Consider the f_i orbits

$$\{H\tau_1\sigma_i, H\tau_1\sigma_i\Phi, \ldots, H\tau_1\sigma_i\Phi^{e_i-1}\},$$
$$\{H\tau_2\sigma_i, H\tau_2\sigma_i\Phi, \ldots, H\tau_2\sigma_i\Phi^{e_i-1}\}, \ldots, \{H\tau_{f_i}\sigma_i, \ldots, H\tau_{f_i}\sigma_i\Phi^{e_i-1}\}.$$

We claim that they are distinct. Suppose $H\tau_u\sigma_i = H\tau_v\sigma_i\Phi^j$. Then

$$\tau_u\sigma_i = h\tau_v\sigma_i\Phi^j$$

which means

$$\tau_u\sigma_i\Phi^{-j}\sigma_i^{-1}\tau_v^{-1} = \tau_u\tau_v^{-1}(\tau_v\sigma_i\Phi^{-j}\sigma_i^{-1}\tau_v^{-1}) = h.$$

But $\tau_v\sigma_i\Phi^{-j}\sigma_i^{-1}\tau_v^{-1} \in D(\mathfrak{B}|q)$ since $D(\tilde{\mathfrak{B}}|q)$ is normal in $D(\tilde{\mathfrak{P}}|p)$. So τ_u and τ_v are in the same coset, contradicting our choice of τ_i's. Notice that $\tau_u\sigma_i\Phi^j(\mathfrak{P}) = \sigma_i(\mathfrak{P}) = \tilde{\mathfrak{P}}$. So the primes corresponding to these orbits will have degree e_i.

To complete the proof we have to show that any coset $H\sigma$ is of the form $H\tau_k\sigma_i\Phi^j$ for some i.

Suppose $\sigma(\mathfrak{P}) \cap k = P_i = \sigma_i(\mathfrak{P}) \cap k$. Then there exists $h \in \mathrm{Gal}(\bar{k}/k)$ such that

$$h\sigma(\mathfrak{P}) = \sigma_i(\mathfrak{P}), \quad \text{i.e.} \quad \sigma_i^{-1}h\sigma(\mathfrak{P}) = \mathfrak{P}.$$

Therefore

$$\sigma_i^{-1}h\sigma = \tau, \quad \tau \in D(\tilde{\mathfrak{P}}|p),$$

which means

$$h\sigma = \sigma_i\tau = \sigma_i\tau\sigma_i^{-1}\sigma_i.$$

Now $\sigma_i\tau\sigma_i^{-1} \in D(\tilde{\mathfrak{P}}|p)$. Therefore there is an h' such that

$$\sigma_i\tau\sigma_i^{-1} = h'\sigma_i\Phi^w\sigma_i^{-1}\tau_k$$

which means $h\sigma = h'\sigma_i\Phi^w\sigma_i^{-1}\tau_k\sigma_i = h'\tau_k\sigma_i\Phi^s\sigma_i^{-1}\sigma_i$ (since $D(\tilde{\mathfrak{B}}|q)$ is normal in $D(\tilde{\mathfrak{P}}|p)$) $= h'\tau_k\sigma_i\Phi^s$. Hence

$$H\sigma = H\tau_k\sigma_i\Phi^s.$$

This completes the proof.

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48, Nalpa Way
Duncraig 6023
Western Australia
THE INSTITUTE OF MATHEMATICAL SCIENCES
Madras–600 113
India

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Polynomials with high multiplicity

by

FRANCESCO AMOROSO (Pisa)

0. Introduction. Let S be a non-empty finite subset of $\mathbf{C}^n$. Following Waldschmidt (see [W2], § 1.3e)) we define $\omega_M(S)$ as the minimum degree of an algebraic hypersurface having a singularity of order $\geq M$ at any point of S. We are looking for inequalities between $\omega_1(S)$ and $\omega_M(S)$, $M > 1$. Trivially, we have

$$\frac{1}{M}\omega_M(S) \leq \omega_1(S). \tag{1}$$

In the opposite sense, using powerful tools from complex analysis, Waldschmidt proved

$$\frac{1}{n}\omega_1(S) \leq \frac{1}{M}\omega_M(S) \tag{2}$$

(see [W2], § 7.5b)). The last inequality follows from Bombieri–Skoda's existence theorem, which in turn derives from some L^2-estimates and from existence theorems for the operator $\bar{\partial}$, due to Hörmander.

Weaker results of the following kind:

$$\frac{1}{c_n}\omega_1(S) \leq \frac{1}{M}\omega_M(S) \tag{2'}$$

where c_n is some constant greater than n, were obtained by Masser and Wüstholz independently (see [M] and [Wu]).

More recently, using deep arguments from projective geometry, Esnault and Viehweg (see [E–V]) have obtained the following improvement of (2):

$$\frac{\omega_1(S)+1}{n} \leq \frac{1}{M}\omega_M(S) \quad \text{for } n > 1.$$

A conjecture of J. P. Demailly asserts that one should have

$$\frac{\omega_1(S)+n-1}{n} \leq \frac{1}{M}\omega_M(S) \quad \text{for } n \geq 1.$$

In this paper we give some results of the type (2′) in the ring $\mathbf{Z}[x_1, \ldots, x_n]$ with explicit bounds for the height of the polynomials.

Given a polynomial $f\in \mathbf{Z}[x_0,\dots,x_n]$ we define its size $t(f)$ as $t(f)=\deg f+\ln H(f)$, where $H(f)$ is the maximum absolute value of its coefficients. For a positive integer M we also define $\bar{\omega}_M(S)$ as the minimum size of a polynomial $f\in \mathbf{Z}[x_1,\dots,x_n]$ such that the hypersurface $\{f=0\}$ has a singularity of order $\geqslant M$ at any point of S (if no such polynomial exists, we let $\bar{\omega}_M(S)=+\infty$). Of course, we have the inequality

$$\bar{\omega}_M(S)\geqslant \omega_M(S).$$

As in the "geometric" case, we have a simple inequality between $\bar{\omega}_1$ and $\bar{\omega}_M$:

$$\frac{1}{M}\bar{\omega}_M(S)\leqslant \bar{\omega}_1(S)+n\log\big(1+\bar{\omega}_1(S)\big).$$

We claim that a relation in the opposite direction exists. In fact we shall prove:

THEOREM 1. *There exists an effective constant $c>0$ depending only on n such that*

$$\frac{1}{c}\bar{\omega}_1(S)\leqslant \frac{1}{M}\bar{\omega}_M(S).$$

A need for results of this kind arises in the study of certain problems connected with relations between transcendence measures in codimension 1 and approximation measures in dimension $n-1$, as we shall show in the last section of this paper.

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I would also like to thank Marc Chardin, Patrice Philippon and Michel Waldschmidt for their useful suggestions. In particular, I am indebted to Philippon for suggesting a new definition for the multiplicity of an ideal at a point.

1. Auxiliary assertions. For the proof of Theorem 1 we use the theory of eliminating forms, as developed by Yu. V. Nesterenko (see [N1], [N2] and [N3]). We work over a ring $\mathbf{R}$ which will be either $\mathbf{Z}$ or $\mathbf{C}$. For an arbitrary polynomial $P\in \mathbf{R}[y_0,\dots,y_m]$ we denote by $d^\circ P$ its total degree. We further denote by A the ring of polynomials in the $n+1$ variables $x_0,\dots,x_n$ over $\mathbf{R}$. We define the rank of a prime ideal $\mathfrak{p}$ of A as the largest integer k for which there exists a strictly increasing chain of length k of prime ideals contained in $\mathfrak{p}$. The rank of an ideal $I\subset A$ will be defined as the minimum rank of the prime ideals containing I. In what follows we denote by I a homogeneous ideal of A with $I\cap \mathbf{R}=(0)$ and such that $I\mathbf{C}[x_0,\dots,x_n]$ is unmixed of rank $n+1-r$. If A and B are polynomial rings over $\mathbf{R}$, $\varrho\colon A\to B$ a homomorphism and A', B' polynomial rings over A and B, we shall denote by the same ϱ the homomorphism $\varrho\colon A'\to B'$ defined in the natural way. Similarly, if v is a valuation over some field K and B is a polynomial ring over K, we shall

denote by the same v the valuation over the quotient field of B defined by taking for $v(P)$, $P\in B$, the minimum value of v on the coefficients of P.

