This group $G$ is the group considered in [11], 6.2. Let $A = \langle a_i \colon i \geqslant 0 \rangle$. Then $A$ is an abelian normal subgroup of $G$, the group $G/A$ is infinite cyclic and

$$A \leqslant \zeta_\omega(G) \leqslant \eta_1(G) \leqslant \bar{\sigma}(G) \leqslant \bar{L}(G),$$

see [11], 6.2. By 3.4 of [10] we have $(v^i, {}_i u) = i!$ and the same inductive proof on $i$ yields that $(v^i, {}_i u^r) = i! r^i$ for all $r \in \boldsymbol{Z}$ and $i \geqslant 0$. Thus $\bar{L}(G) \cap \langle g \rangle = \langle 1 \rangle$ and $A = \bar{L}(G)$. Also by [11], 6.2 the group $G$ is hypocentral of central depth at most $\omega$ and $G$ is not nilpotent so the central depth is exactly $\omega$. Finally the claim concerning $\gamma^i G$ follows easily.

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*Received 26 December 1988*

# The structure of compacta satisfying $\dim(X \times X) < 2 \dim X$

by

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*Dedicated to Professor Yukihiro Kodama on his Sixtieth Birthday*

**Abstract.** Let $X$ be a 2-dimensional compactum. In this note we prove that the condition $\dim(X \times X) < 4$ is satisfied iff any mapping from an arbitrary closed subset $A$ of $X$ into a circle $S$ admits an extension $X \to B_n$ for some $n$, where $B_n$ is a certain 2-dimensional CW-complex defined in Introduction. As a corollary we obtain that if $\dim(X \times X) < 4$, then any mapping $X \to \boldsymbol{R}^4$ can be approximated arbitrarily closely by imbeddings. This together with results of [Kr] and [Sp] shows that for an $m$-dimensional compactum $X$ the condition $\dim(X \times X) < 2m$ is satisfied if and only if the set of all imbeddings $X \to \boldsymbol{R}^{2m}$ is dense in the space of all mappings $X \to \boldsymbol{R}^{2m}$.

**Introduction.** In this note we prove the case $m = 2$ of the following

THEOREM 1. *Let $X$ be an $m$-dimensional compactum. If $\dim(X \times X) < 2m$, then the set $E(X, \boldsymbol{R}^{2m})$ of all imbeddings $X \to \boldsymbol{R}^{2m}$ is dense in the space $C(X, \boldsymbol{R}^{2m})$ of all mappings $X \to \boldsymbol{R}^{2m}$.*

The above implication for $m > 2$ was established in [Sp]. The inverse implication was proved by J. Krasinkiewicz in [Kr] for all $m$. Theorem 1 and a result of [Kr] imply the following

THEOREM 2. *For an $m$-dimensional compactum $X$ the condition $\dim(X \times X) < 2m$ is satisfied if and only if the set $E(X, \boldsymbol{R}^{2m})$ is dense in the space $C(X, \boldsymbol{R}^{2m})$.*

The above result was conjectured by J. Krasinkiewicz (cf [M–R2]). For other related results the reader is referred to [M–R1], [K–L], [M–R2], [Kr], [Sp] and [K–K].

The case $m = 2$ of Theorem 1 is a consequence of the following main result of our paper. In the statement we need the following notion. Let $S \vee T$ be the one point union, with the base-point $*$, of the circles $S$ and $T$. Let $a$ and $b$ be generators of the 1-homotopy groups $\pi_1(S, *)$ and $\pi_1(T, *)$, respectively. By a 2-*dimensional Boltyanskii–Kodama bubble* $B_n^2$ we understand the CW-complex obtained by attaching two 2-cells to $S \vee T$ by mappings corresponding to the element $a^n b^{n^2} \in \pi_1(S \vee T, *)$ and the commutator $[a, b] \in \pi_1(S \vee T, *)$, respectively. The reason for using this name for $B_n^2$ is that Boltyanskii and Kodama applied a similar CW-complex in their constructions of a 2-dimensional compactum $X$ such that $\dim(X\times X)<4$ (see [Bo] and [Ko]).

THEOREM 3. *Let $X$ be a (2-dimensional) compactum. Then $\dim(X\times X)<4$ if and only if for an arbitrary closed subset $A$ of $X$ any mapping $A\to S$ admits an extension $X\to B^2_n$ for some $n$.*

In the proof of the case $m=2$ of Theorem 1 an essential role is played by the fact that two linked 1-spheres in $S^3=\partial D^4$ bound a certain pair of disjoint membranes in the ball $D^4$. This fact, first discovered in [K–L] and further developed in [M–R], is expressed here in Proposition (4.8). Combined with (4.8), Theorem 3 implies (see § 4, compare [K–K]) the following corollary conjectured in [Kr]:

COROLLARY 1. *Let $X$ be a (2-dimensional) compactum such that $\dim(X\times X)<4$. Then for an arbitrary closed subset $A$ of $X$ and any mapping $f\colon A\to S$ there exist mappings*

$$F_1\colon X\to D^2\times D^2 \quad \text{and} \quad F_2\colon X\to D^2\times D^2$$

*with disjoint images such that*

$$F_1(x)=(f(x),0) \quad \text{and} \quad F_2(x)=(0,f(x))$$

*for each $x\in A$. This means that any mapping $g\colon X\to D^2$ is transversely trivial in the sense of* [Kr].

