The upper central series of some matrix groups

by

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Abstract. In this paper we make a detailed study of the nilpotent structure (specifically of the upper central series, central height, Hirsch–Plotkin radical and Engel elements) of an arbitrary group of matrices over a specific type of division ring. These division rings are characterized by a local residual property and are quite common. They include the following: all fields, division rings of finite index a power of their positive characteristics, the universal division ring of fractions of the group algebra of a free group, division rings of quotients of group algebras of certain groups, including, in particular, the free soluble groups, and division rings of quotients of the universal enveloping algebras of certain Lie algebras, including, with certain reservations, all finite-dimensional Lie algebras.

In [11] we analysed in detail the structure of nilpotent and locally nilpotent groups of matrices over certain division rings, which, as a temporary measure, we called special. (We remind the reader of the precise definition of special later in this introduction.) In this we built on work of Lichtman [6]. In this present paper we attempt to describe the upper central series and the Engel structure of an *arbitrary* group of matrices over a special division ring. Our results are comparatively complete, much more so than seemed possible when [11] was written. In particular the hypercentrality of the locally nilpotent groups, a major gap in [11], is settled here positively (and easily) for a large class of special division rings. However, we do give below examples of locally nilpotent matrix groups over certain special division rings that are not hypercentral.

If D is any division ring we define as follows the function $u(n, D)$ of the positive integer variable n into $N \cup \{\infty\}$. Consider subgroups G of $\mathrm{GL}(n, D)$ and normal *unipotent* subgroups N of G consisting of right Engel elements of G. (We follow the Engel terminology of [7], [8] and [9]; in particular our multiple commutators are all left normed.) Then $u(n, D)$ is the least non-negative integer e for which $[N, {}_eG] = \langle 1 \rangle$ for all such N and G, or ∞ if no such e exists. For all fields D and many division rings D we have $u(n, D) = n-1$, see [10]. In [11] we gave an example of a special division ring D with $u(n, D) = \infty$ for all $n \geqslant 2$, see [11], 6.3. What we missed in [11] is that this example seems somewhat exceptional. A simple argument shows that $u(n, D)$ is finite for very many special divisions rings D; indeed frequently we have the best possible result, namely $u(n, D) = n-1$.

If G is any subgroup of $\mathrm{GL}(n, D)$, where n is a positive integer and D is a special division ring, our description of the upper central series and the Engel elements of G are essentially complete if either the unipotent radical of G is trivial or $u(n, D)$ is finite. By complete we mean our information is as detailed as the linear case. The word 'essentially' refers to the case $n = \operatorname{char} D = 2$, for which we leave some niggling gaps.

We summarize here our main conclusions. For Theorems A to C below D *is a special division ring of characteristic $p \geqslant 0$, n is a positive integer and G is some subgroup of* $\mathrm{GL}(n, D)$. Let $u(G)$ denote the unipotent radical of G. For any prime q define $e(n, q)$ by $q^{e(n,q)}$ is the largest power of q to divide $n!$. Note that $e(n, q) \leqslant (q-1)^{-1}(n-1)$, see [8], p. 112.

THEOREM A1. *Suppose either that $u(G) = \langle 1\rangle$ or that $u(n, D) = m$ is finite. Then G has central height at most*

$$\begin{array}{lll} 1 & \textit{if } n = 1, \textit{ or if } n = p = 2 \textit{ with } u(G) = \langle 1\rangle, & \\ m+1 & \textit{if } n = p = 2 \textit{ and } m < \infty, & \\ \omega+e(n,3) \leqslant \omega+[\frac{1}{2}(n-1)] & \textit{if } n > 2 = p & \textit{and} \\ \omega+e(n,2) \leqslant \omega+n-1 & \textit{if } n \geqslant 2 \neq p. & \end{array}$$

Here ω denotes the first infinite ordinal. These bounds are not far from the best possible; indeed they are (very close to) the best known (at time of writing) bounds in the linear case ([9], 8.6) and in the locally finite-dimensional case ([8], 3.4.13), except for the anomalous situation $n = p = 2$, $m < \infty$, where the bound involves m. We give an example to show that m is attainable in this exceptional case. We do not know whether the true bound is m or $m+1$.

Our results are even more striking if G in Theorem A1 is also locally nilpotent. There, except for the anomalous $n = p = 2$, $m < \infty$ case the bounds are the same as for the linear case and consequently are attained. Even the locally finite-dimensional case, where the exact bounds are also known, see [8], 3.4.14, does not parallel the linear case quite so closely. These precise bounds are as follows.

THEOREM A2. *Assume G is locally nilpotent and suppose either that $u(G) = \langle 1\rangle$ or that $u(n, D) = m < \infty$. Then G is hypercentral and has central height at most*

$$\begin{array}{lll} 1 & \textit{if } n = 1, \textit{ or if } n = p = 2 \textit{ with } u(G) = \langle 1\rangle, & \\ m+1 & \textit{if } n = p = 2 \textit{ and } m < \infty, & \\ \omega+\max\{q^{[\log_q n]}-1\colon q = 3, 5, 7\} & \textit{if } n > 2 = p & \textit{and} \\ \omega+\max\{q^{[\log_q n]}-1\colon q = 2, 3\} & \textit{if } n \geqslant 2 \neq p. & \end{array}$$

Further G never has central height ω.

In the anomalous case of Theorem A2 again m is attainable and we leave open whether the true bound is m or $m+1$ if $n = p = 2$ and $m < \infty$. For any group H

let $\{\zeta_\alpha(H)\colon \alpha \geqslant 0\}$ denote the upper central series and $\{\gamma^\alpha H\colon \alpha \geqslant 1\}$ the lower central series of H and set $\zeta(H) = \bigcup \zeta_\alpha(H)$, the hypercentre of H.

THEOREM B. *Suppose either that $u(G) = \langle 1\rangle$ or that $u(n, D) < \infty$. In the former case set $m = 0$. Otherwise set $m = u(n, D)$. Let e be the largest p'-divisor of $n!$ (meaning $n!$ if $p = 0$) and put $\pi = \{\text{primes } q\colon p \neq q \leqslant n\}$. Then:*

1. $[\zeta(G), {}_{m+1}G]$ *is a π-group.*
2. $[\zeta(G), {}_{\max\{m+1, n-1\}}G]$ *is abelian; if $m = 0$ even $[\zeta(G), {}_{n-1}G]$ is abelian.*
3. $\zeta(G)/\zeta_{m+1}(G)$ *is a π-group.*
4. $\zeta(G)/\zeta_\omega(G)$ *is finite of order dividing e.*

The conclusions of Theorem B are much as in the linear case. There is one obvious difference; m is involved. The linear case (and the locally finite-dimensional case) suggests that in Theorem B the $m+1$ should be replaced by m. The difficulties with settling this point are related to the problems associated with the anomalous case of Theorem A (1 and 2).

As a minor corollary of Theorem B we have the following, see Section 15 below. *Suppose G is locally nilpotent and either $u(G) = \langle 1\rangle$ or $u(n, D) < \infty$. If also G satisfies the maximal condition on abelian characteristic subgroups then G is nilpotent.* In general, a locally nilpotent group with the maximal condition on abelian normal subgroups need not be nilpotent, since there are locally finite q-groups, q a prime, whose only ascendent abelian subgroup is trivial, see [7], Vol. 2, p. 29. We also prove the following, see Section 14 below. *Suppose G is finitely generated and either $u(G) = \langle 1\rangle$ or $u(n, D) < \infty$. Then G has finite central height and nilpotent Hirsch–Plotkin radical.* All this mirrors the linear case.