DEFINITION 1. Let $U=\{u_j^i,\ i=1,\dots,r;\ j=0,\dots,n\}$ be a set of independent variables and let

$$L_i=\sum_{j=0}^{n}u_j^i x_j,\qquad i=1,\dots,r$$

be r linear forms. We define the ideal $\bar{I}$ of $\mathbf{R}[U]$ as the set of polynomials $G\in \mathbf{R}[u_j^i]$ for which there exists a natural number M such that

$$Gx_j^M\in(I,L_1,\dots,L_r)\qquad\text{for } j=0,\dots,n.$$

$\bar{I}$ is a principal ideal (see [N1], Prop. 2). We say that a generator F of $\bar{I}$ is an *eliminating form of I* and we define $N(I)$ as $\frac{1}{r}d^\circ F$. If $\mathbf{R}=\mathbf{Z}$ we define the *size* $t(I)$ *of* I as $t(I)=N(I)+\ln H(F)$.

The following factorization formula is available (see [N2], Lemma 2):

PROPOSITION 1. *Let F be an eliminating form of I. Then*

$$F=a\prod_{h=1}^{N(I)}L_r(\alpha^h)$$

where $a\in \mathbf{R}[u^1,\dots,u^{r-1}]$ and

$$\alpha^h=(\alpha_0^h,\dots,\alpha_n^h)\quad \textit{with } \alpha_j^h\in\overline{\mathbf{Q}(u^1,\dots,u^r)}\quad \textit{for } h=1,\dots,N(I),\ j=0,\dots,n.$$

Moreover, if $x_j\notin\mathfrak{p}$ for any prime ideal $\mathfrak{p}$ of I, we may assume $\alpha_j^h=1$ for $h=1,\dots,N(I)$.

Let $S^1,\dots,S^r$ be skew-symmetric matrices in the new variables s_{kl}^i, $1\leqslant i\leqslant r$; $0\leqslant k,l\leqslant n$ which are connected only by the relations

$$s_{kl}^i+s_{lk}^i=0.$$

We denote by S the corresponding set of independent variables, $S=\{s_{kl}^i,\ 1\leqslant i\leqslant r;\ 0\leqslant k<l\leqslant n\}$. Let $\theta\colon \mathbf{C}[U]\to\mathbf{C}[S,x]$ be the homomorphism given on each u^i by $u^i\mapsto S^i\cdot x$. For $\omega\in\mathbf{C}^{n+1}\setminus\{0\}$ we further denote by $\varrho_\omega\colon \mathbf{C}[x]\to\mathbf{C}$ the homomorphism which maps x to ω; the composed homomorphism $\varrho_\omega\circ\theta$ will be denoted by θ_ω.

If $\mathbf{R}=\mathbf{Z}$ we define the norm $\|I\|_\omega$ as

$$\|I\|_\omega=|\omega|^{-rN(I)}H(\theta_\omega F)$$

where F is an eliminating form of I.

For any $f\in A$ we define its multiplicity $m_\omega(f)$ at $\omega\in\mathbf{C}^{n+1}\setminus\{0\}$ in the usual way,

$$m_\omega(f)=\min\left\{a\mid\ \exists j_1,\dots,j_a\in[0,\dots,n]\ \text{such that}\ \varrho_\omega\frac{\partial^a f}{\partial x_{j_1}\dots\partial x_{j_a}}\neq 0\right\}.$$

If $F \in \mathbf{R}[U]$ we define $i_\omega(F)$ as

$$i_\omega(F) = m_\omega(\theta F) = \min_{f \in J_F} m_\omega(f)$$

where $J_F \subset A$ is the ideal generated by the coefficients of the products of power of the independent variables $s^i_{ik} \in S$ in θF. It is the same as taking

$$i_\omega(F) = \min\left\{a \mid \exists j_1, \ldots, j_a \in [0, \ldots, n] \text{ such that } \varrho_\omega \frac{\partial^a \theta F}{\partial x_{j_1} \ldots \partial x_{j_a}} \not\equiv 0\right\}.$$

Notice that i_ω defines a valuation over $\mathbf{R}(U)$.

Now we want to make clear some important properties of i_ω. First of all, it would be very agreeable to show that $i_\omega(F) = i_\omega(F(u^1, \ldots, u^{r-1}, T\omega))$ for "almost-all" skew-symmetric matrices T, if F is an eliminating form. The geometric meaning of this is that the generic hyperplane section through ω of some algebraic variety V has the same order of multiplicity at ω as V. We begin with a simple lemma:

LEMMA 1. *Let ν_1, ν_2 be two valuations over $\mathbf{C}(U)$. Assume that the following assertions hold:*

1) *for any eliminating form F there exist $r-1$ vectors $v^2, \ldots, v^r \in \mathbf{C}^{n+1} \backslash \{0\}$ such that*

$$\nu_i(F) = \nu_i(F(u^1, v^2, \ldots, v^r)), \quad i = 1, 2;$$

2) *for any $\alpha \in \mathbf{C}^{n+1} \backslash \{0\}$ we have:*

$$\nu_1(L^1(\alpha)) \geqslant \nu_2(L^1(\alpha)).$$

Then $\nu_1(F) \geqslant \nu_2(F)$ for any eliminating form F.

Proof. Let F be an eliminating form of an ideal I. We have, with 1),

$$\nu_i(F) = \nu_i(F(u^1, v^2, \ldots, v^r)) = \nu_i(G_1^{e_1} \ldots G_l^{e_l}), \quad i = 1, 2,$$

where $G_1, \ldots, G_l \in \mathbf{C}[u^1]$ are eliminating forms of the prime ideals of codimension n associated to $(I, v^2, \ldots, v^r)$. Thus it is enough to prove Lemma 1 for an eliminating form of a prime ideal $\mathfrak{p} \subset \mathbf{C}[x]$ of codimension n, hence for a linear form, but this follows obviously from 2). ∎

For $\omega \in \mathbf{C}^{n+1} \backslash \{0\}$ we define three other functions $\nu_{i,\omega}$: $\mathbf{C}[U] \to \mathbf{N} \cup \{+\infty\}$, $i = 1, 2, 3$:

$$\nu_{1,\omega}(F) = \min\left\{a \mid \exists j_1, \ldots, j_a \in [0, \ldots, n] \text{ such that } \varrho_\omega \frac{\partial^a \theta^1_\omega F}{\partial x_{j_1} \ldots \partial x_{j_a}} \not\equiv 0\right\},$$

$$\nu_{2,\omega}(F) = \min\left\{a \mid \exists j \in [0, \ldots, n] \text{ such that } \theta_\omega \frac{\partial^a F}{\partial (u^1_j)^a} \not\equiv 0\right\},$$

$$\nu_{3,\omega}(F) = \min\left\{a \mid \exists j_1, \ldots, j_a \in [0, \ldots, n],\ \exists i_1, \ldots, i_a \in [1, \ldots, r] \text{ such that } \tilde{\varrho}_\omega \frac{\partial^a \tilde{\theta} F}{\partial x^{(i_1)}_{j_1} \ldots \partial x^{(i_a)}_{j_a}} \not\equiv 0\right\},$$

where θ^1_ω, $\tilde{\theta}$, $\tilde{\varrho}_\omega$ are the homomorphisms defined as follows:

$$\theta^1_\omega\colon \mathbf{C}[U] \to \mathbf{C}[S, x],$$

$$u^i \mapsto \begin{cases} S^1 x & \text{if } i = 1, \\ S^i \omega & \text{if } i = 2, \ldots, r; \end{cases}$$

$$\tilde{\theta}\colon \mathbf{C}[U] \to \mathbf{C}[S, x^{(1)}, \ldots, x^{(r)}],$$

$$u^i \mapsto S^i x^{(i)}, \quad i = 1, \ldots, r;$$

$$\tilde{\varrho}_\omega\colon \mathbf{C}[x^{(1)}, \ldots, x^{(r)}] \to \mathbf{C},$$

$$x^{(i)} \mapsto \omega, \quad i = 1, \ldots, r.$$

The following proposition, which is due to P. Philippon, shows that these functions take the same values as i_ω on the eliminating forms.