Above $D^2$ denotes a 2-dimensional disc with the boundary $S$. By Theorem (2.2) in [Kr] (compare also [M–R2]. Corollary 1 implies the case $m=2$ of Theorem 1.

The strategy of the proof of Theorem 3 and Fundamental Lemma is as follows. Let $A$ be a closed subset of a 2-dimensional compactum $X$ and let $g\colon (X,A)\to(D^2,S)$ be a mapping such that

(a) the homomorphism $H^2(g)\colon H^2(D^2,S)\to H^2(X,A)$ has the property $H^2(g)\otimes H^2(g)=0$

(observe that (a) holds if $\dim(X\times X)<4$). Then the mapping $g$ is homotopic to a composition

$$(X,A)\to(L,L_0)\xrightarrow{g'}(D^2,S), \quad \text{where}$$

(b) $H^2(g')\otimes H^2(g')=0$, and

(c) $(L,L_0)$ is a pair of finite polyhedra, $\dim L\leqslant 2$.

Now our task is to change the diagram above to the form where $L_0=S$, $g'|L_0=\mathrm{id}_S$ and $L$ has a possibly simple structure; it is important in this simplification process not to loose property (b). To do so we introduce in § 1 the following notion.

We say that a mapping $f\colon(L,L_0)\to(K,S)$ is *admissible*, if for any homomorphism $h$ of abelian groups the following condition is satisfied

$$(H^2(f)\circ\delta^*)\otimes h \text{ is trivial iff } \delta^*\otimes h \text{ is trivial},$$



where $\delta^*\colon H^1(S)\to H^2(K,S)$ is the coboundary homomorphism. Since

$$(H^2(f)\circ\delta^*)\otimes h=(H^2(f)\otimes\mathrm{id})\circ(\delta^*\otimes h)$$

thus only one implication in the above condition is essential.

After discussing in § 1 and § 2 some preliminary results and examples concerning admissible mappings we prove in § 3 a sequence of lemmas leading to the following main step of the proof of 2-dimensional case of Main Theorem:

FUNDAMENTAL LEMMA (3.9). *Let $(L,L_0)$ be a pair of finite polyhedra such that $\dim L=2$. Then any mapping $L_0\to S$ can be extended to an admissible mapping $f\colon(L,L_0)\to(K,S)$ where $K$ has the following form:*

*(i) $K$ is a CW-complex obtained by adjoining 2-cells to a one-point union $S\vee S_1\vee\ldots\vee S_k$ of circles by attaching mappings corresponding to words of the form*

$$b_0^{n(j)}b_j^{m(j)} \quad \text{for} \quad 1\leqslant j\leqslant k \quad \text{and}$$
$$[b_i,b_j] \quad \text{for} \quad 0\leqslant i<j\leqslant k,$$

*where $b_0$ is a generator of $\pi_1(S,*)$ and $b_j$ is a generator of $\pi_1(S_j,*)$, for $1\leqslant j\leqslant k$.*

We note that the condition (b) for a mapping

$$(K,S)\to(D^2,S),$$

which is an identity on $S$, implies that

(ii) $m(j)$ divides $(n(j))^2$ for each $j$, $1\leqslant j\leqslant k$.

Using this we prove in § 4 the following

PROPOSITION 1. *Let $A$ be a closed subset of a 2-dimensional compactum $X$. Then, a mapping $g\colon(X,A)\to(D^2,S)$ satisfies $H^2(g)\otimes H^2(g)=0$ if and only if $g|A$ admits an extension $(X,A)\to(K,S)$, where $K$ is a CW-complex satisfying conditions (i) and (ii) above.*

Finally, if $A$ is a closed subset of a compactum $X$ such that $\dim(X\times X)<4$, then we prove that the assertions of Proposition 1 are satisfied with $k=1$, thereby completing the process of simplifying the polyhedral structure of the complex $K$ to the final form $B^2_n$.

In a similar way as in our paper one can prove a higher dimensional version of Fundamental Lemma (3.9) and then:

THEOREM 3′. *Let $X$ be an $m$-dimensional compactum. Then, $\dim(X\times X)<2m$ if and only if for an arbitrary closed subset $A$ of $X$ any mapping $A\to S^{m-1}$ admits an extension $X\to B^m_n$ for some $n$, where $B^m_n$ is an $m$-dimensional Boltyanskii–Kodama bubble.*

By an *$m$-dimensional Boltyanskii–Kodama bubble* $B^m_n$, where $m\geqslant 3$, we understand the $m$-dimensional CW-complex obtained by attaching one $m$-cell to the one-point union $S^{m-1}\vee T^{m-1}$ of $(m-1)$-spheres by the mapping corresponding to the element $a^nb^{n^2}$ of the group $\pi_{m-1}(S^{m-1}\vee T^{m-1},*)$, where $a$ and $b$ are generators of the groups $\pi_{m-1}(S^{m-1}, *)$ and $\pi_{m-1}(T^{m-1}, *)$, respectively. (We use multiplicative notation for the higher homotopy groups also.)

The case $m \geqslant 3$ of Theorem 1 follows from Theorem 3′. We will not give details of the proof, since they are analogous to the ones discussed before in the case $m = 2$. An alternative proof for the case $m \geqslant 3$ was in given [Sp] using the Whitney lemma.

Using the methods of this paper one can prove the following characterization of compacta with the property which occures in the famous construction of Pontrjagin [Po].