In order to discuss Engel structure we have unfortunately to introduce a substantial amount of notation. Again we follow in the main [7], [8] or [9]. For any group H,

$L(H)$ denotes the *set* of left Engel elements of H,
$\bar{L}(H)$ the *set* of bounded left Engel elements of H,
$R(H)$ the *set* of right Engel elements of H,
$\bar{R}(H)$ the *set* of bounded right Engel elements of H,
$\eta(H)$ the Hirsch–Plotkin radical of H,
$\eta_1(H)$ the Fitting subgroup of H,
$\sigma(H)$ the Gruenberg radical of $H = \{x \in H\colon \langle x\rangle \operatorname{asc} H\}$,
$\bar{\sigma}(H)$ the Baer radical of $H = \{x \in H\colon \langle x\rangle \lhd\lhd H\}$,
$\varrho(H) = \{x \in H\colon \forall g \in H, g \in \sigma(\langle g, x^H\rangle)\}$ and
$\bar{\varrho}(H) = \{x \in H\colon (\exists k \in N)(\forall g \in H), \langle g\rangle \text{ is subnormal } \langle g, x^H\rangle \text{ in } k \text{ steps}\}$

Although the first four here are in general only subsets of H the remaining six are always subgroups; this remark is important in the proofs. For an arbitrary group H we have (see [7], Vol. 2, p. 63)

$$\sigma(H) \leqslant \eta(H) \subseteq L(H), \qquad \eta_1(H) \leqslant \bar{\sigma}(H) \subseteq \bar{L}(H),$$
$$\zeta(H) \leqslant \varrho(H) \subseteq R(H), \qquad \zeta_\omega(H) \leqslant \bar{\varrho}(H) \subseteq \bar{R}(H).$$

We also have $\eta_1(H) \leqslant \eta_1\eta(H) \leqslant \bar{\sigma}(H)$. (Here $\eta_1\eta(H)$ denotes $\eta_1(\eta(H))$ etc.) Thus the following is as strong a result as we have any right to expect.

THEOREM C. *For our arbitrary subgroup G of $\mathrm{GL}(n, D)$, where D is special,*

$$\sigma(G) = \eta(G) = L(G), \quad \eta_1\eta(G) = \bar{\sigma}(G) = \bar{L}(G),$$
$$\varrho(G) = R(G), \quad \bar{\varrho}(G) = \bar{R}(G).$$

In particular the four sets of Engel elements of G are subgroups.

If, further, $u(G) = \langle 1 \rangle$ or $u(n, D) < \infty$ then

$\eta_1(G) = \bar{\sigma}(G) = \bar{L}(G)$, which is nilpotent,

$\zeta(G) = \varrho(G) = R(G)$ and $\zeta_\omega(G) = \bar{\varrho}(G) = \bar{R}(G)$.

It seems possible that $\eta_1(G) = \eta_1\eta(G)$ for all G as in Theorem C. Note that it follows from Theorem C that if D is special then any locally nilpotent subgroup of $\mathrm{GL}(n, D)$ is a Gruenberg group and any Baer subgroup of $\mathrm{GL}(n, D)$ is a Fitting group.

Now we discuss specific examples of special division rings. We begin by reminding the reader of the definition of special. A division ring D is *special* if for every finite subset X of D there is a ring homomorphism φ of the subring R of D generated by X into a division ring D_Φ such that D_Φ has finite (Schur) index a power of its positive characteristic. If Y is any finite subset of R then φ can always be chosen to be one-to-one on Y, see the introduction to [11].

The examples of special division rings discussed in [11] (and by implication in [6]) are of five broad types.

(a) Any field.

(b) A division ring of finite index a power of its positive characteristic.

(c) Division rings of quotients of the universal enveloping algebras of certain Lie algebras, including all finite-dimensional Lie algebras.

(d) The universal division ring of fractions (in the sense of P. M. Cohn, see [1], p. 254) of the group algebra of a free group.

(e) Division rings of quotients of group algebras over certain groups, including, in particular, the free soluble groups.

Here types (a) and (b) are obviously special, while (c), (d) and (e) cover the main division rings considered in [6] and [11].

If F is a field then $u(n, F) = n-1$ and if D is a division ring of type (b), say of characteristic p and index p^m, then

$$n-1 \leqslant u(n, D) \leqslant p^m(n-1) < \infty.$$

This follows from 1.2 of [10], although the finiteness of $u(n, D)$ is easy; for D is a vector space of dimension p^m over some maximal subfield of D and so

$$u(n, D) \leqslant p^m n - 1 < \infty$$

by the field case. As we pointed out above [11] contains an example of a special division ring D of type (c) for which $u(n, D) = \infty$ for all $n \geqslant 2$. No such examples exist of types (d) or (e). In the following two results the fact that D is special is already given in [11], see especially [11], 1.4 and 1.5, and in many cases in [6]; it is the computation of $u(n, D)$ that we wish to record here.

PROPOSITION D. *Let F be a field, L a free group and D the universal division ring of fractions of the group algebra FL. Then D is special with $u(n, D) = n-1$ for all $n \geqslant 1$.*

To state the next result adequately we need P. Hall's calculus of group classes, see [7], Vol. 1, pp. 1–4. In particular $\mathfrak{A}$, $\mathfrak{F}$, $\mathfrak{F}^{-s}$ and $\mathfrak{F}_\pi$ denote the classes of abelian, finite, torsion-free and, for some set π of primes, finite-π groups. Also P, R and L denote the poly, residual and local operators.

PROPOSITION E. *Let F be a field of characteristic $p \geqslant 0$ and S a soluble group. Under any one of the following four conditions the group algebra FS has a division ring D of quotients that is special with $u(n, D) = n-1$ for all $n \geqslant 1$.*

1. *$p = 0$ and $S \in P(\mathfrak{A} \cap \mathfrak{F}^{-s}) \cap \bigcap_{q\in\pi} R(\mathfrak{A}\mathfrak{F}_q \cap \mathfrak{F}^{-s})$ for some infinite set π of primes.*
2. *$p > 0$ and $S \in P(\mathfrak{A} \cap \mathfrak{F}^{-s}) \cap R(\mathfrak{A}\mathfrak{F}_p \cap \mathfrak{F}^{-s}) \cap R(\mathfrak{A}\mathfrak{F}_{p'} \cap \mathfrak{F}^{-s})$.*
3. *$S = L/M$ for some free group L and some term $M \neq N$, of the derived series of the normal subgroup N of L such that either $p = 0$ and $L/N \in \bigcap_{q\in\pi} R\mathfrak{F}_q$ for some infinite set π of primes or $p > 0$ and $L/N \in R\mathfrak{F}_p \cap R\mathfrak{F}_{p'}$.*
4. *S is free soluble.*

The conditions on S in Proposition E are the same as those required by 1.4 and 1.5 of [11] to derive that D is special, except for 2 and the case $p > 0$ of 3, where they are slightly stronger. Note that in both propositions we obtain the best possible result, namely $u(n, D) = n-1$. For the division rings D of Propositions D and E we settle the queries left by Theorem A in the anomalous case $n = p = 2$, $m < \infty$. For the same division rings D we also make some progress with the queries left by Theorem B concerning $m+1$. See (20) and (21) below.