PROPOSITION 2. *For any eliminating form F*

$$\nu_{1,\omega}(F) = \nu_{2,\omega}(F) = \nu_{3,\omega}(F) = i_\omega(F).$$

Proof. Let F be an eliminating form of I. First we prove the equality $\nu_{1,\omega}(F) = \nu_{2,\omega}(F)$. For this we apply Lemma 1 for $j = 0, \ldots, n$ to the valuations $\nu_{1,\omega}$ and

$$\nu_{2,\omega,j}(F) = \min\left\{a \mid \theta_\omega \frac{\partial^a F}{\partial (u^1_j)^a} \not\equiv 0\right\}.$$

Assertion 1 is obviously satisfied. Further we observe that

$$\nu_{1,\omega}(L^1(\alpha)) = \begin{cases} 0 & \text{if } \alpha \not\equiv \omega, \\ 1 & \text{if } \alpha \equiv \omega, \end{cases}$$

$$\nu_{2,\omega,j}(L^1(\alpha)) = \begin{cases} 0 & \text{if } \alpha \not\equiv \omega, \\ 1 & \text{if } \alpha \equiv \omega \text{ and } \omega_j \neq 0, \\ \infty & \text{if } \alpha \equiv \omega \text{ and } \omega_j = 0, \end{cases}$$

where $\alpha \equiv \beta$ means that $\alpha, \beta \in \mathbf{C}^{n+1} \backslash \{0\}$ define the same point in the projective space. Hence Lemma 1 leads to

$$\nu_{1,\omega}(F) = \nu_{2,\omega}(F) = \min_{j=0,\ldots,n} \nu_{2,\omega,j}(F).$$

To prove $\nu_{2,\omega}(F) \geqslant i_\omega(F)$, we recall that Proposition 1 of [P2] implies

$$x_j^M \theta \frac{\partial^a F}{\partial (u^1_j)^a} \in \left(\frac{\partial^a \theta f}{\partial x_{j_1} \ldots \partial x_{j_a}} \mid f \in J_F, j_1, \ldots, j_a \in [0, \ldots, n]\right)$$

for some integer $M \geqslant 1$.

The inequality $\nu_{3,\omega}(F) \geqslant \nu_{1,\omega}(F)$ derives immediately from Proposition 2 of [P2], as explained there.

Finally, the relation $i_\omega(F) \geqslant \nu_{3,\omega}(F)$ is obvious. ∎

COROLLARY 1. *For any eliminating form F we have*

$$i_\omega(F) = i_\omega\big(F(u^1, \dots, u^{r-1}, T\omega)\big)$$

for a generic skew-matrix T.

Now we may define the multiplicity of I at ω.

DEFINITION 2. Let $\omega \in C^{n+1} \setminus \{0\}$ and I be as in Definition 1. Let F be an eliminating form of I; we define the *multiplicity* $i_\omega(I)$ *of* I *at* ω as $i_\omega(I) = i_\omega(F)$.

Remark. It is easy to see that $i_\omega(I) = 0$ if and only if ω is in the projective variety generated by I. It is also possible to prove that $i_\omega(I) = 1$ for a prime ideal I if and only if the projective variety generated by I is smooth at ω (see [A], Lemma 2.2).

The following lemma shows the equivalence between $i_\omega((f))$ and the usual notion of multiplicity of an algebraic hypersurface at a point.

LEMMA 2. *Let* $f \in R[x_0, \dots, x_n]$ *and* $\omega \in C^{n+1} \setminus \{0\}$. *Then* $i_\omega((f)) = m_\omega(f)$.

Proof. Let us assume $\omega_0 \neq 0$, and let $\Delta_0, \Delta_1, \dots, \Delta_n$ be the cofactors of $x_0, x_1, \dots, x_n$ in the matrix

$$\begin{bmatrix} x_0 & x_1 & \cdots & x_n \\ u_0^1 & u_1^1 & \cdots & u_n^1 \\ \cdot & \cdot & \cdots & \cdot \\ u_0^n & u_1^n & \cdots & u_n^n \end{bmatrix}.$$

Clearly $F(u) = f(\Delta_0, \dots, \Delta_n)$ is an eliminating form of (f) (see [N3], Lemma 2). Moreover, $\theta_\omega \Delta_j = Ax_j$ for some $A \in C[s_{kl}^i, x_0, \dots, x_n]$ with $A(\omega) \not\equiv 0$ (see [N3], p. 432). Hence

$$i_\omega((f)) = i_\omega(F) = m_\omega(A^{d^\circ f} f) = m_\omega(A^{d^\circ f}) \cdot m_\omega(f) = m_\omega(f). \ \blacksquare$$

Let

$$g \in A \setminus \bigcup_{h=1}^{t} \mathfrak{p}'_h$$

where $\mathfrak{p}'_1, \dots, \mathfrak{p}'_t$ are the prime ideals associated to I. We define the resultant $\mathrm{Res}(F, g)$ of F and g as

$$\mathrm{Res}(F, g) = a^{d^\circ g} \prod_{h=1}^{N(I)} g(\alpha^h).$$

Lemma 4 of [N2] ensures $\mathrm{Res}(F, g) \in R[u^1, \dots, u^{r-1}]$. Moreover, $\mathrm{Res}(F, g) = bE_1^{e_1} \dots E_s^{e_s}$ where $b \in R$ and $E_1, \dots, E_s$ are eliminating forms of the minimal prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_s$ of (I, g) such that $\mathfrak{p}_l \cap R = (0)$ for $l = 1, \dots, s$ (see [N2], Lemma 6). We define $\mathrm{Res}(I, g)$ as the corresponding intersection of symbolic powers

$$\mathrm{Res}(I, g) = \mathfrak{p}_1^{(e_1)} \cap \dots \cap \mathfrak{p}_s^{(e_s)}.$$

The following propositions show the behaviour of the quantities $N(I)$, $i_\omega(I)$, $t(I)$ and $\|I\|_\omega$ with respect to the primary decomposition and the resultant operation.

PROPOSITION 3. *Let* $I = Q_1 \cap \dots \cap Q_t$ *be an irreducible primary decomposition in which for* $l \leqslant s$ *we have* $Q_l \cap R = (0)$ *and* $Q_{s+1} \cap \dots \cap Q_t \cap R = (b)$, $b \in R \setminus \{0\}$. *Furthermore, for* $l \leqslant s$ *suppose that* $\mathfrak{p}_l = \sqrt{Q_l}$ *and* e_l *is the exponent of the ideal* Q_l. *Let* $E_1, \dots, E_s$ *be eliminating forms of* $\mathfrak{p}_1, \dots, \mathfrak{p}_s$. *Then* $F = bE_1^{e_1} \dots E_s^{e_s}$ *is an eliminating form of I. Hence*

(i) $$N(I) = \sum_{l=1}^{s} e_l N(\mathfrak{p}_l);$$

(ii) $$i_\omega(I) = \sum_{l=1}^{s} e_l i_\omega(\mathfrak{p}_l).$$

Moreover, if $R = Z$,

(iii) $$\log|b| + \sum_{l=1}^{s} e_l t(\mathfrak{p}_l) - cN(I) \leqslant t(I) \leqslant \log|b| + \sum_{l=1}^{s} e_l t(\mathfrak{p}_l) + cN(I);$$

(iv) $$\log|b| + \sum_{l=1}^{s} e_l \|\mathfrak{p}_l\|_\omega - cN(I) \leqslant \|I\|_\omega \leqslant \log|b| + \sum_{l=1}^{s} e_l \|\mathfrak{p}_l\|_\omega + cN(I)$$

where c is some positive constant depending only on n.