THEOREM 4. *Let $X$ and $Y$ be compacta such that* $\dim X = m$ *and* $\dim Y = m'$. *Then,* $\dim(X\times Y) < m+m'$ *if and only if for arbitrary closed subsets $A$, $B$ of $X$, $Y$, respectively, any mappings*

$$A \to S^{m-1} \quad \text{and} \quad B \to S^{m'-1}$$

*admit extensions*

$$X \to P^m_{k,l,r} \quad \text{and} \quad Y \to P^{m'}_{k',l',r'}$$

*such that the common greatest divisor of $l$ and $l'$ divides the product $k\cdot k'$, $l$ divides $k\cdot r'$, $l'$ divides $k'\cdot r$ and $r\cdot r' = 0$.*

Above, $P^m_{k,l,r}$ denotes the CW-complex obtained by attaching three $m$-cells to $S^{m-1}\vee T^{m-1}$ by the mappings corresponding to $a^k b^l$, $a^r$ and $[a, b]$, respectively. (If $m \geqslant 2$ then the cell corresponding to $[a, b]$ is clearly redundant.)

In this note we will not give the proof of the above theorem, since it is similar (however slightly more elaborate) to that of Theorem 3. The main step is a version of Proposition 1 that characterizes the property $H^m(f)\otimes H^n(g) = 0$ of mappings

$$f\colon (X, A)\to (D^m, S^{m-1}) \quad \text{and} \quad g\colon (Y, B)\to (D^n, S^{n-1})$$

in terms of extendibility of the mappings $f|A$ and $g|B$ into certain polyhedra of the form considered in (3.9) or its higher dimensional analogue.

The paper is concluded with some remarks stated in § 5.

We would like to add that recently (during a visit to Warsaw in the middle of December 1988) A. N. Dranishnikov has informed that he, E. V. Shchepin and D. Repovš have also been working on problems similar to those discussed in the papers [Kr] and [Sp] (submitted to Fundamenta Mathematicae in early Spring of 1988) and those of the present note.

**1. Admissible mappings.** By $D^n$ we denote the unit $n$-ball, i.e. $D^n = \{x\in R^n|\ ||x|| \leqslant 1\}$, and by $S^{n-1}$ we denote the unit $(n-1)$-sphere, i.e. the boundary of the ball $D^n$.

Let $(X, A)$ and $(Y, B)$ be pairs of compacta and let

$$p\colon A\to S^{n-1} \quad \text{and} \quad q\colon B\to S^{n-1}$$

be (continuous) mappings. Let

$$p'\colon (X, A)\to (D^n, S^{n-1}) \quad \text{and} \quad q'\colon (Y, B)\to (D^n, S^{n-1})$$



extend $p$ and $q$, respectively. By the convexity of $D^n$, the homomorphisms $H^n(p')$ and $H^n(q')$ depend on $p$ and $q$ only. We say that a mapping $f\colon X\to Y$ is *admissible with respect to $p$ and $q$* if the following two conditions are satisfied

(A1) $f(A)\subseteq B$ and $q\circ f|A = p$, and

(A2) for any homomorphism $h$ of abelian groups, the tensor product $h\otimes H^n(p')$ is trivial if and only if the tensor product $h\otimes H^n(q')$ is trivial.

Observe that the composition of admissible mappings is admissible.

If $A = B = S^{n-1}$ and $p$, $q$ are the identities on $S^{n-1}$ then $f\colon X\to Y$ which is admissible with respect to $p$ and $q$ we will simply call *admissible with respect to* $S^{n-1}$. Observe that in this case the conditions (A1) and (A2) are equivalent, respectively, to the following two conditions

(A3) $f|A$ is the identity on $S^{n-1}$, and

(A4) for any homomorphism $h$ of abelian groups, the tensor product $h\otimes \delta^*$ is trivial if and only if the tensor product $h\otimes(\delta')^*$ is trivial; here $\delta^*$ and $(\delta')^*$ are the connecting homomorphisms from $H^{n-1}(S^{n-1})$ into $H^n(X, S^{n-1})$ and $H^n(Y, S^{n-1})$, respectively

The following lemmas will be useful.

(1.1) LEMMA. *Suppose, for $i = 1, 2$, there are mappings $X_i\to Y_i$ which are admissible with respect to the mappings $p_i\colon A_i\to S^{n_i-1}$ and $q_i\colon B_i\to S^{n_i-1}$. Let*

$$p_i'\colon (X_i, A_i)\to (D^{n_i}, S^{n_i-1}) \quad \text{and} \quad q_i'\colon (Y_i, B_i)\to (D^{n_i}, S^{n_i-1})$$

*extend $p_i$ and $q_i$, respectively. Then the homomorphism $H^{n_1}(p_1')\otimes H^{n_2}(p_2')$ is trivial if and only if the homomorphism $H^{n_1}(q_1')\otimes H^{n_2}(q_2')$ is trivial.*

(1.2) LEMMA. *Suppose that $S^{n-1}\subseteq X\cap Y$. If the mappings $f\colon X\to Y$ and $g\colon Y\to X$ are both identities on $S^{n-1}$ then they are admissible with respect to $S^{n-1}$.*

(1.3) COROLLARY. *Let $X'$ be obtained from $X$ by adjoining a cell by a homotopically trivial attaching map $S^r\to X$. Then there exist mappings $X\to X'$ and $X'\to X$ which are admissible with respect to any sphere contained in $X$.*

Proof. It is well known and easy to proove that there is a retraction $X'\to X$. The corollary follows by (1.2).