Finally we turn to our examples of locally nilpotent groups that are not hypercentral. For each prime p we construct groups of 2 by 2 matrices over special division rings of characteristic p that are locally nilpotent (and hence Gruenberg groups by Theorem C) but are not Baer groups, are Baer (and hence Fitting) groups but are not Baer groups, are Baer (and hence Fitting) groups but are not hypercentral, are hypercentral but are not Fitting groups and are hypercentral Fitting groups but are not nilpotent. Clearly these examples cannot arise from special division rings of types (a), (b), (d) or (e) considered above. In fact they arise from the following sixth method for constructing special division rings.

LEMMA F. *Let F be a field of characteristic $p > 0$ and S a torsion-free group acting as a group of field automorphisms of F. Suppose for every finite subset X of F and finitely generated subgroup Y of S there is a Y-invariant subfield of F containing X and centralised by some abelian normal subgroup of Y of finite index a power of p.*

Let D be the division ring of quotients of the skew group ring FS of S over F. Then D is special.

Note that $S \in \langle P, L\rangle(\mathfrak{A}\mathfrak{F})$, so D does exist by 4.1 of [5]. We also use Lemma F to give examples to show that the condition $u(G) = \langle 1\rangle$ or $u(n, D) < \infty$ cannot be completely removed from Theorem B. The obvious question arises as to whether there are characteristic-zero versions of all these counter examples. We show that this is the case for those counter examples preventing the strengthening of Theorem B. However, it remains open as to whether a locally nilpotent group of matrices over a special division ring of characteristic zero necessarily is hypercentral and I think this must be counted the most pressing open question in this area.

The proofs. Until we come to the proofs of Propositions D and E, we *let D denote a special division ring of characteristic $p \geqslant 0$* (even for these propositions and also Lemma F this is true, but by conclusion rather than direct hypothesis). Recall that unipotent subgroups of $\mathrm{GL}(n, D)$ are always unitriangularizable ([11], 4.2b).

(1) *Let G be a locally nilpotent subgroup of $\mathrm{GL}(n, D)$. Then $\{g \in G: g \text{ unipotent}\}$ is the unipotent radical $u(G)$ of G.*

A subgroup of $\mathrm{GL}(n, D)$ in which 1 is the only unipotent element we call *unipotent-free*. Thus by (1) a locally nilpotent subgroup G of $\mathrm{GL}(n, D)$ is unipotent-free if and only if $u(G) = \langle 1\rangle$.

Proof. Let $h \in \langle g \in G: g \text{ unipotent}\rangle$. If D is finite-dimensional over its centre then h is unipotent by the linear case ([9],7.11). Suppose φ is any ring homomorphism as in the definition of special for which

$$h \in \langle g \in G \cap \mathrm{GL}(n, \mathrm{dom}\,\varphi): g \text{ unipotent}\rangle .$$

If φ_n denotes the map induced by φ on n by n matrices, then $h\varphi_n$ is unipotent. Enough such φ exist to apply ([11], 4.2a). Consequently h is unipotent and the lemma follows.

(2) *Let F be any central subfield of D and G a subgroup of $\mathrm{GL}(n, D)$ with $G/u(G)$ locally finite. Then the F-subalgebra $F[G]$ of the full matrix ring $D^{n\times n}$ generated by G is locally finite-dimensional over F.*

Proof. Suppose G is finitely generated, so now $G/u(G)$ is finite. Let $\mathfrak{n}$ denote the nilpotent radical of $F[G]$. Since $u(G)$ is unitriangularizable we have $u(G) = G\cap(1+\mathfrak{n})$. Hence $\dim_F(F[G]/\mathfrak{n})$ is finite. But $\mathfrak{n}$ is nilpotent and $F[G]$ is a finitely generated F-algebra. Therefore $\dim_F F[G]$ is finite, cf. [8], 4.3.13 or [9], p. 22, Point 1. The general case follows.

(3) *Let L be a locally nilpotent subgroup of $\mathrm{GL}(n, D)$ with $L/u(L)$ periodic. Then there is a unipotent subgroup H and a unipotent-free subgroup K of $\mathrm{GL}(n, D)$ with $[H, K] = \langle 1\rangle$ and $HL = KL = HK \leqslant \mathrm{GL}(n, D)$, such that $N_{\mathrm{GL}(n,D)}(L)$ normalizes both H and K.*

This result has a number of immediate consequences. Clearly $H\cap K = \langle 1\rangle$, so $HK = H\times K$ and $H \cong HK/K = KL/K \cong L/(K\cap L)$ and $K \cong L/(H\cap L)$ are both locally nilpotent. Further $u(HK) = H$ and $u(L) = H\cap L$, for example by (1), so $K \cong L/u(L)$.

Proof. Suppose $p = 0$ and let F denote the centre of D. Then F is trivially perfect and $F[L] \leqslant D^{n\times n}$ is locally finite-dimensional over F by 2. Hence there are by [8], 3.1.7, subgroups H and K of $\mathrm{GL}(n, D)$ exactly as required; in the notation of that result they are the Jordan components $L_u = H$ and $L_d = K$ of L.

Now suppose that $p > 0$. Then L is periodic and locally nilpotent, so $L = 0_p(L)\times 0_{p'}(L)$. Set $H = 0_p(L)$ and $K = 0_{p'}(L)$. In this case H and K are actually characteristic subgroups of L.

(4) *Suppose $G = \langle L(G), R(G)\rangle \leqslant \mathrm{GL}(n, D)$ is a subgroup of $\mathrm{GL}(n, D)$ generated by its Engel elements. Then G is soluble and locally nilpotent.*

Actually every locally nilpotent subgroup of $\mathrm{GL}(n, D)$ is soluble with bounded derived length, see [11].

Proof. G has a local system of finitely generated subgroups H each generated by its own Engel elements. Consider such an H. For each ring homomorphism φ as in the definition of special with $H \leqslant \mathrm{GL}(n, \mathrm{dom}\,\varphi)$, the group $H\varphi_n$, for φ_n the map on n by n matrices induced by φ, is nilpotent by the linear case ([9], 8.15). Then $H\varphi_n$ is soluble of derived length at most $2n$, for example by [11], 1.1 a). (Actually its derived length is at most $1-[-\log_2 n]$ by [8], 3.3.8 and, for $n = 1$, by [11], 1.1a)). Thus G is locally residually, soluble of derived length at most $2n$, and consequently G is soluble. But G is generated by its Engel elements. Hence G is also locally nilpotent by a theorem of Gruenberg ([2], Theorem 4).

(5) *Let G be any subgroup of $\mathrm{GL}(n, D)$. Then*

$$\sigma(G) = \eta(G) = L(G), \quad \eta_1\eta(G) = \bar{\sigma}(G) = \bar{L}(G),$$
$$\varrho(G) = R(G), \quad \bar{\varrho}(G) = \bar{R}(G).$$

In particular the four sets of Engel elements of G are subgroups.

Proof. By (4) the subgroup $\langle L(G), R(G)\rangle$ of G is soluble and locally nilpotent. All claims now follow from another theorem of Gruenberg ([3], Theorem 1.5) with the exception of the equality $\eta_1\eta(G) = \bar{\sigma}(G)$. Always $\eta_1\eta(G) \leqslant \bar{\sigma}(G)$. Set $M = \eta(G)$, $N = \bar{\sigma}(G)$ and $U = u(M) \leqslant N$. By [11], 1.3b) the group M/U is isomorphic to a linear group. Therefore N/U is nilpotent. Also $M'U/U$ is periodic by [11], 1.3c), so $[N, M]U/U$ too is periodic. Apply (3) to $[N, M]U$. Thus we have $[N, M]U \leqslant H\times K \leqslant \mathrm{GL}(n,D)$, where H is unipotent and $K \cong [N, M]U/U$. Hence H, K and so $[N, M]$ are nilpotent. Let $x \in N$. Certainly $\langle x\rangle[N, M]$ is normal in M. But N is a Baer group, so $\langle x\rangle$ is subnormal in N and $[N, M, {}_kx] = \langle 1\rangle$ for some positive integer k. It follows that x acts unipotently on the upper central factors of the nilpotent group $[N, M]$ and so $\langle x\rangle[N, M]$ too is nilpotent. This proves that $N \leqslant \eta_1(M)$ and the proof is complete.