Proof. For (i), (iii) and (iv) see [N3], Proposition 2 and [W1], Lemma 4.2.14; (ii) is obvious. $\blacksquare$

PROPOSITION 4. *Let g be as above. Then*

(i) $$N(\mathrm{Res}(I, g)) \leqslant N(I) d^\circ g;$$

(ii) $$i_\omega(\mathrm{Res}(I, g)) \geqslant i_\omega(I) i_\omega((g)).$$

Moreover, if $R = Z$,

(iii) $$t(\mathrm{Res}(I, g)) \leqslant (3 + n + r\ln(n+1))\, t(I)\, t(g);$$

(iv) $$\log\|(\mathrm{Res}(I, g))\|_\omega \leqslant ct(I)t(g) + \log\max(\|I\|_\omega, |\omega|^{-d^\circ g}|g(\omega)|)$$

where c is some positive constant depending only on n.

Proof. (i) See [N3], Lemma 5.

(ii) We assume $\omega_t \neq 0$; let $N = i_\omega((g))$, $\delta = N(I)$, $D = d^\circ g$ and let F be an eliminating form of I. According to Proposition 1, we have

$$F = a \prod_{h=1}^{\delta} L_r(\alpha^h).$$

We may extend the valuation

$$v\colon C(u^1, \dots, u^{r-1}) \to Z$$

defined by $v(F/G) = i_\omega(F) - i_\omega(G)$ to a valuation over $K = C(u^1, \dots, u^{r-1}, \alpha_i^h)$ which we still denote by v. Moreover, we may extend v to the polynomial ring $K[u^r]$ in the following way. Let $P \in K[u^r]$ and assume

$$P(S^r\omega) = \sum_{m \in \Lambda} b_m m$$

where $\Lambda \subset C[s_{kl}^r]$ is a finite set of monomials and $b_m \in K$ $\forall m \in \Lambda$. Then we define $v(P)$ as

$$v(P) = \min_{m \in \Lambda} v(b_m).$$

Lemma 1 gives $v(G) = i_\omega(G)$ for any $G \in C[u^1, \ldots, u^r]$. We have

$$i_\omega(\mathrm{Res}(F,g)) = v(\mathrm{Res}(F,g)) = v(a^D \prod_{h=1}^{\delta} g(\alpha^h)) = Dv(a) + \sum_{h=1}^{\delta} v(g(\alpha^h)).$$

The Taylor expansion of g gives

$$g(x) = \sum_{\substack{\lambda = (\lambda_0, \ldots, \hat{\lambda}_t, \ldots, \lambda_n) \\ N \leqslant |\lambda| \leqslant D}} c_\lambda x_t^{D-|\lambda|} \prod_{\substack{j=1 \\ j \neq t}}^{n} (x_t \omega_j - x_j \omega_t)^{\lambda_j}, \quad c_\lambda \in C.$$

Hence

$$v(g(\alpha^h)) \geqslant N \min_{\substack{1 \leqslant j \leqslant n \\ j \neq t}} v(\alpha_t \omega_j - \alpha_j \omega_t)$$

$$\geqslant N \min_{1 \leqslant t < j \leqslant n} v(\alpha_t \omega_j - \alpha_j \omega_t) = Nv(S^r \omega \cdot \alpha^h).$$

Thus

$$i_\omega(\mathrm{Res}(F,g)) \geqslant Dv(a) + N \sum_{h=1}^{\delta} v(S^r \omega \cdot \alpha^h)$$

$$\geqslant Nv(F(u^1, \ldots, u^{r-1}, S^r \omega)) = Ni_\omega(F).$$

(iii) See [N3], Lemma 5.

(iv) See [N3], Proposition 3. ∎

For the proof of Theorem 1 we should find a lower bound for the exponent of some primary components associated with I. This is the aim of the following lemma:

LEMMA 3. *We use the same notations as in Proposition 3. Assume $i_\omega(I) \geqslant M$ for the generic point ω of $V_{P(C^n)}(\mathfrak{p}_1)$. Then $e_1 \geqslant M$.*

Proof. We observe that $\partial E_1/\partial u_0^1 \notin \bar{\mathfrak{p}}_1$ since its total degree is less than $d^\circ E_1$. Thus, taking into account Proposition 1, we have

$$i_\omega(I) \geqslant M;$$

$$i_\omega(\mathfrak{p}_h) = 0 \quad \text{for } h = 2, \ldots, l;$$

$$i_\omega(\mathfrak{p}_1) = 1$$

for the generic point ω of $V_{P(C^n)}(\mathfrak{p}_1)$. Hence by Proposition 3(ii)

$$M \leqslant i_\omega(I) = e_1 i_\omega(\mathfrak{p}_1) + \ldots + e_l i_\omega(\mathfrak{p}_l) = e_1. \ ∎$$

2. Proof of Theorem 1. Now we assume $R = Z$. For a homogeneous prime ideal $\mathfrak{p} \subset A$ we define $S_\mathfrak{p}(H, s)$ as the set of residues modulo $\mathfrak{p}$ of homogeneous polynomials $g \in Z[x_0, \ldots, x_n]$ of degree s whose coefficients do not exceed H in absolute value. Using an upper bound for the growth of $S_\mathfrak{p}(H, s)$ due to Yu. V. Nesterenko (see [N2], Thorem 3) it is easy to prove the following

COROLLARY 2. *There exists $g \in \sqrt{I}$ such that*

$$t(g) \leqslant 3(6n)^{n+4} t(I)^{1/(n+1-r)}.$$

Proof of Theorem 1. Let S be a non-empty subset of C^n and let $P \in Z[x_1, \ldots, x_n]$ with $t(P) = \tilde{\omega}_M(S) = T$ such that $D^\mu P(\alpha) = 0$ for $\alpha \in S$ and for any multiindex $\mu \in N^n$ such that $|\mu| < M$. Let $f = {}^hP$ be the homogenization of P. Clearly, it is enough to give a homogeneous polynomial g with $t(g) \leqslant cT/M$ such that $g(\alpha) = 0$ for any $\alpha \in V_M$, where

$$V_M = \{\alpha \in P(C^n) \text{ such that } D^\lambda f(\alpha) = 0 \text{ for any } \lambda \in N^{n+1} \text{ with } |\lambda| < M\}.$$

We assume $V_M \neq \emptyset$ and we denote by $c_1, \ldots, c_8$ positive constants depending only on n. Let $t_0, \ldots, t_n \in [0, 1]$ be defined by

$$t_0 = 0, \quad t_k = (n+1-k)^{-1} \quad \text{for } k = 1, \ldots, n.$$

Let $k_0 \leqslant n$ be a natural number which will be specified later. By induction we define a sequence $\{I_k\}_{k=1,\ldots,k_0}$ of pure ideals of rank k:

$k = 1$:

$$I_1 = (f).$$

$k \to k+1$: Let

$$I_k = Q_{1,k} \cap \ldots \cap Q_{l_k,k}$$

be an irreducible primary decomposition of I_k. Let us put $\mathfrak{p}_{j,k} = \sqrt{Q_{j,k}}$ and let us denote by $e_{j,k}$ the exponent of $Q_{j,k}$. After a permutation of $1, \ldots, l_k$, we may assume that there exists an integer $s_k \in [0, \ldots, l_k]$ such that:

$$\begin{cases} D^\lambda f \in \mathfrak{p}_{j,k} & \text{for any } \lambda \in N^{n+1} \text{ with } |\lambda| \leqslant t_k M, \text{ if } j = 1, \ldots, s_k; \\ D^{\lambda^j} f \notin \mathfrak{p}_{j,k} & \text{for some } \lambda^j \in N^{n+1} \text{ with } |\lambda^j| \leqslant t_k M, \text{ if } j = s_k + 1, \ldots, l_k. \end{cases}$$

Let

$$J_k = \bigcap_{j > s_k} Q_{j,k}.$$

If $V_{P(C^n)}(J_k) \cap V_M = \emptyset$ we let $k_0 = k$ (this certainly occurs if $k = n$, since otherwise there would exist an index $j > s_n$ such that $V_{P(C^n)}((\mathfrak{p}_{j,s_n}, D^{\lambda^j} f)) \neq \emptyset$, which is impossible because the homogeneous ideal $(\mathfrak{p}_{j,s_n}, D^{\lambda^j} f)$ has codimension $n+1$).