**2. Admissible transformations of 2-dimensional CW-complexes.** First let us introduce some conventions. In the sequel of the paper we fix a copy of the unit circle and denote it by $S_0$ or by $S$. Let $S_0\vee\ldots\vee S_k$ denote the one point union, with the base point $*$, of the circles $S_0, \ldots, S_k$ and let $a_j$ be a generator of $\pi_1(S_j, *)$ for $0\leqslant j\leqslant k$. If $\{v_i\colon i\in I\}$ is any collection of words in the symbols $a_0, \ldots, a_k$ then one can form a 2-dimensional CW-complex whose 1-skeleton is $S_0\vee\ldots\vee S_k$ and whose 2-cells $\{\sigma_i\colon i\in I\}$ are such that the attaching map of $\sigma_i$ is given by $v_i\in\pi_1(S_0\vee\ldots\vee S_k, *)$ for each $i\in I$. We call

$$P = \{a_0, \ldots, a_k;\ \{v_i\colon i\in I\}\}$$

the *presentation* of the arising complex $K(P)$ and identify that complex with $P$ often. Given presentations

$$P = \{a_0, \dots, a_k;\ v_1, \dots, v_p\} \quad \text{and} \quad Q = \{b_0, \dots, b_l;\ w_1, \dots, w_q\}$$

we say therefore, with some abuse of the language, that $f\colon P \to Q$ is a *mapping fixed on* $a_0$ (resp. $f\colon P \to Q$ is an *admissible mapping with respect to* $a_0$ or shortly an *admissible mapping*) if $a_0 = b_0$ correspond to the same distinguished circle $S_0 \subseteq K(P) \cap K(Q)$ and $f\colon K(P) \to K(Q)$ is a mapping such that $f(x) = x$ for $x \in S_0$ (resp. $f\colon K(P) \to K(Q)$ is an admissible mapping with respect to $S_0$). Similarly, we sometimes speak about mappings $P \to Q$ which are fixed on a specified set of generators belonging to $\{a_0, \dots, a_k\} \cap \{b_0, \dots, b_l\}$.

Let us recall (see [M-K-S]) that an *elementary Tietze transformation* of the presentation

$$P = \{a_0, \dots, a_k;\ v_1, \dots, v_p\}$$

is one of the following transformations applied to $P$.

(T.1) If the word $w$ is derivable from $v_1, \dots, v_p$, then add $w$ to the words in $P$.

(T.2) If the word $v_i$, for some $1 \leqslant i \leqslant p$, is derivable from the other words in $P$, then delete $v_i$ from the words in $P$.

(T.3) If $w$ is any word in $a_0, \dots, a_k$, then adjoin the symbol $a$ to the generators in $P$ and adjoin the word $a^{-1}w$ to the words in $P$.

(T.4) If some of the words in $P$ take the form $a_j^{-1}w$, for some $0 \leqslant j \leqslant k$, and $w$ is a word in generators of $P$ other than $a_j$, then delete $a_j$ from the generators in $P$, delete $a_j^{-1}w$ from the words in $P$ and replace $a_j$ by $w$ in the remaining words in $P$.

If in (T.4) we additionally assume that $a_j \neq a_0$ then we say that the above transformations are *elementary Tietze transformations fixed on* $a_0$.

(2.1) Lemma. *If there exists an elementary Tietze transformation from $P$ to $Q$ which is fixed on $a_0$ then there exists an admissible mapping $f\colon P \to Q$.*

Proof. If the given transformation is of type (T.1) or (T.2) then the lemma follows by (1.3). If it is of type (T.3) then it is fairly easy to construct mappings $f\colon P \to Q$ (the inclusion) and $g\colon Q \to P$ (a retraction) which are fixed on $a_0$; thus (1.2) applies.

Now, assume that for some $j$ and $i$, $0 < j \leqslant k$ and $1 \leqslant i \leqslant p$, we have $v_i = a_j^{-1}w$, where $w$ is a word in generators in $P$ other than $a_j$. For simplicity we assume that $j = k$ and $i = p$. Let $v_s'$ denote the word obtained from the word $v_s$ by replacing $a_k$ by $w$, for $1 \leqslant s < p$. Then the presentation

$$P' = \{a_0, \dots, a_k;\ v_1', \dots, v_{p-1}', v_p\}$$

can be obtained from $P$ by a repeated application of the Tietze transformations (T.1) and (T.2). Namely, first add the words $v_1', \dots, v_{p-1}'$ to the words in $P$ and then delete the words $v_1, \dots, v_{p-1}$. Thus there is an admissible mapping $P \to P'$. Observe that also there is an admissible mapping (a retraction) $P' \to P''$, where $P''$ is of the form

$$P'' = \{a_0, \dots, a_{k-1};\ v_1', \dots, v_{p-1}'\}$$

This finishes the proof of the lemma.

Remark. Observe that, if the given transformation in (2.1) is of type (T.1), (T.2) or (T.3) then there is a mapping $f\colon P \to Q$ which is admissible with respect to any $a_j$, $0 \leqslant j \leqslant k$.

The following two transformations of $P$ are compositions of the Tietze transformations (T.1) and (T.2) and thus, by the above remark, they correspond to mappings of $P$ which are admissible with respect to any $a_j$, $0 \leqslant j \leqslant k$.