The following lemma is in some ways the main result of this paper.

(6) *Let N be a normal subgroup of right Engel elements of the subgroup G of* $\mathrm{GL}(n, D)$ *and suppose that* $u(N) = \langle 1\rangle$.

(a) $N \leqslant \zeta(G)$.

(b) *N has central height in G at most $\omega + e(n, 2)$ and at most $\omega + e(n, 3)$ if $p = 2$.*

(c) *Let $\pi = \{\text{primes } q\colon p \neq q \leqslant n\}$. Then $[N, G]$ is a π-group and $[N, {}_{n-1}G]$ is abelian. In particular $[N, G] = \langle 1\rangle$ if $n = 1$ or if $n = p = 2$.*

(d) *$N/(N \cap \zeta_1(G))$ is a π-group. If e is the largest p'-divisor of $n!$ then $N/(N\cap \zeta_\omega(G))$ is a finite group of order dividing e.*

(e) *If $N|eG$, i.e. if for all $g \in G$ there is an integer k such that $[x, {}_kg] = 1$ for all $x \in N$, then $N \leqslant \zeta_\omega(G)$.*

(f) *If N is nilpotent then $N \leqslant \zeta_\omega(G)$.*

Proof. (a) Clearly $N = \langle L(N)\rangle$, so by (4) the group N is locally nilpotent. Hence N is abelian-by-finite by [11], 1.3a). Therefore N has a characteristic abelian subgroup A_0 of finite index. By the finite case $[N, {}_rG] \leqslant A_0$ for some positive integer r.

Let $g \in G$. Then $\langle g\rangle A_0$ is hypercentral. Hence by [11], 1.3c) it is unipotent-by-periodic-by-abelian. Then $[A_0, g] \leqslant A_0$ is unipotent-by-periodic, and A_0 is unipotent-free. Consequently $[A_0, g]$ lies in the maximal periodic subgroup T_0 of A_0. This is for all $g \in G$ and so $[A_0, G] \leqslant T_0$.

Finally T_0 is periodic abelian of rank at most n, see [8], 2.3.1 and 2.5.1. Then T_0 has an ascending characteristic series with finite factors and length ω. Hence $T_0 \leqslant \zeta_\omega(G)$ by the finite case and $N \leqslant \zeta(G)$ as required.

(b) The proof of (a) yields that N has G-central height at most $\omega + r + 1$. We must be more careful in the choice of A_0 if we wish to bound r.

Let A be an abelian subgroup of N that is normal in G and is maximal with these properties. Then $C_N(A) \geqslant A$ is also normal in G. If $C_N(A) \neq A$, then since $N \leqslant \zeta(G)$ by Part (a), there exists $x \in C_N(A)\setminus A$ with $[x, G] \leqslant A$. Hence $\langle x\rangle A \leqslant N$ is abelian and normal in G. This contradiction of the maximality of A shows that $A = C_N(A)$. In particular A is also a maximal abelian normal subgroup of the unipotent-free, locally nilpotent group N.

By ([11], 1.3a) the group N/A is isomorphic to a p'-subgroup of $\mathrm{Sym}(n)$. For any prime q the composition length of the Sylow q-subgroup of N/A is at most $e(n, q)$. But

$$e(n, q) \leqslant e(n, 3) \leqslant e(n, 2)$$

for all primes $q \geqslant 5$ and integers $n \geqslant 1$, see [8], p. 112. Consequently N/A has G-central height at most $e(n, 2)$ in general and at most $e(n, 3)$ if $p = 2$. As in the proof of Part (a), with A for A_0, the G-central height of A is at most $\omega + 1$. Once we have proved (e) (in fact (d) will do) we will have $A \leqslant \zeta_\omega(G)$, and the proof of (b) will be complete.

(c) Let $g \in G$. Then $\langle g\rangle N$ is locally nilpotent by (a) and so $[N, g]$ is unipotent by a π-group by [11], 1.3c). But $[N, g] \leqslant N$, which is unipotent-free. Consequently $[N, g] \leqslant 0_\pi(N)$. This is for all $g \in G$ and so $[N\, G] \leqslant 0_\pi(N)$ is a π-group. If $n = 1$ or if $n = p = 2$ then π is empty and $[N, G] = \langle 1\rangle$.

Let A be as in the proof of (b). Then $[N, {}_{e(n,2)}G] \leqslant A$. But A is abelian and $e(n, 2) \leqslant n-1$. Therefore $[N, {}_{n-1}G]$ is abelian.

(d) Let A be as in the proof of (b). Then N/A if a finite π-group of order dividing e, so we need to prove that $A/C_A(G)$ is a π-group. If $g \in G$ then $\langle g\rangle A$ is locally nilpotent, so $[A, g^e]$ is unipotent by [11], 1.3a). But $[A, g^e] \leqslant N$, which is unipotent-free. Therefore $[A, g^e] = \langle 1\rangle$, for all $g \in G$, and we have shown that $G^e \leqslant C_G(A)$.

By (c) the group $T = [A, G] \leqslant A$ is a π-group, and it is also abelian of rank at most n. Thus T is a Černikov group and hence its (outer) automorphism group is isomorphic to some linear group of characteristic zero ([4], 3.38). Thus $H = C_G(T)$ is normal of finite index in G by Burnside's theorem ([9], 1.23). Let $h \in H$. Clearly $A \leqslant \zeta_2(\langle h\rangle A)$. Hence $A^e \leqslant \zeta_1(\langle h\rangle A)$ by [11], 4.5 and the unipotent-freeness of A. Consequently $A^e \leqslant B = C_A(H)$ and so A/B is a π-group.

Again let $g \in G$. Then $\langle g\rangle B$ is hypercentral and [11], 4.5 and the unipotent-freeness of B yields that $(B\cap\zeta_2(G))^e \leqslant \zeta_1(\langle g\rangle B)$. Consequently $(B\cap \zeta_2(G))^e \leqslant B\cap\zeta_1(G)$ and a simple induction yields that

$$(B\cap\zeta_i(G))^e \leqslant B\cap\zeta_{i-1}(G) \quad \text{for all } 2\leqslant i<\omega .$$

But $C_G(B) \geqslant H$ has finite index in G, so $B \leqslant \zeta_\omega(G)$, for example by [9], 8.1. Therefore $B/C_B(G) = (B\cap\zeta_\omega(G))/(B\cap\zeta_1(G))$ is a π-group and hence so is $N/(N\cap\zeta_1(G))$.

In fact $A \leqslant \zeta_\omega(G)$, for if $a \in A$ then $\langle a\rangle T$ is an abelian normal subgroup of G. Now T is Černikov and we have just shown that $A/C_A(G)$ is a π-group. Hence $\langle a\rangle T/C_{\langle a\rangle T}(G)$ is also an abelian Černikov group. As such it is a union of its finite characteristic subgroups. Therefore $\langle a\rangle T \leqslant \zeta_\omega(G)$, for all $a \in A$, and so $A \leqslant \zeta_\omega(G)$ as claimed. But $(N : A)$ is finite of order dividing e. Consequently so too is $(N\colon N\cap\zeta_\omega(G))$.