A classical trick (see for instance [M–W], Ch. 4, Lemma 2 or [P1], Lemma 1.9) allows us to find $\lambda^1, \ldots, \lambda^a \in N^{n+1}$ with $|\lambda^i| \leqslant t_k M$ and $\phi_1, \ldots, \phi_a \in A$ with $d^\circ \phi_i = |\lambda^i|$ and $t(\phi_i) \leqslant c_1 T$ such that

$$\psi_k = \phi_1 \frac{D^{\lambda^1} f}{\lambda^1 !} + \ldots + \phi_a \frac{D^{\lambda^a} f}{\lambda^a !} \notin \mathfrak{p}_{j,k}$$

for any $j > s_k$. We observe that $D^\lambda \psi_k(\alpha) = 0$ for $\alpha \in V_N$ and $N > |\lambda| + t_k M$. Besides

$$t(\psi_k) \leqslant c_2 T. \tag{3}$$

Then we define
$$I_{k+1} = \mathrm{Res}(J_k, \psi_k).$$
We claim that the following three assertions hold:

(4) $$V_M \subseteq \bigcup_{k=1}^{k_0} \bigcup_{j=1}^{s_k} V_{P(C^n)}(\mathfrak{p}_{j,k});$$

(5) $$e_{j,k} \geqslant M^k \prod_{h=0}^{k-1} (t_k - t_h) \geqslant n^{-2k} M^k \quad \text{for } j = 1, \ldots, s_k \text{ and } k = 1, \ldots, k_0;$$

(6) $$\sum_{j=1}^{s_k} e_{j,k}\, t(\mathfrak{p}_{j,k}) \leqslant c_3 T^k \quad \text{for } k = 1, \ldots, k_0.$$

Assume for the moment (4), (5), (6) proved. For any $k = 1, \ldots, k_0$, Corollary 2 ensures the existence of $g_k \in \bigcap_{j=1}^{s_k} \mathfrak{p}_{j,k}$ such that
$$t(g_k) \leqslant c_4 \Big(\sum_{j=1}^{s_k} t(\mathfrak{p}_{j,k})\Big)^{1/k}.$$
Using (5) and (6) we obtain
$$t(g_k) \leqslant c_5 M^{-1} \Big(\sum_{j=1}^{s_k} e_{j,k}\, t(\mathfrak{p}_{j,k})\Big)^{1/k} \leqslant c_6 \frac{T}{M}.$$
Let $g = \prod_{k=1}^{k_0} g_k$: relation (4) ensures that g is zero over V_M and we have $t(g) \leqslant c_7 T/M$. Hence it is enough to prove (4), (5) and (6).

:(4) By induction we have
$$V_M \subseteq \Big(\bigcup_{h=1}^{k} \bigcup_{j=1}^{s_h} V_{P(C^n)}(\mathfrak{p}_{j,h})\Big) \cup V_{P(C^n)}(J_k)$$
and $V_{P(C^n)}(J_{k_0}) \cap V_M = \emptyset$.

:(5) By induction we prove the following

LEMMA 4. *Let* $N > t_{k-1} M$ *and*
$$\omega \in V_N \setminus \bigcup_{h=1}^{k-1} \bigcup_{j=1}^{s_h} V_{P(C^n)}(\mathfrak{p}_{j,h}).$$
Then
$$i_\omega(I_k) \geqslant \prod_{h=0}^{k-1} (N - t_h M).$$

Proof. $k = 1$: Lemma 2 ensures that $i_\omega(I_1) = N$ for any $\omega \in V_M$.
$k \Rightarrow k+1$: By the inductive hypothesis, for
$$\omega \in V_N \setminus \bigcup_{h=1}^{k} \bigcup_{j=1}^{s_h} V_{P(C^n)}(\mathfrak{p}_{j,h})$$
we have
$$i_\omega(J_k) \geqslant \prod_{h=0}^{k-1} (N - t_h M)$$

(if G is an eliminating form of J_k and F is an eliminating form of I_k then, by Proposition 1, $G = EF$ and $\theta_\omega E \not\equiv 0$, hence $i_\omega(J_k) = i_\omega(I_k)$); besides,
$$i_\omega((\psi_k)) \geqslant N - t_k M.$$
Hence, using Proposition 4(ii),
$$i_\omega(I_{k+1}) \geqslant \prod_{h=0}^{k} (N - t_h M). \ \blacksquare$$

Lemma 4 allows us to prove (5). In fact,
$$V_{P(C^n)}(\mathfrak{p}_{j,h}) \subseteq V_{t_k M} \quad \text{for } j = 1, \ldots, s_k.$$
Hence, using the lemma above and Lemma 3,
$$\begin{aligned} e_{j,k} &\geqslant \prod_{h=0}^{k-1} (t_k M - t_h M) \\ &= \frac{M^k}{n-k+1} \prod_{h=1}^{k-1} \frac{k-h}{(n-k+1)(n-h+1)} \geqslant n^{-2k+1} M^k, \quad j = 1, \ldots, s_k, \end{aligned}$$
and (5) is proved.

:(6) Using Proposition 4 and inequality (3), it is easy to see $t(I_k) \leqslant c_7 T^k$. Hence, by Proposition 3(iii),
$$\sum_{j=1}^{s_k} e_{j,k}\, t(\mathfrak{p}_{j,k}) \leqslant c_8 T^k. \ \blacksquare$$

Remark. Our method says something about the relation between $\omega_1(S)$ and $\omega_M(S)$, but we obtain only

(7) $$4^{-n} n^{-n-3} \omega_1(S) \leqslant \frac{1}{M} \omega_M(S).$$

Using Chardin's bound for Hilbert's function (see [CH]), we may improve (7) to
$$n^{-4} \omega_1(S) \leqslant \frac{1}{M} \omega_M(S).$$

3. Some applications. Let $\xi = (\xi_1, \ldots, \xi_n)$ be a n-uple of complex numbers. We define its transcendence type $\tau(\xi)$ as the infimum of the set of real numbers τ for which there exists a positive constant c_τ such that
$$\log|P(\xi)| > -c_\tau\, t(P)^\tau$$
for any non-zero polynomial P with integer coefficients. Using the box-principle, it is easy to see that $\tau(\xi) \geqslant n+1$.

Similarly we define $\eta(\xi)$ as the infimum of the set of real numbers η for which there exists a positive constant c_η such that
$$\log|\alpha - \xi| > -c_\eta\, \bar{\omega}_1(\alpha)^\eta$$
for any $\alpha \in C^n$.

We have the trivial inequality

$$\eta(\xi) \leqslant \tau(\xi)$$

which reposes on the following lemma:

LEMMA 5. *Let $\xi \in C^n$. For any $P \in C[x_1, \dots, x_n]$ and for any $\alpha \in C^n$ with $P(\alpha) = 0$ and $|\alpha - \xi| \leqslant 1$ we have*

$$|P(\xi)| \leqslant |\alpha-\xi| \, [(2+|\xi|)(n+1)^2]^{d^\circ P} H(P).$$

Proof.

$$|P(\xi)| \leqslant \sum_{1 \leqslant |\lambda| \leqslant d^\circ P} \frac{|D^\lambda P(\alpha)|}{\lambda!} |\alpha_1-\xi_1|^{\lambda_1} \dots |\alpha_n-\xi_n|^{\lambda_n}$$

$$\leqslant |\alpha-\xi| \sum_{0 \leqslant |\lambda| \leqslant d^\circ P} \frac{|D^\lambda P(\alpha)|}{\lambda!}$$

$$\leqslant |\alpha-\xi| (n+1)^{d^\circ P} \sup_{|x|=1} |P(x+\alpha)|$$

$$\leqslant |\alpha-\xi| \, [(1+|\alpha|)(n+1)^2]^{d^\circ P} H(P)$$

$$\leqslant |\alpha-\xi| \, [(2+|\xi|)(n+1)^2]^{d^\circ P} H(P). \ \blacksquare$$

In the opposite sense, using Lemma 2.7 of [P2], it is possible to prove

$$\tau(\xi) \leqslant \eta(\xi)+1.$$

It seems to be natural to expect

$$\tau(\xi) = \eta(\xi) \quad \text{for } \tau(\xi) > n+1 \tag{8}$$

(notice that (8) holds if $n = 1$: see for instance [W1], p. 133).