Procedure R. If $v_i$ and $v_j$ are two words in $P$, then replace the word $v_i$ by $v_i g v_j^n g^{-1}$ or by $v_i^{-1} g v_j^n g^{-1}$, where $n$ is an integer and $g$ is a word in the generators $a_0, \dots, a_k$.

Procedure C. Let $v_i = g_1 g_2 g_3 g_4$ and $v_j = [g_2, g_3]$ be two words in $P$, where $g_s$ is a word in the generators $a_0, \dots, a_k$, for $s \in \{1, 2, 3, 4\}$. Then replace the word $v_i$ by $v_i' = g_1 g_3 g_2 g_4$.

We will also need the following transformation of $P$.

Procedure A. Let $g_1$ and $g_2$ be a word in the generators $a_0, \dots, a_k$. Then adjoin the commutator $[g_1, g_2]$ to the words in $P$.

(2.2) Lemma. *If a presentation $Q$ is obtained from $P$ by Procedure* A, *then there is a mapping $P \to Q$ (the inclusion) which is admissible with respect to any $a_j$, $0 \leqslant j \leqslant k$.*

Proof. We will denote by $i\colon (K(P), S_j) \to (K(Q), S_j)$ the inclusion map. Observe that the cohomology group $H^2(K(Q), S_j)$ contains a direct summand $G$ such that

(i) $G$ contains the image of the connecting homomorphism $(\delta')^*\colon H^1(S_j) \to H^2(K(Q), S_j)$, and

(ii) the homomorphism $H^2(i)\colon H^2(K(Q), S_j) \to H^2(K(P), S_j)$ maps isomorphically $G$ onto $H^2(K(P), S_j)$.

The connecting homomorphism is a natural transformation, thus we have $H^2(i) \circ (\delta')^* = \delta^*$, where $\delta^*\colon H^1(S_j) \to H^2(K(P), S_j)$ is the connecting homomorphism. Since the tensor product commutes with direct sums, we obtain that the condition (A4) of the definition of admissible mappings is satisfied. Thus the inclusion $K(P) \to K(Q)$ is an admissible mapping with respect to any $S_j$, $1 \leqslant j \leqslant k$.

The following two examples of mappings of a presentation $P$ will be useful.

(2.3) Example. Suppose a word $v_i$ in $P$ has the form $w^n$, where $w$ is a word in the generators $a_0, \dots, a_k$. If $Q$ is the presentation obtained from $P$ by replacing the word $v_i$ by the word $w$, then there is a mapping $P \to Q$ which is fixed on the set $\{a_0, \dots, a_k\}$.

(2.4) Example. Let $w$ be a word in the generators $a_0, \dots, a_k$. For some $j_0 \in \{1, \dots, k\}$, replace $a_{j_0}$ by $w$ in each word in $P$; additionally, if $w$ is the empty word, delete $a_{j_0}$ from the generators in $P$. If $Q$ denotes the resulting presentation, then there exist a mapping $P \to Q$ which is fixed on each $a_j$, $j \neq j_0$.

Now we will give some examples of admissible transformations of 2-dimensional CW-complexes.

(2.5) LEMMA. *Let us consider two presentations*

$$P = \{a, b;\ a^k b^m, [a, b]\}, \qquad Q = \{a, b;\ a^l b^m, [a, b]\}$$

*of 2-dimensional* CW-*complexes, where $k$, $l$ and $m$ are nonzero integers.*

(a) *If $l$ divides $k$ then there exists a mapping $P \to Q$ fixed on $a$.*

(b) *If $l = (k, m)$, the greatest common divisor of $k$ and $m$, then there exists a mapping $Q \to P$ fixed on $a$.*

Proof. (a) Let $k = l \cdot r$. First, apply (2.4) to the presentation $P$; namely, replace $b$ by $b^r$ in each word in $P$. Next, applying Procedure A, adjoin $[a, b]$ to the words in $P$ and then delete the word $[a, b^r]$ applying the Tietze transformation (T.2). Finally, replace the word $a^k b^{m \cdot r}$ by $a^l b^m$ applying Procedure C and (2.3). It finishes the proof of part (a) since to each of the above transformations corresponds a mapping which is fixed on $a$.

(b) Let $s$ and $t$ be integers such that $t \cdot k = l + s \cdot m$. There is a mapping

$$Q \to Q' = \{a, b;\ a^l (a^s b^t)^m, [a, b]\}.$$

fixed on $a$, which can be obtained by using the following sequence of transformations. First, replace $b$ by $a^s b^t$ in each word of $Q$ applying (2.4). Then, adjoin the commutator $[a, b]$ to the words in the resulting presentation and delete the superfluous word applying the Tietze transformation (T.2).

Observe also that there is a mapping $Q' \to P$ fixed on $a$. Namely; first, replace the word $a^l (a^s b^t)^m$ by the word $(a^k b^m)^t$ applying Procedure C and then, the last word by the word $a^k b^m$ applying (2.3).

By Lemma (2.5), we obtain the following

(2.6) COROLLARY. *In notation of* (2.5), *if $(k, m) = (l, m)$ then there exists a mapping $P \to Q$ which is admissible with respect to $a$.*

Proof. Let $P' = \{a, b;\ a^{(k,m)} b^m, [a, b]\}$. Then use (2.5) and (1.2) to obtain the desired map as a composition of admissible mappings $P \to P' \to Q$.