(e) We may assume that $N = \langle x^G\rangle$ for some x, for example by (1). Let P be the maximal periodic subgroup of N. By (c) we have $[N, G] \leqslant P$ and hence $N = \langle x\rangle P$. By (d) the group $\bar N = N/C_N(G)$ is a π-group. If A is as in the proof of (b) then $(P\colon P\cap A)$ is finite and $P\cap A$ has rank at most n. Therefore $\bar N$ is a Černikov group.

Now $\bar N$ is also nilpotent. Thus the maximal divisible abelian subgroup J of $\bar N$ lies in the centre of $\bar N$ ([4] 1.F.1). Let K denote the kernel of the transfer map of $\bar N$ into J. Then $JK = \bar N$, see [4] 3.9 and proof, J and K are both characteristic in $\bar N$, J is the union of its finite characteristic subgroups and K is finite. Therefore $\bar N$ is the union of its finite characteristic subgroups.

But $N = \langle x^G\rangle$. Consequently $\bar N$ is finite and so $[\bar N, {}_rG] = \langle 1\rangle$ for some integer r. But then $[N, {}_{r+1}(G)] = \langle 1\rangle$ and $N \leqslant \zeta_{r+1}(G) \leqslant \zeta_\omega(G)$. The proof is complete.

(f) Trivially $N = \bar L(N)$. Also N is isomorphic to a linear group by [11], 1.3(b). Thus N is nilpotent by the linear case ([9], 8.15ii). Now apply (e).

The parts of Theorems A1 and B with $u(G) = \langle 1\rangle$ follow at once from (6) by choosing $N = \zeta(G)$. Clearly then $u(N) \leqslant u(G) = \langle 1\rangle$ and $N \leqslant R(G)$. Slightly more generally we have the following.

(7) *Let G be any subgroup of* $\mathrm{GL}(n, D)$.

(a) $\zeta(G)u(G)/u(G)$ *has G-central height at most* $\omega+e(n, 2)$ (*at most* $\omega+e(n, 3)$ *if* $p = 2$).

(b) $[\zeta(G), G]$ *is unipotent by a π-group (π as in (6)(c) and* $[\zeta(G), {}_{n-1}G]$ *is unipotent-by-abelian.*

(c) $\zeta(G)/C_{\zeta(G)}(G/u(G))$ *is a π-group.*

Proof. Since $u(G)$ is un triangularizable, $G/u(G)$ is isomorphic to a completely reducible subgroup of $\mathrm{GL}(n, D)$, see [8], 1.1.2 and proof and 1.1.7. Now apply (6) to $G/u(G)$ and its normal subgroup $N = \zeta(G)u(G)/u(G)$.

(8) *Let G be a subgroup of* $\mathrm{GL}(n, D)$ *with* $u(G) = \langle 1\rangle$. *Then*

$$\eta_1(G) = \bar{L}(G)\,, \text{ which is nilpotent}\,,$$

$$\zeta(G) = R(G) \quad \zeta_\omega(G) = \bar{R}(G)\,.$$

Note that (5) and (8) deal with all but the $u(n, D) < \infty$ case of Theorem C.

Proof. By (5) the set $R(G)$ is a subgroup of G, so by (6) (a) we have $R(G) \leqslant \zeta(G)$. The reverse is always true. Also by (5) we have $\bar{R}(G) = \bar{\varrho}(G)$. If $x \in \bar{\varrho}(G)$ then $\langle x^G\rangle |\,\mathrm{e}G$ and (6) (f) yields that $x \in \zeta_\omega(G)$. It follows that $\bar{\varrho}(G) = \zeta_\omega(G)$.

Again by (5) we have that $\bar{L}(G) = \bar{\sigma}(G)$. Always $\eta_1(G) \leqslant \bar{\sigma}(G)$ and $\bar{\sigma}(G) |\,\mathrm{e}\bar{\sigma}(G)$. Hence by (6) (f) we have $\bar{\sigma}(G) \leqslant \zeta_\omega\bar{\sigma}(G) \leqslant \eta_1(G)$. Finally $\eta_1(G)$ is isomorphic to a linear Fitting group by [11], 1.3b). Therefore $\eta_1(G)$ is nilpotent.

(9) *The proof of Theorems* A1, B *and* C. In view of (5), (6) and (8) we have only to consider the case where $u(n, D) = m < \infty$. Then by hypothesis $\zeta(G) \cap u(G) \leqslant \zeta_m(G)$. Now $G/u(G)$ is isomorphic to a completely reducible subgroup of $\mathrm{GL}(n, D)$. Apply (6) (b) and (6) (c) to its normal subgroup $\zeta(G)u(G)/u(G)$. Then Theorem A1 follows.

By (7) (b) the group $L = [\zeta(G), G]$ is unipotent by a π-group. Apply (3) to L; so let H and K be as in that result. Then $H \cong_G L/(K\cap L)$ is unipotent and hypercentral in GH. Hence $[H, {}_mG] = \langle 1\rangle$ and so

$$[\zeta(G), {}_{m+1}G] \leqslant K \cong L/(L\cap u(G))$$

is a π-group. This proves Theorem B1. If $n = 1$ then $\zeta(G)$ is abelian by (7) (b). If $n > 1$ then $[L, {}_{n-2}G] = [\zeta(G), {}_{n-1}G]$ is unipotent-by-abelian. Thus $[K, {}_{n-2}G]$ is abelian, since $K \cong_G L/u(L)$, and so

$$[L, {}_{\max(m, n-2)}G] \leqslant [H, {}_mG][K, {}_{n-2}G]$$

is also abelian. Theorem B2 follows. Theorems B3 and B4 follow at once from (6) (d) and the containment $\zeta(G) \cap u(G) \leqslant \zeta_m(G)$.

Suppose N is a normal subgroup of G with $N \leqslant R(G)$. Then by (6) (a) and hypothesis $Nu(G)/u(G) \leqslant \zeta(G/u(G))$ and $N \cap u(G) \leqslant \zeta_m(G)$.

It follows that $N \leqslant \zeta(G)$. If in fact $N |\,\mathrm{e}G$ then $Nu(G)/u(G) \leqslant \zeta_\omega(G/u(G))$ by (6) (f) and so $N \leqslant \zeta_\omega(G)$. The remaining case of Theorem C follows from this, cf. the deduction of (8) from (6) (a) and (6) (f).

(10) *The proof of Theorem* A2. Clearly $G = R(G)$ here, so by (6) (a) the group $G/u(G)$ is hypercentral. By hypothesis $u(G) = \langle 1\rangle$ or $u(G) \leqslant \zeta_m(G)$. Therefore G is hypercentral. By (6) (c) we have G abelian if $n = 1$ or if $n = p = 2$ with $u(G) = \langle 1\rangle$. By the same result G has central height at most $m+1$ if $n = p = 2$ with $m < \infty$.

Suppose $n > 2$ or $n = 2 \neq p$. Then we may pass to $G/u(G)$ and assume that $u(G) = \langle 1\rangle$. But then G is isomorphic to a linear group of degree n and characteristic p and so the central height of G is bounded by the maximum central height of such a linear group. For the same reason G cannot have central height ω, see [9], 8.3.

(11) *For every positive integer h there are D and G as in Theorem* A2, *with* $n = p = 2$, $u(2, D) = 2^h$ *and G nilpotent of class exactly* 2^h.