(8) implies the following conjecture of G. V. Chudnovsky (see [C], Problem 1.3, p. 178):

CONJECTURE. *For almost all (in the sense of Lebesgue measure in R^{2n}) n-uples ξ of complex numbers we have*

$$\tau(\xi) \leqslant n+1.$$

The link between (8) and the conjecture above is given by the following proposition:

PROPOSITION 5. *The set of n-uples of complex numbers ξ for which*

$$\eta(\xi) > n+1$$

has Lebesgue measure 0.

Proof. We denote by λ the Lebesgue measure in C^n. Let $B = \{\xi \in C^n$ such that $|\xi| \leqslant 1\}$. It is enough to prove that

$$\Lambda = \{\xi \in B \text{ such that } \eta(\xi) > n+1\}$$

has Lebesgue measure 0. From the definition of Λ we have

$$\Lambda \subset \bigcap_{s=2}^{+\infty} \bigcup_{N \in N} \bigcup_{\substack{f \in Z[x_1,\dots,x_n] \\ [t(f)]=N}} A_f(\exp(-sN^{n+1}))$$

where

$$A_f(\varepsilon) = \{\xi \in B| \ \operatorname{dist}(\xi, \{f=0\}) \leqslant \varepsilon\}.$$

We need the following lemma from measure theory:

LEMMA 6. *Let V be a pure algebraic variety in C^n of codimension k and degree d. Then for any $\varepsilon \in (0, 1)$*

$$\lambda(\{\xi \in B| \ \operatorname{dist}(\xi, V) \leqslant \varepsilon\}) \leqslant c(n, k)\varepsilon^{2k} d$$

where $c(n, k)$ is some positive constant depending only on n and k.

Proof. We denote by H^k the $2k$-dimensional Hausdorff measure and by $B_x(r)$ the ball of C^n with centre at x and radius r. We also denote by $c_9, \dots, c_{13}$ effective positive constants depending only on n and k.

We begin with a bound for the area of $V \cap B_0(r)$. Using Theorem 3.2.22(4) of [F1], a Fubini–Tonelli argument yields

$$H^{n-k}(V \cap B_0(r)) = c_9 \int_{G(n,n-k)} d\nu(p) \int_{p(V \cap B_0(r))} \operatorname{card}(V \cap B_0(r) \cap p^{-1}(y)) \, dH^{n-k}(y)$$

where $G(n, n-k)$ is the set of $(n-k)$-dimensional complex subvector spaces of C^n (which are in turn identified with the set of orthogonal projections p over these spaces) and ν is the only measure on $G(n, n-k)$ with unitary mass and invariant by the action of $U(n)$. For ν-almost all p and for all $y \in p(V \cap B_0(r))$

$$\operatorname{card}(V \cap B_0(r) \cap p^{-1}(y)) \leqslant d.$$

Hence

$$H^{n-k}(V \cap B_0(r)) \leqslant c_9 d \int_{G(n,n-k)} d\nu(p) \int_{p(V \cap B_0(r))} dH^{n-k}(y) \leqslant c_{10} d r^{2(n-k)}. \tag{9}$$

The link between the growth of the area and the measure of the set of points which are close to V is given by the following formula which derives from Theorem 6.2 of [F2]:

$$H^{n-k}(V \cap B_0(r)) H^n(B_0(s)) = \int_{C^n} H^{n-k}(V \cap B_0(r) \cap B_\xi(s)) \, d\lambda(\xi).$$

Using the formula above with $r = 1+2\varepsilon$ and $s = 2\varepsilon$ and the bound (9) we find

$$c_{11} d\varepsilon^{2n} \geqslant \int_{\{\xi \in B| \operatorname{dist}(\xi, V) < \varepsilon\}} H^{n-k}(V \cap B_\xi(2\varepsilon)) \, d\lambda(\xi). \tag{10}$$

For $\xi \in B$, $\operatorname{dist}(\xi, V) < \varepsilon$, let $\xi^* \in V$ be such that $\operatorname{dist}(\xi, V) = \operatorname{dist}(\xi, \xi^*)$. Then

$$V \cap B_\xi(2\varepsilon) \supset V \cap B_{\xi^*}(\varepsilon).$$

The function

$$\varepsilon \to \frac{H^{n-k}(V\cap B_{\xi^*}(\varepsilon))}{\varepsilon^{2(n-k)}}$$

is monotonically increasing and is bounded from below by some positive constatnt c_4 (see [L], Theorem 2.23). Hence

$$H^{n-k}(V\cap B_\xi(2\varepsilon)) \geqslant H^{n-k}(V\cap B_{\xi^*}(\varepsilon)) \geqslant c_{12}\varepsilon^{2(n-k)}.$$

Combining with (10) we have

$$\lambda(\{\xi\in B|\ \operatorname{dist}(\xi, V)\leqslant\varepsilon\})\leqslant c_{13}d\varepsilon^{2k}. \ \blacksquare$$

From the lemma above with $V=\{f=0\}$, we obtain

$$\lambda(A_f(\exp(-sN^{n+1})))\leqslant c(n,1)N\exp(-2sN^{n+1}).$$

The number of polynomials in n variables with integer coefficients and size $\leqslant N$ is bounded by $\exp(2N^{n+1})$, hence for all $s\geqslant 2$

$$\lambda(\Lambda)\leqslant\lambda\Big(\bigcup_{N\in\mathbf{N}}\ \bigcup_{\substack{f\in\mathbf{Z}[x_1,\dots,x_n]\\ [t(P)]=N}} A_f(\exp(-sN^{n+1}))\Big)$$

$$\leqslant\sum_{N\geqslant 1} c(n,1)N\exp(-2(s-1)N^{n+1})=\psi(s)$$

and

$$\psi(s)\to 0 \quad \text{as } s\to+\infty. \ \blacksquare$$

Let $\tau=\tau(\xi)$ and $\eta=\eta(\xi)$. As an application of the method of the proof of Theorem 1, we shall prove:

THEOREM 2. *Assume $\tau>n+1$, $n\geqslant 2$. Then*

$$\tau\leqslant\eta+\max\left(\frac{n-1}{\eta+1},\frac{2n-\eta}{n+1}\right).$$

Moreover, if $n=2$,

$$\tau\leqslant\eta+\max(0,(4-\eta)/3).$$

For example, if $n=2$ we find:

$$\tau\leqslant 3.34 \text{ for } \eta\leqslant 3;\quad \tau=\eta \text{ for } \eta>4.$$

If $n=3$ the situation is a little worse:

$$\tau\leqslant 4.5 \text{ for } \eta\leqslant 4;\quad \tau\leqslant 6.34 \text{ for } \eta\leqslant 6.$$

We observe that for any fixed n our result approaches (8) when η (or τ) $\to+\infty$:

COROLLARY 3.

$$\eta\leqslant\tau\leqslant\eta+o(1/\eta) \quad \text{for } \eta\to+\infty.$$

Proof of Theorem 2. Let us assume $\tau>n+1$. We choose a real number ϱ with $n+1\leqslant\varrho<\tau$. By hypothesis, for any positive constant C there exists a polynomial P with integer coefficients such that

$$\log|P(\xi)|<-CT^\varrho, \tag{11}$$

where T is the size of P. Let $d=d^\circ P$; in what follows we denote by $c_{14},\dots,c_{25}$ positive constants depending only on n and $|\xi|$.