We shall also employ

(2.7) LEMMA. *Let us consider two presentations*

$$P = \{a, b;\ ab^k, b^m, [a, b]\}, \qquad Q = \{a, b;\ ab^l, b^m, [a, b]\}$$

*of 2-dimensional* CW-*complexes. If $(k, m) = (l, m)$ then there exists a mapping $P \to Q$ which is admissible with respect to $a$.*

Proof. As in the proof (2.6), it is sufficient to show that there exist fixed on $a$ mappings $Q \to P$ provided $l$ divides $k$ and $P \to Q$ provided $l = (k, m)$.

The mapping $Q \to P$ is easily obtained by substituting $b$ by $b^{k/l}$ in each word in $Q$ and then, using (2.3) and applying procedure A and the Tietze transformation (T.2).



Let $l = t \cdot k + s \cdot m$. To get $P \to Q$ use the following sequence of transformations of the presentation $P$. First, replace $b$ by $b^t$ in each word in $P$. Then, replace the word $b^{t \cdot m}$ by the word $b^m$ applying (2.3). Next, replace the word $ab^{t \cdot k}$ by the word $ab^l = ab^{t \cdot k} b^{s \cdot m}$ applying Procedure R. Finally, adjoin the commutator $[a, b]$ and apply the Tietze transformation (T.2).

**3. Simplifying 2-dimensional CW-complexes by using admissible transformations.** First we will prove the following lemma.

(3.1) LEMMA. *Let $p$: $L_0 \to S$ be a map of a subpolyhedron of a connected compact 2-dimensional polyhedron $L$. Then there exists a finite 2-dimensional* CW-*complex $K$ which has a single 0-cell and whose 1-skeleton contains the circle $S$, and there exists a map $f$: $(L, L_0) \to (K, S)$ such that*

(i) $f | L_0 = p$,

(ii) *$H^2(f)$: $H^2(K, S) \to H^2(L, L_0)$ is an isomorphism.*

Proof. We may assume that the map $p$ is simplicial. First let us consider the polyhedron $K'$ obtained by attaching $L$ to $S$ by the map $p$. By $g_1$ we denote the projection of $L$ onto $K'$. Observe that the map

$$g_1\colon (L, L_0) \to (K', S)$$

induces an isomorphism $H^2(g_1)$ of the cohomology groups.

There is a simply connected 1-dimensional subpolyhedron (a tree) $T$ of $K'$ which contains all vertices of $K'$ and all 1-simplexes of $S$ except exactly one. Let $K$ be the 2-dimensional CW-complex obtained from $K'$ by shrinking $T$ to a point. By $g_2$ we denote the projection from $K'$ onto $K$. Then $g_2(S)$ is a circle which we identify with $S$. The 0-skeleton of $K$ is the base point $* = g_2(T)$. Let us note that $g_2$, considered as the map of pairs $(K', S) \to (K, S)$, is a homotopy equivalence. Thus the mapping

$$f = g_2 \circ g_1\colon (L, L_0) \to (K, S)$$

satisfies the condition (ii) of our lemma and the following condition

$$\text{the maps } p \text{ and } f | L_0\colon L_0 \to S \text{ are homotopic}.$$

Thus by the Borsuk homotopy extension theorem we may assume that also the condition (i) of the lemma is satisfied.

(3.2) Remark. The map $f$: $L \to K$ asserted in (3.1) to exist is admissible with respect to $p$ and the identity map on $S$.

Now we consider a finite 2-dimensional CW-complex $K$ with a presentation

$$P = \{a_0, \ldots, a_k;\ w_0, \ldots, w_r\}.$$

In this section by an admissible mapping of a presentation we will always understand one with respect to the first generator which appears in the presentation.

(3.3) Lemma. *There is an admissible mapping*

$$P \to P_1 = \{a_0, \ldots, a_k;\ w'_0, \ldots, w'_r, [a_i, a_j] \text{ for } 0 \leqslant i < j \leqslant k\}.$$

*where*

$$w'_i = a_0^{m(i)} a_1^{m(i,1)} \ldots a_k^{m(i,k)} \quad \text{for } 0 \leqslant i \leqslant r,$$

*with* $m(i) = 0$ *if* $i \neq 0$.

Proof. Procedure A yields an admissible mapping $P \to P'$, where

$$P' = \{a_0, \ldots, a_k;\ w_0, \ldots, w_r, [a_i, a_j] \text{ for } 0 \leqslant i < j \leqslant k\}.$$

Using Procedure C allows us to assume that

$$w_i = a_0^{n(i)} a_1^{n(i,1)} \ldots a_k^{n(i,k)} \quad \text{for } 0 \leqslant i \leqslant r.$$

Moreover, we may assume that $n(i) \neq 0$ if $0 \leqslant i \leqslant p$ and $n(i) = 0$ otherwise.

An *elementary operation* on a sequence $m_0, \ldots, m_p$ of integers is the replacement in that sequence of some $m_i$, $0 \leqslant i \leqslant p$, by $m_i + m_j$ or by $m_i - m_j$, where $j \neq i$. Suppose that $n$ is the greatest common divisor of the integers $n(0), \ldots, n(p)$. If $p > 0$, then we can find (using the Euclidean algorithm) a sequence of elementary operations on the sequence $n(0), \ldots, n(p)$ in order to get finally the sequence $n, 0, \ldots, 0$.