Proof. The construction works for any prime p. Set $q = p^h$ and let $K = P(x_1, \ldots, x_q)$ be the rational function field in the q exhibited variables over the field P of p elements. Let y be an indeterminate acting on K as the field automorphism defined by $y\colon x_i \mapsto x_{i+1}$ for all i modulo q. The skew polynomial ring $K[y]$ is an Ore domain; let D denote its division ring of quotients. Then $C_K(y)(y^q)$ is the centre of D and Galois theory yields that D has index $q = p^h$. By [10], 1.2 we have $u(n, D) \leqslant q(n-1)$ for all $n \geqslant 1$, so certainly $u(2, D) \leqslant p^h$. Set

$$G = \left\langle \begin{pmatrix} y & 0\\ 0 & y\end{pmatrix}, \begin{pmatrix} 1 & 0\\ x & 1\end{pmatrix}\colon x \in D\right\rangle \leqslant \mathrm{GL}(2, D)\,.$$

Since y^q centralizes D, the group G is nilpotent of class at most q. But G has a section isomorphic to the wreath product of a cyclic group of order p by a cyclic group of order q, so the class of G is exactly $q = p^h$ and $u(2, D) = p^h$.

Theorems A2 and B immediately yield the following

(12) *Let G be a locally nilpotent subgroup of* $\mathrm{GL}(n, D)$, *with* $u(G) = \langle 1\rangle$ *or* $u(n, G) < \infty$.

If m, π and e are as in Theorem B *then*:

(a) $\gamma^{m+2}G$ *is a π-group*;

(b) $\gamma^{\max(m+2, n)}G$ *is abelian*; *if $m = 0$ then γ^nG is abelian*;

(c) $G/\zeta_{m+1}(G)$ *is a π-group*;

(d) $G/\zeta_\omega(G)$ *is finite of order dividing e.*

(13) *Let G be a subgroup of* $\mathrm{GL}(n, D)$ *and assume* $u(G) = \langle 1\rangle$ *or* $u(n, D) < \infty$. *Then* $\eta_1(G)/u(G) = \eta_1(\eta(G)/u(G))$ *and* $(\eta(G)\colon \eta_1(G))$ *divides e (e as in Theorem* B).

Note that $u(G)$ is nilpotent, so $u(G) \leqslant \eta_1(G)$ and the statement of (13) makes sense. This little result parallels closely the locally finite-dimensional case, see [8], 3.5.5. It is not true in general, see the examples of (22) and (23) below. Actually by Theorems A2 and B4 we have $\zeta\eta(G) = \eta(G)$, $\zeta_\omega\eta(G) \leqslant \eta_1(G)$ always and $(\zeta\eta(G)\colon \zeta_\omega\eta(G))$ divides e, but the simple proof below avoids using these results. Since

$$\eta_1(G)/u(G) \leqslant \eta_1\eta(G)/u(G) \leqslant \eta_1(\eta(G)/u(G))$$

it also furnishes an alternative proof of a small part of Theorem C, namely that $\eta_1(G) = \eta_1\eta(G)$ is nilpotent.

Proof. Always $\eta_1(G) \leqslant K$ for $K/u(G) = \eta_1(\eta(G)/u(G))$. By [11], 1.3(b) the group $\eta(G)/u(G)$ is isomorphic to a linear group of degree n and characteristic p. In particular by the linear case $K/u(G)$ is nilpotent and $(\eta(G): K)$ divides e, see [9], 8.2ii). Thus all parts of (13) follow from the nilpotence of K. If $u(G) = \langle 1 \rangle$ this is clear. If $u(n, D) = m < \infty$ then $[u(G), {}_m\eta(G)] = \langle 1 \rangle$, so $u(G) \leqslant \zeta_m(K)$ and K is nilpotent.

We conclude this present discussion with two simple conditions that force the central height to be finite.

(14) *Let G be a subgroup of* $\mathrm{GL}(n, R)$, *for some finitely generated subring R of D. Suppose $u(G) = \langle 1 \rangle$ or $u(n, D) < \infty$. Then G has finite central height and $\eta(G)$ is nilpotent.*

If G is any finitely generated subgroup of $\mathrm{GL}(n, D)$ then clearly an R exists as in (14).

Proof. We may assume that $u(G) = \langle 1 \rangle$. By definition of specia R is residually a finite-dimensional algebra. Therefore G is residually finite. But $[\zeta(G), {}_nG]$ is an abelian π-group by Theorem B, and necessarily it has finite rank. Consequently $[\zeta(G), {}_nG]$ is finite and G has finite central height. In particular taking $\eta(G)$ for G and using Theorem A2, the group $\eta(G)$ is nilpotent.

(15) *Let G be a subgroup of* $\mathrm{GL}(n, D)$ *satisfying the maximal condition on abelian characteristic subgroups. If either $u(G) = \langle 1 \rangle$ or $u(n, D) < \infty$ then G has finite central height and $\eta(G)$ is nilpotent.*

Proof. By Theorems B1 and B2 there is a positive integer k such that $[\zeta(G), {}_kG]$ is an abelian π-group of finite rank. It must satisfy the maximal condition on characteristic subgroups. Therefore $[\zeta(G), {}_kG]$ is finite and the result follows.

The following result is the key to Propositions D and E.

(16) *Let F be a field of characteristic $p \geqslant 0$: S a group in $P(\mathfrak{A} \cap \mathfrak{F}^{-s})$ and D the division ring of quotients of the group algebra FS. If $p = 0$ assume $S \in R(\mathfrak{A}\mathfrak{F} \cap \mathfrak{F}^{-s})$. If $p > 0$ assume $S \in R(\mathfrak{A}\mathfrak{F}_{p'} \cap \mathfrak{F}^{-s})$. Then $u(n, D) = n-1$ for all $n \geqslant 1$.*

Note that D in (16) does exist, for example by 1.4.4 of [8].

Proof. Suppose $p > 0$ and $S \in \mathfrak{A}\mathfrak{F}_{p'} \cap \mathfrak{F}^{-s}$. Let A be an apelian normal subgroup of S with $a = (S : A)$ finite and prime to p. Let K be the subfield of D generated by F and A. Then S normalizes K and $(K[S] : K) = a$. In particular it is finite, so $K[S]$ is a division subring of D. Therefore $K[S] = D$ and $(D : K) = a$. By Galois theory $(K : C_K(S)) = (S : C_S(A))$, which divides a. Also $C_K(S)$ lies in the centre Z of D. Hence $(D : Z)$ divides a^2 and in particular is prime to p. Consequently $u(n, D) \leqslant n-1$ by 1.2 of [10]. But $n-1 \leqslant u(n, F) \leqslant u(n, D)$. Therefore $u(n, D) = n-1$ in this case.

Now suppose $p > 0$ and $S \in R(\mathfrak{A}\mathfrak{F}_{p'} \cap \mathfrak{F}^{-s})$. Then by [11], 5.1 the ring D is locally residually a division ring of the type just considered. It follows easily that

$u(n, D) = n-1$. Finally if $p = 0$ a slight simplification of the above proof settles this case too.

(17) *The proof of Proposition* E. Part 1 follows from (16) and [11], 1.4b, Part 2 from (16) and [11], 1.4a) and Part 4 immediately from Part 3. Thus consider Part 3. If $p = 0$ then S satisfies the hypothesis of Part 1 by [11], 5.6. Suppose $p > 0$. The same result yields that

$$S \in P(\mathfrak{A} \cap \mathfrak{F}^{-s}) \cap R(\mathfrak{A}\mathfrak{F}_p \cap \mathfrak{F}^{-s}).$$

But also $S \in R(\mathfrak{A}\mathfrak{F}_{p'} \cap \mathfrak{F}^{-s})$ by [7], Vol. 2, 9.41 and 9.43. Now we can apply Part 2.