For any multiindex $\lambda\in\mathbf{N}^n$ we define the real number $\phi(\lambda)$ as

$$\phi(\lambda)=\frac{1+\operatorname{card}\{h\in[1,\dots,n] \text{ such that } \lambda_h=0\}}{n+1};$$

we have $\phi((0,\dots,0))=1$ and $\phi(\lambda)\geqslant 1/(n+1)$ for any multiindex $\lambda\in\mathbf{N}^n$. Let $\bar\lambda\in\mathbf{N}^n$ be a multiindex with $|\bar\lambda|=d$ such that the monomial $x_1^{\bar\lambda_1}\dots x_n^{\bar\lambda_n}$ has non-zero coefficient in $P(x)$; then, using (11),

$$\left|\frac{1}{\bar\lambda!}D^{\bar\lambda}P(\xi)\right|\geqslant 1>|P(\xi)|^{\phi(\bar\lambda)}.$$

Hence we can define an integer $M\in(0,d)$ as the first integer for which there exists $\tilde\lambda\in\mathbf{N}^n$ with $|\tilde\lambda|=M+1$ such that

$$\left|\frac{1}{\tilde\lambda!}D^{\tilde\lambda}P(\xi)\right|>|P(\xi)|^{\phi(\tilde\lambda)}. \tag{12}$$

We can find $h\in[1,\dots,n]$ such that $\tilde\lambda_h\neq 0$; let

$$\mu=(\tilde\lambda_1,\dots,\tilde\lambda_{h-1},0,\tilde\lambda_{h+1},\dots,\tilde\lambda_n).$$

We have $|\mu|\leqslant M$ and $\phi(\mu)-\phi(\tilde\lambda)=1/(n+1)$. Let us consider

$$Q(t)=\frac{1}{\mu!}D^\mu P(\xi_1,\dots,\xi_{h-1},t,\xi_{h+1},\dots,\xi_n),$$

which is a polynomial in one variable of degree $\delta\leqslant d-|\mu|$; let $\alpha_1,\dots,\alpha_\delta$ be its roots. We need the following lemma:

LEMMA 7. *For any $s\geqslant 0$ there exists a homogeneous polynomial $R_s\in\mathbf{C}[y_1,\dots,y_\delta]$ of degree s and height $\leqslant\delta^{s-1}s!$ such that*

$$\frac{\partial^s Q(t)}{\partial t^s}=Q(t)R_s((t-\alpha_1)^{-1},\dots,(t-\alpha_\delta)^{-1}). \tag{13}$$

Proof. Let

$$Q(t) = a\prod_{h=1}^{\delta}(t-\alpha_h)$$

and let $\sigma: C[y_1, \ldots, y_\delta] \to C(t)$ be the homomorphism defined by $y_h \mapsto (t-\alpha_h)^{-1}$ for $h = 1, \ldots, \delta$. We prove our assertion using induction on s; we define R_0 as $R_0 = 1$ and R_1 as $R_1 = y_1 + \ldots + y_\delta$. It is easy to verify that relation (13) holds for $s = 0, 1$. Let us assume (13) holds for some s for a polynomial R_s of degre s and height $\leqslant \delta^{s-1}s!$; then

$$\frac{\partial^{s+1}Q(t)}{\partial t^{s+1}} = \frac{\partial Q(t)}{\partial t}\sigma R_s - Q(t)\sum_{h=1}^{\delta}(t-\alpha_h)^{-2}\sigma\frac{\partial R_s}{\partial y_h} = Q(t)\sigma\left(R_1R_s - \sum_{h=1}^{\delta} y_h^2\frac{\partial R_s}{\partial y_h}\right).$$

Hence we can define R_{s+1} as

$$R_{s+1} = R_1R_s - \sum_{h=1}^{\delta} y_h^2\frac{\partial R_s}{\partial y_h}.$$

Using the inductive hypothesis we see that R_{s+1} is a homogeneous polynomial of degree $s+1$ and height

$$H(R_{s+1}) \leqslant \delta H(R_s) + \delta s H(R_s) \leqslant \delta^s(s+1)!. \ \blacksquare$$

Now we assume

$$|\alpha_1 - \xi_h| \leqslant \ldots \leqslant |\alpha_\delta - \xi_h|;$$

then, by Lemma 7,

$$\left|\frac{\partial^{\tilde{\lambda}_h}Q(\xi_h)}{\partial t^{\tilde{\lambda}_h}}\right| \leqslant |Q(\xi_h)|(d-|\mu|)^{\tilde{\lambda}_h-1}\tilde{\lambda}_h!\,|\alpha_1-\xi_h|^{-\tilde{\lambda}_h}. \tag{14}$$

By the definition (12) of M we have $|Q(\xi_h)| \leqslant |P(\xi)|^{\phi(\mu)}$ and

$$\left|\frac{\partial^{\tilde{\lambda}_h}Q(\xi_h)}{\partial t^{\tilde{\lambda}_h}}\right| = \left|\frac{1}{\mu!}D^{\tilde{\lambda}}P(\xi)\right| > \tilde{\lambda}_h!\,|P(\xi)|^{\phi(\tilde{\lambda})}.$$

Combining with (14) we find

$$|\alpha_1-\xi_h|^{\tilde{\lambda}_h} < (d-|\mu|)^{\tilde{\lambda}_h-1}|P(\xi)|^{\phi(\mu)-\phi(\tilde{\lambda})}.$$

Let $\alpha = (\xi_1, \ldots, \xi_{h-1}, \alpha_1, \xi_{h+1}, \ldots, \xi_n)$; taking the logarithms in the last inequality and using our upper bound (11) for $\log|P(\xi)|$ we find

$$\log|\alpha-\xi| < \log d - \frac{C}{(M+1)(n+1)}T^{\varrho}. \tag{15}$$

Moreover, $D^{\mu}P(\alpha) = 0$, hence

$$\bar{\omega}_1(\alpha) \leqslant t\left(\frac{1}{\mu!}D^{\mu}P\right) \leqslant 2T. \tag{16}$$

Let $u \in [0, 1]$ be defined by

$$u = \frac{\log(M+1)}{\log T}.$$

From relations (15) and (16) (with a suitable choice of C) we have

$$\varrho \leqslant \eta + u. \tag{17}$$

Now we apply the machinery of Theorem 1 to find another bound for ϱ which becomes better for large u. We closely follow the pattern of the proof of Theorem 1. Let f be the homogenization ${}^hP$ of P; for simplicity we shall consider $C^n \subset P^n$ via the canonical map

$$(x_1, \ldots, x_n) \mapsto (1 : x_1 : \ldots : x_n).$$

Using the definition (12) of M and the inequality $\phi(\lambda) \geqslant 1/(n+1)$ we find

$$\max_{\substack{\lambda\in N^{n+1}\\ |\lambda|\leqslant M, \lambda_0 = 0}}\left|\frac{1}{\lambda!}D^{\lambda}f(\xi)\right| \leqslant |P(\xi)|^{1/(n+1)}.$$

We prove by induction that

$$\left|\frac{1}{\lambda!}D^{\lambda}f(\xi)\right| \leqslant \frac{[(d+n)|\xi|]^{\lambda_0}}{\lambda_0!}|P(\xi)|^{1/(n+1)} \tag{18}$$

for any $\lambda \in N^{n+1}$ such that $|\lambda| \leqslant M$. Let us assume (18) holds for any λ with $\lambda_0 = k-1$ and let $\tilde{\lambda} \in N^{n+1}$ be a multiindex with $\lambda_0 = k$; by Euler's formula we have

$$\sum_{t=0}^{n}\left[\frac{\partial}{\partial x_t}D^{\mu}f\right]x_t = (d-|\mu|)D^{\mu}f$$

where $\mu = (\lambda_0 - 1, \tilde{\lambda}_1, \ldots, \lambda_n)$. Hence

$$\begin{aligned}|D^{\lambda}f(\xi)| &\leqslant (d-|\mu|)\mu!\left|\frac{1}{\mu!}D^{\mu}f(\xi)\right| \\ &\quad + (n+|\mu|)\mu!\,|\xi|\max_{1\leqslant t\leqslant n}\left\{\frac{1}{\mu_0!\ldots(\mu_t+1)!\ldots\mu_n!}\left|\frac{\partial}{\partial x_t}D^{\mu}f(\xi)\right|\right\} \\ &\leqslant \frac{\lambda!}{k}(d+n)|\xi|\frac{[(d+n)|\xi|]^{k-1}}{(k-1)!}|P(\xi)|^{1/(n+1)} \\ &= \lambda!\frac{[(d+n)|\xi|]^{k}}{k!}|P(\xi)|^{1/(n+1)},\end{aligned}$$

so (18) is proved. Combining this with (11) we obtain

$$\max_{\lambda\leqslant M}\log|D^{\lambda}f(\xi)| < -c_1CT^{\varrho}. \tag{19}$$

