)$ is ineffective in the parameter s. In order to make it effective, we prove an estimate for the orders of zeros of such functions. By using our estimate, Becker [1] shows that the right side of the estimate (0) can be replaced by $\exp(-\gamma s^m(\log H + s^{2m+2}))$ for any H and $s \geqslant 1$. (See also Becker and Nishioka [2].)

For a formal power series $f(z)$, we denote by $\operatorname{ord} f(z)$ the order of zeros of $f(z)$ at $z = 0$.

THEOREM. *Let $f_1(z), \ldots, f_m(z) \in C[[z]]$ be formal power series with coefficients in a field C of characteristic 0 and satisfy*

$$f_i(z^d) = \frac{A_i(z, f_1(z), \ldots, f_m(z))}{A_0(z, f_1(z), \ldots, f_m(z))} \quad (1 \leqslant i \leqslant m),$$

where $d \geqslant 2$ is a rational integer and $A_i(z, x_1, \ldots, x_m) \in C[z, x_1, \ldots, x_m]$ $(0 \leqslant i \leqslant m)$ are polynomials with $\deg_z A_i \leqslant s$ and $\operatorname{tot.deg}_x A_i \leqslant t$. Suppose that