(18) *The proof of Proposition* D. For each finite subset X of D and finite subset Y of the ubring R of D generated by X, there is a positive integer d and a ring homomorphism of R into the division ring of quotients of the group algebra over F of L modulo the dth term of the derived series of L, that is one-to-one on Y, see [6] Section 6.1. Thus Proposition D follows from Part 4 of Proposition E.

(19) *Let F be a field of characteristic* 2, *S a group in $P(\mathfrak{A} \cap \mathfrak{F}^{-s}) \cap (\mathfrak{A}\mathfrak{F}_{2'} \cap \mathfrak{F}^{-s})$ and D the division ring of quotients of FS. Suppose G is a subgroup of* $\mathrm{GL}(2, D)$ *and N is a normal subgroup of G with $N \leqslant \zeta(G) \cap \mathrm{Tr}(2, D)$. Then $[N, G]$ is unipotent-free.*

Proof. By [11], 5.1 and a simple local residual argument we may assume that $S \in \mathfrak{A}\mathfrak{F}_{2'} \cap \mathfrak{F}^{-s}$. Let Z denote the centre of D. As in the proof of (16) we have that $(D : Z)$ is finite and odd. Hence if $x \in D$ then the minimal poynomial of x over Z has odd degree and in particular is separable. It follows easily that if $g \in N$ then the characteristic polynomial of g over Z in the obvious action of g on the Z-space $D \oplus D$ is also separable. Therefore the Jordan decomposition of g over Z takes place in $\mathrm{GL}(2, Z[g, g^{-1}]) \leqslant \mathrm{GL}(2, D)$, cf. [8], 3.1.6 and proof. Thus $N \leqslant N_u \times N_d \leqslant \mathrm{GL}(2, D)$ by [8], 3.1.7, where N_u is unipotent, N_d is unipotent-free, and both are acted on hypercentrally by G, see [8], 3.1.8. Then $[N_u, G] = \langle 1 \rangle$ by (16) and so $[N, G] \leqslant N_d$. Therefore $[N, G]$ is unipotent-free.

(20) *Let D be any of the division rings considered in Propositions* D *and* E. *A locally nilpotent subgroup of* $\mathrm{GL}(n, D)$ *has central height at most the greatest central height of any locally nilpotent linear group of degree n and characteristic* $\mathrm{char}\, D$. *If* $\mathrm{char}\, D = 2$ *then every subgroup of* $\mathrm{GL}(2, D)$ *has central height at most* 1.

Proof. By Theorem A2 we have only to consider the case $n = \mathrm{char}\, D = 2$. Thus let G be a subgroup of $\mathrm{GL}(2, D)$ and set $N = \zeta(G)$. If N is irreducible then $N \cap u(G) = \langle 1 \rangle$ and $[N, G] = \langle 1 \rangle$ by Theorem B1. If N is not irreducible then by replacing G by a conjugate we may assume that $N \leqslant \mathrm{Tr}(2, D)$ Also $[N, G]$ is unipotent by (7) (b). If D is as in Propositon E then (19) appli.s directly, $[N, G]$ is unipotent-free and so $[N, G] = \langle 1 \rangle$. If D is as in Proposition D then as in (18) we obtain that $[N, G]$ is locally residually trivial. Hence again $[N, G] = \langle 1 \rangle$.

(21) *Let D be any division ring as in Propositions* D *or* E *with* $\mathrm{char}\, D = 0$ *and suppose G is a subgroup of* $\mathrm{GL}(n, D)$. *Then $[\zeta(G), {}_{n-1}G]$ is abelian and unipotent-free and $[\zeta(G), {}_{\max(1, n-1)}(G)]$ and $\zeta(G)/\zeta_{\max(1, n-1)}(G)$ are locally residually π-groups.*

Note that for D as in (21), Theorem B only yields that $[\zeta(G), {}_nG]$ is an abelian π-group and that $\zeta(G)/\zeta_n(G)$ is a π-group.

Proof. As in previous proofs we reduce to the case where D is as in (16) with $\operatorname{char} D = 0$ and $S \in \mathfrak{A}\mathfrak{F} \cap \mathfrak{F}^{-s}$. Then D is finite-dimensional over its centre and $\zeta(G)$ has a Jordan decomposition

$$\zeta(G) \leqslant \zeta(G)_u \times \zeta(G)_d \leqslant \mathrm{GL}(n, D),$$

on the factors of which G acts hypercentrally, see [8], 3.1.8. Since $u(n, D) = n-1$ by (16) we have $[\zeta(G)_{u}, {}_{n-1}G] = \langle 1 \rangle$. Also $[\zeta(G)_d, G]$ is a π-group and $[\zeta(G)_{d}, {}_{n-1}G]$ is abelian by (6) (c). Therefore $[\zeta(G), {}_{\max\{1, n-1\}}G]$ is a π-group and $[\zeta(G), {}_{n-1}G]$ is abelian. Further (6) (d) applied to $\zeta(G)_d$ yields that $\zeta(G)/\zeta_{\max\{1, n-1\}}(G)$ is also a π-group. The proof is complete.

(22) *The proof of Lemma* F. By a simple localization argument we may assume that S has an abelian normal subgroup $T \leqslant C_S(F)$ such that $(S : T)$ is finite and a power of p. Let K be the subfield of D generated by F and T. Then $D = K[S]$ and $(D : K) = (S : T)$. By Galois theory $(K: C_K(S)) = (S: C_S(K))$, which divides $(S : T)$. Clearly $C_K(S)$ lies in the centre Z of D. Consequently $(D : Z)$ is a finite power of p and D is special.

(23) *Some examples in positive characteristic.* Let p be a prime and $P = A]B$ the split extension of the elementary abelian p-group A by the abelian group B. Suppose that for each finite subset X of A and finitely generated subgroup Y of B there is a subgroup of Y of finite index a power of p centralizing $\langle X, Y \rangle$. Then in particular P is locally nilpotent, being locally, centre by a metabelian p-group.

Pick a basis for A and let F be the field of rational functions in the elements of this basis over the field of p elements. Then A can be regarded as an additive subgroup of F and the action of B on A defines an action of B on F. Let S be a free abelian group mapping (homomorphically) onto B with kernel T. Use this map of S to B to lift the action of B to one of S on F. Then F and S satisfy the hypotheses of Lemma F.

Let D be the special division ring of characteristic p so determined by F and S. Set

$$G = \left\langle \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}, \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} : a \in A, s \in S \right\rangle \leqslant \mathrm{GL}(2, D).$$

Then G is isomorphic to a split extension $A]S$ of A by S and so G is a central extension of T by P. (This step is the critical use of the hypothesis that B is abelian.) In particular G is locally nilpotent. Moreover G will be hypercentral (resp. Baer, Fitting or Gruenberg) if and only if P is. We now make different choices for P.

(a) Suppose P is the (standard, restricted) wreath product of a cyclic group of order p by a countably infinite, elementary abelian p-group. Then P is a Fitting group but is not hypercentral (it does not even have the normalizer condition, see [7], Vol. 2, p. 3). *Thus G is a Fitting group but is not hypercentral.* Also the centre of P is trivial, so $\zeta_1(G) = \zeta(G)$, and γ [illegible] is an infinite p-group for $1 \leqslant i < \omega$, so $\gamma^i G$ is an infinite p-group for $2 \leqslant i < \omega$. In particular $\gamma^i G$ is never a π-group for $i < \omega$.