From this point on, we closely follow the pattern of the proof of Theorem 1. We define I_1 as usual; let us assume $I_1, \ldots, I_k$ defined. If

$$\log\|J_k\|_{\xi} \geqslant \tfrac{1}{2}\log\|I_k\|_{\xi}$$

we let $k_0 = k$ and we stop here. Otherwise we construct I_{k+1} as in the proof of Theorem 1. Inequalities (5) and (6) are still true. Moreover, repeatedly applying Proposition 4(iii) and (iv) with the bounds (19) for the value of $D^\lambda f$ at ξ, we obtain

$$t(I_{k_0}) < c_{14} T^{k_0}, \quad \log \|I_{k_0}\|_\xi < -c_{15} C T^\varrho$$

(we remember that $\varrho \geqslant n+1$ and $C \gg 1$). This implies $k_0 \leqslant n$, since otherwise we would find an ideal I_{n+1} of codimension $n+1$ which satisfies $\log \|I_{n+1}\|_\xi < 0$. Notice that $k_0 \geqslant 2$ too (f is irreducible and, *a fortiori*, square-free). Hence, using Proposition 3(iv) and relation (6),

$$\sum_{j=1}^{s_{k_0}} e_{j,k_0} \log \|\mathfrak{p}_{j,k_0}\|_\xi \leqslant \log \|I_{k_0}\|_\xi - \log \|J_{k_0}\|_\xi + c_{16} T^{k_0} \tag{20}$$

$$< -c_{17} C T^\varrho \leqslant -c_{18} C \Big(\sum_{j=1}^{s_{k_0}} e_{j,k_0} t(\mathfrak{p}_{j,k_0}) \Big)^{\varrho/k_0}.$$

Let us assume

$$\log \|\mathfrak{p}_{j,k_0}\|_\xi \geqslant -c_{19} C t(\mathfrak{p}_{j,k_0})^{(\varrho - u k_0)/((1-u)k_0)} \quad \text{for } j = 1, \ldots, s_k.$$

By the two inequalities above,

$$c_{18} C \Big(\sum_{j=1}^{s_{k_0}} e_{j,k_0} t(\mathfrak{p}_{j,k_0}) \Big)^{\varrho/k_0} < c_{19} C \sum_{j=1}^{s_{k_0}} e_{j,k_0} t(\mathfrak{p}_{j,k_0})^{(\varrho - u k_0)/((1-u)k_0)}$$

$$\leqslant c_{19} C \Big(\sum_{j=1}^{s_{k_0}} e_{j,k_0} t(\mathfrak{p}_{j,k_0}) \Big) \Big(\sum_{j=1}^{s_{k_0}} t(\mathfrak{p}_{j,k_0}) \Big)^{(\varrho - k_0)/((1-u)k_0)}.$$

Hence

$$\sum_{j=1}^{s_{k_0}} e_{j,k_0} t(\mathfrak{p}_{j,k_0}) < (c_{19}/c_{18})^{k_0/(\varrho - k_0)} \Big(\sum_{j=1}^{s_{k_0}} t(\mathfrak{p}_{j,k_0}) \Big)^{1/(1-u)}. \tag{21}$$

On the other hand, using (5) and (6) we obtain

$$\sum_{j=1}^{s_{k_0}} e_{j,k_0} t(\mathfrak{p}_{j,k_0}) \geqslant n^{-2k_0} M^{k_0} \sum_{j=1}^{s_{k_0}} t(\mathfrak{p}_{j,k_0}) = n^{-2k_0} T^{u k_0} \sum_{j=1}^{s_{k_0}} t(\mathfrak{p}_{j,k_0})$$

$$\geqslant n^{-2k_0} c_3^{-u} \Big(\sum_{j=1}^{s_{k_0}} e_{j,k_0} t(\mathfrak{p}_{j,k_0}) \Big)^u \sum_{j=1}^{s_{k_0}} t(\mathfrak{p}_{j,k_0}).$$

Hence

$$\sum_{j=1}^{s_{k_0}} e_{j,k_0} t(\mathfrak{p}_{j,k_0}) \geqslant (n^{2k_0} c_3^u)^{-1/(1-u)} \Big(\sum_{j=1}^{s_{k_0}} t(\mathfrak{p}_{j,k_0}) \Big)^{1/(1-u)}. \tag{22}$$

Comparing (21) and (22) we find

$$c_{19} > c_{20} := c_{18} (n^{2k_0} c_3^u)^{(-\varrho + k_0)/((1-u)k_0)}.$$

Hence by (20) there exists some prime ideal $\mathfrak{p}$ of I_{k_0} such that

$$\log \|\mathfrak{p}\|_\xi < -c_{20} C t(\mathfrak{p})^{(\varrho - u k_0)/((1-u)k_0)} < 0. \tag{23}$$

Corollary 2 ensures the existence of $g \in \mathfrak{p}$ with $t(g) \leqslant c_{21} t(\mathfrak{p})^{1/k_0}$. Hence for any zero $\alpha \in C^n$ of $\mathfrak{p}$ we have

$$\bar{\omega}_1(\alpha) \leqslant c_{21} t(\mathfrak{p})^{1/k_0}. \tag{24}$$

We distinguish two cases:

Case 1. Let us assume $2 \leqslant k_0 \leqslant n-1$ (hence this case does not occur if $n = 2$). Then Lemma 2.7 of [P 1] and inequalities (23)–(24) ensure the existence of a zero $\alpha \in C^n$ in the projective variety defined by $\mathfrak{p}$ such that

$$\log |\alpha - \xi| < c_{22} t(\mathfrak{p})^{-1} \log \|\mathfrak{p}\|_\xi$$

$$\leqslant -c_{23} C \bar{\omega}_1(\alpha)^{(\varrho - k_0)/(1-u)} \leqslant -c_{23} C \bar{\omega}_1(\alpha)^{(\varrho - n+1)/(1-u)}.$$

We conclude

$$\varrho \leqslant \eta(1-u) + n - 1. \tag{25}$$

Case 2. Let us assume $k_0 = n$. The set of projective zeros of $\mathfrak{p}$ is a zero-dimensional variety, hence smooth. Theorem 1.1 of [A] asserts that we can find a zero $\alpha \in C^n$ in the projective variety defined by $\mathfrak{p}$ such that

$$\log |\alpha - \xi| < \log \|\mathfrak{p}\|_\xi + c_{24} t(\mathfrak{p})^2.$$

Thus if

$$\frac{\varrho - un}{(1-u)n} \geqslant 2 \quad \text{and} \quad C \geqslant \frac{2c_{24}}{c_{20}}$$

we have (using (23) and (24))

$$\log |\alpha - \xi| < -\tfrac{1}{2} c_{20} C \bar{\omega}_1(\alpha)^{(\varrho - un)/(1-u)} \leqslant -\tfrac{1}{2} c_{20} C \bar{\omega}_1(\alpha)^\varrho.$$

Hence we conclude

$$\varrho \leqslant \max((2-u)n, \eta). \tag{26}$$

Collecting (17), (25) and (26) we find

$$\varrho \leqslant \min(\eta + u, \eta(1-u) + n - 1) \leqslant \eta + \frac{n-1}{\eta+1}$$

for $2 \leqslant k_0 \leqslant n-1$, and

$$\varrho \leqslant \min(\eta + u, \max((2-u)n, \eta)) \leqslant \eta + \max\left(0, \frac{2n-\eta}{n+1}\right)$$

for $k_0 = n$. In any case

$$\varrho \leqslant \eta + \max\left(\frac{n-1}{\eta+1}, \frac{2n-\eta}{n+1}\right).$$

If $n = 2$ case 1 does not occur and we have the better result

$$\varrho \leqslant \eta + \max\left(0, \frac{2n-\eta}{n+1}\right).$$

Theorem 2 is proved. ∎

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SCUOLA NORMALE SUPERIORE
Piazza dei Cavalieri 7
56100 Pisa, Italia

UNIVERSITÉ PARIS VI
URA DU CNRS n. 763 "PROBLÈMES DIOPHANTIENS"
INSTITUT HENRI POINCARÉ
11 rue Pierre et Marie Curie
75231 Paris Cedex 05, France

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