Thus, if $p > 0$, we apply to the presentation $P'$ the Procedures R and C finitely many times (each corresponding to the above elementary operations) and we get finally the required presentation $P_1$.

Next we will prove the following lemma.

(3.4) Lemma. *Let $P_1$ be given by* (3.3). *Then there is an admissible mapping $P_1 \to P_2$, where the presentation $P_2$ is of the following form*

$$P_2 = \{b_0, b_1, \ldots, b_q;\ v, v_1, \ldots, v_p, [b_i, b_j] \text{ for } 0 \leqslant i < j \leqslant q\},$$

*where $q \geqslant p$, $b_0 = a_0$, and moreover*

$$v = b_0^n b_1^{l(1)} \ldots b_q^{l(q)},$$

$$v_j = b_j^{k(j)} \quad \text{for } 1 \leqslant j \leqslant p.$$

*Additionally we may require that each $l(j)$ is nonzero and that each $k(j)$ is a power of a prime.*

Proof. Since we have $m(i) = 0$ for $1 \leqslant i \leqslant r$ thus

$$P' = \{a_1, \ldots, a_k;\ w'_1, \ldots, w'_r, [a_i, a_j] \text{ for } 1 \leqslant i < j \leqslant k\}$$

is a presentation of a 2-dimensional CW-complex as well as a presentation of an (abelian) group $G$. Let

$$P'' = \{b_1, \ldots, b_q;\ v_1, \ldots, v_p, [b_i, b_j] \text{ for } 1 \leqslant i < j \leqslant q\}$$

be another presentation of the group $G$, such that $v_j = b^{k(j)}$. By the Tietze theorem ([M-K-S], Corollary 1.5), the presentation $P'$ may be changed into $P''$ by a finite sequence of elementary Tietze transformations. Observe that this sequence of transformations applied to $P_1$ changes $P_1$ into the following presentation

$$P'_1 = \{b_0, \ldots, b_q;\ v_0, \ldots, v_p, [b_i, b_j] \text{ for } 1 \leqslant i < j \leqslant q,\ v'_{0,j} \text{ for } 1 \leqslant j \leqslant k\},$$

where $b_0 = a_0$ and $v'_{0,j}$ is an element of the commutant of the free group generated by $b_0, \ldots, b_q$, for $1 \leqslant j \leqslant k$. Now, adjoin the commutators $[b_0, b_j]$ to the words in $P'_1$ and delete all $v'_{0,j}$ applying (T.2). Finally we obtain the required presentation $P_2$ applying Procedure C to the word $v_0$. The above transformations yield an admissible mapping $P_1 \to P_2$.

We may assume that all integers $l(1), \ldots, l(q)$ are nonzero. Otherwise, if $l(j) = 0$ for some $j \in \{1, \ldots, q\}$ then there is an obvious retraction of $P_2$ onto the presentation obtained from $P_2$ by deleting $b_j$, $v_j$ and the commutators with $b_j$.

(3.5) Lemma. *In* (3.4) *we may achieve that* $p \in \{q, q-1\}$.

Proof. Suppose that $p < q$. For $p < j \leqslant q$, let $l(j) = l \cdot l'(j)$, where $l$ is the greatest common divisor of the integers $l(p+1), \ldots, l(q)$. Then, by using Procedure C, the word $v$ in $P_2$ can be replaced by the word $b_0^n b_1^{l(1)} \ldots b_p^{l(p)} c_{p+1}^l$, where $c_{p+1} = b_{p+1}^{l'(p+1)} \ldots b_q^{l'(q)}$. Since the greatest common divisor of the integers $l'(p+1), \ldots, l'(q)$ is equal to 1 thus there exist words $c_{p+2}, \ldots, c_q$ in terms of $b_{p+1}, \ldots, b_q$ such that $c_{p+1}, \ldots, c_q$ generate the free abelian group given by the presentation $\{b_{p+1}, \ldots, b_q, [b_i, b_j] \text{ for } p < i < j \leqslant q\}$.

Therefore, there exist mappings

$$P_2 \to P' \quad \text{and} \quad P' \to P_2,$$

which are fixed on $b_0$ and thus admissible by (1.2), where $P'$ is the following presentation

$$P' = \{c_0, \ldots, c_q;\ v', v'_1, \ldots, v'_p, [c_i, c_j] \text{ for } 0 \leqslant i < j \leqslant q\},$$

where

$$c_j = b_j \quad \text{for } 0 \leqslant j \leqslant p,$$

$$v' = c_0^n c_1^{l(1)} \ldots c_p^{l(p)} c_{p+1}^l \quad \text{and}$$

$$v'_j = c_j^{k(j)} \quad \text{for } 1 \leqslant j \leqslant p.$$

Finally, there exists an obvious retraction $P' \to P''$, where

$$P'' = \{c_0, \ldots, c_{p+1};\ v', v'_1, \ldots, v'_p, [c_i, c_j] \text{ for } 0 \leqslant i < j \leqslant p+1\}.$$

(3.6) Lemma. *Let $P_2$ be the presentation given by* (3.4) *and let $P'_2$ be defined in the same way as $P_2$ except that we replace $l(j)$ by $r(j)$ for some $j \in \{1, \ldots, p\}$. If $(r(j), k(j)) = (l(j), k(j))$, then there exists an admissible mapping $P_2 \to P'_2$.*

Proof. For simplicity assume that $j = 1$. The following presentation

$$P' = \{b_0, \ldots, b_{q+1};\ w,\ b_{q+1}^{-1} b_1^{l(1)}, v_1, \ldots, v_p, [b_i, b_j] \text{ for } 0 \leqslant i < j \leqslant q+1\},$$

where

$$w = b_0^n b_{q+1} b_2^{l(2)} \ldots b_q^{l(q)},$$

can be obtained from $P_2$ by applying transformations of the type (T.3), R and A successively. Thus there exists an admissible mapping $P_2 \to P'$.