(b) Suppose P is the direct product of groups P_i for $i = 1, 2, \ldots$, where P_i is the wreath product of a cyclic group of order p by a cyclic group of order p^i. Then *G is hypercentral and hypocentral, and a Fitting group, but G is not nilpotent.* Also G has central height and central depth ω and $\gamma^i G$ is an infinite p-group for $2 \leqslant i < \omega$.

(c) Suppose $P = A]B$, where $B = \langle b \rangle$ is infinite cyclic, A is the direct product of groups A_j for $j = 1, 2, \ldots$ where $A_j = \langle a_{1j} \rangle \times \ldots \times \langle a_{p^j, j} \rangle$ is elementary abelian of order p^{p^j} and $a_{ij}^b = a_{i+1, j} (= a_{1j}$ if $i = p^j)$ for all i and j. Then P is not a Baer group, for if it were we would have B subnormal in P, say in k steps, and then for $p^j > k+1$ we would have

$$1 = [a_{1j, k+1} b] = a_{1j}^{?} a_{2j}^{?} \ldots a_{k+1, j}^{?} a_{k+2, j} \neq 1.$$

But P is hypercentral; clearly A has P-central height exactly ω. If $b^r \in \zeta_\omega(P)$ for some r then $b^r \in \zeta_s(P)$ for some $s < \omega$ and $\langle b^r \rangle$ is subnormal in $A\langle b^r \rangle$. A similar argument to that above (showing $\langle b \rangle$ is not subnormal) shows that this is false. Therefore $\zeta_\omega(P) = A$. For the same reason $\eta_1(P) = A$.

Here B is free abelian, so we can choose $S = B$, $T = \langle 1 \rangle$ and $G \cong P$. Then *G is hypercentral but not a Baer group. Also G has central height $\omega+1$ and $G/\zeta_\omega(G) = G/\eta_1(G)$ is infinite cyclic. Further G is hypocentral of central depth ω and $\gamma^i G$ is an infinite p-group for $2 \leqslant i < \omega$. In particular $(\eta(G) : \eta_1(G))$ is infinite.*

(d) Suppose P is the direct product of the three groups P considered in (a), (b) and (c) above. The resulting group G then exhibits the above phenomena in a single group. In particular

$$\zeta_0(G) < \zeta_1(G) < \ldots < \zeta_i(G) < \ldots < \zeta_\omega(G),$$

$$\zeta_{\omega+1}(G) = \zeta(G), \quad \eta_1(G) = \bar{\sigma}(G) \quad \text{and} \quad \sigma(G) = \eta(G) = G$$

are all distinct.

(23) *Some examples in characteristic zero.* We know of no example of a locally nilpotent group of matrices over a special division ring of characteristic zero that is not hypercentral. However we do have the following.

Let F be a field of characteristic zero, $L = Fu \oplus Fv$, the Lie F-algebra determined by $[v, u] = u$, and D the division ring of quotients of the universal enveloping algebra of L. Then *D is a special division ring of characteristic zero.* Set

$$G = \left\langle a_i = \begin{pmatrix} 1 & 0 \\ v^i & 1 \end{pmatrix}, g = \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} : i = 0, 1, 2, \ldots \right\rangle \leqslant \mathrm{GL}(n, D).$$

Then G is hypercentral of central height $\omega+1$ and hypocentral of central depth ω. Also $\zeta_\omega(G) = \eta_1(G) = \bar{\sigma}(G)$, $\zeta(G) = \sigma(G) = \eta(G) = G$ and $\zeta(G)/\zeta_\omega(G) = \eta(G)/\eta_1(G)$ is infinite cyclic. $\gamma^i G$ is infinite and torsion-free for $1 \leqslant i < \omega$.

This group G is the group considered in [11], 6.2. Let $A = \langle a_i \colon i \geqslant 0\rangle$. Then A is an abelian normal subgroup of G, the group G/A is infinite cyclic and

$$A \leqslant \zeta_\omega(G) \leqslant \eta_1(G) \leqslant \bar{\sigma}(G) \leqslant \bar{L}(G),$$

see [11], 6.2. By 3.4 of [10] we have $(v^i, {}_iu) = i!$ and the same inductive proof on i yields that $(v^i, {}_iu^r) = i!r^i$ for all $r \in \mathbf{Z}$ and $i \geqslant 0$. Thus $\bar{L}(G) \cap \langle g\rangle = \langle 1\rangle$ and $A = \bar{L}(G)$. Also by [11], 6.2 the group G is hypocentral of central depth at most ω and G is not nilpotent so the central depth is exactly ω. Finally the claim concerning $\gamma^i G$ follows easily.

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The structure of compacta satisfying $\dim(X\times X) < 2 \dim X$

by

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Dedicated to Professor Yukihiro Kodama on his Sixtieth Birthday

Abstract. Let X be a 2-dimensional compactum. In this note we prove that the condition $\dim(X\times X) < 4$ is satisfied iff any mapping from an arbitrary closed subset A of X into a circle S admits an extension $X \to B_n$ for some n, where B_n is a certain 2-dimensional CW-complex defined in Introduction. As a corollary we obtain that if $\dim(X\times X) < 4$, then any mapping $X \to \mathbf{R}^4$ can be approximated arbitrarily closely by imbeddings. This together with results of [Kr] and [Sp] shows that for an m-dimensional compactum X the condition $\dim(X\times X) < 2m$ is satisfied if and only if the set of all imbeddings $X \to \mathbf{R}^{2m}$ is dense in the space of all mappings $X \to \mathbf{R}^{2m}$.

Introduction. In this note we prove the case $m = 2$ of the following

THEOREM 1. *Let X be an m-dimensional compactum. If $\dim(X\times X) < 2m$, then the set $E(X, \mathbf{R}^{2m})$ of all imbeddings $X \to \mathbf{R}^{2m}$ is dense in the space $C(X, \mathbf{R}^{2m})$ of all mappings $X \to \mathbf{R}^{2m}$.*

The above implication for $m > 2$ was established in [Sp]. The inverse implication was proved by J. Krasinkiewicz in [Kr] for all m. Theorem 1 and a result of [Kr] imply the following

THEOREM 2. *For an m-dimensional compactum X the condition $\dim(X\times X) < 2m$ is satisfied if and only if the set $E(X, \mathbf{R}^{2m})$ is dense in the space $C(X, \mathbf{R}^{2m})$.*

The above result was conjectured by J. Krasinkiewicz (cf [M–R2]). For other related results the reader is referred to [M–R1], [K–L], [M–R2], [Kr], [Sp] and [K–K].

The case $m = 2$ of Theorem 1 is a consequence of the following main result of our paper. In the statement we need the following notion. Let $S \vee T$ be the one point union, with the base-point $*$, of the circles S and T. Let a and b be generators of the 1-homotopy groups $\pi_1(S, *)$ and $\pi_1(T, *)$, respectively. By a *2-dimensional Boltyanskii–Kodama bubble* B_n^2 we understand the CW-complex obtained by attaching two 2-cells to $S\vee T$ by mappings corresponding to the element $a^n b^{n^2} \in \pi_1(S\vee T, *)$ and the commutator $[a, b] \in \pi_1(S \vee T, *)$, respectively. The reason for using this name for B_n^2 is that Boltyanskii and Kodama applied a similar CW-complex in