Using (2.7), and applying the transformations A and (T.2), it follows that there exists an admissible mapping $P' \to P''$, where $P''$ is the presentation defined in the same way as $P'$ except that we replace $l(1)$ by $r(1)$.

Finally, there exists an admissible mapping $P'' \to P_2'$ which can be defined using the Tietze transformations (T.4) and (T.2).

Now we will prove the following lemma.

(3.7) Lemma. *Let $P_2$ be the presentation given by* (3.4), *where $p \in \{q, q-1\}$ and each $k(j)$ is a power of a prime and each $l(j)$ is nonzero. Then there is an admissible mapping $P_2 \to Q$, where $Q$ is of the form*

$$(3.8)\qquad Q = \{b_0, \ldots, b_k;\ b_0^{n(j)} b_j^{m(j)} \text{ for } 1 \leqslant j \leqslant k,\ [b_i, b_j] \text{ for } 0 \leqslant i < j \leqslant k\}.$$

*Additionally, we may require that $n(j) \neq 0$ for each $j$.*

Proof. The proof is by induction with respect to the number $q$ of the generators $b_j$ in $P_2$ decreased by 1. The cases

(i) $q = 0$ (then $P_2 = \{b_0; b_0^n\}$) and

(ii) $q = 1$ and $p = 0$ (then $P_2 = \{b_0, b_1; b_0^n b_1^{l(1)}, [b_0, b_1]\}$)

are obvious. Now suppose $q \geqslant 1$ and $p \geqslant 1$. By (3.6), we may assume that $l(1)$ divides $k(1)$.

We consider the following 4 cases.

(a) $l(1)$ divides each $l(j)$.

Applying Procedure C, we can replace the word $v$ in the presentation $P_2$ by the word of the form

$$v' = b_0^n (b_1 b_2^{l'(2)} \ldots b_q^{l'(q)})^{l(1)}.$$

Next, replace the word $v_1 = b_1^{k(1)}$ by $v_1' = v_1^{-1}(v')^{k(1)/l(1)}$ and then $v_1'$ by the word

$$v_1'' = b_0^{n \cdot k(1)/l(1)} (b_2^{l'(2)} \ldots b_q^{l'(q)})^{k(1)}.$$

Thus, there exists an admissible mapping $P_2 \to P'$, where

$$P' = \{b_0, \ldots, b_q;\ v', v_1'', v_j \text{ for } 2 \leqslant j \leqslant p,\ [b_i, b_j] \text{ for } 0 \leqslant i < j \leqslant q\}.$$

Let $P''$ be obtained from $P'$ by replacing the word $v'$ by the word $v'' = b_0^n b_1^{l(1)}$. There exist mappings $P' \to P''$ and $P'' \to P'$, which are fixed on $b_0$ and thus admissible by (1.2). The mapping $P' \to P''$ can be defined by using the following sequence of transformations. First, applying (2.4), replace $b_1$ by $b_1(b_2^{l'(2)} \ldots b_q^{l'(q)})^{-1}$ in each word in $P'$. Then adjoin the required commutators and delete superfluous words by applying the Tietze transformation (T.2). The mapping $P'' \to P'$ can be obtained in the same way except that we replace $b_1$ by $b_1 b_2^{l'(2)} \ldots b_q^{l'(q)}$ in each word in $P''$.

Now, it is easy to complete the inductive step in case (a).



By (3.6), we may assume that $l(j)$ divides $k(j)$ for each $j$, $1 \leqslant j \leqslant p$. Therefore $l(j)$ is a power of a prime. Thus we may assume this property in the remaining three cases.

(b) $p = q$.

We may assume (changing the order of generators if necessary) that $l(1)$ divides each $l(j)$ which is a power of the same prime as $l(1)$ is. By (3.6), we can replace each $l(j)$, $2 \leqslant j \leqslant q$, by $r(j)$ which is divisible by $l(1)$; thus we reduced this case to (a).

(c) $p = q-1$ and $l(q)$ is divisible by some $l(j)$, $1 \leqslant j \leqslant p$.

There is $i_0$, $1 \leqslant i_0 \leqslant p$, such that $l(i_0)$ divides each $l(i)$ which is a power of the same prime as $l(j)$ is, $1 \leqslant i \leqslant p$. For convenience we assume that $i_0 = 1$. Again, by (3.6), we can reduce this case to (a).

(d) $p = q-1$ and $l(q)$ is not divisible by any $l(j)$, $1 \leqslant j \leqslant p$.

By (3.6), we can replace each $l(j)$, $1 \leqslant j \leqslant p$, by $r(j)$ which is divisible by $l(q)$. Then we can replace in the presentation $P_2$ the word $v$ by the word

$$b_0^n (b_1^{l'(1)} \ldots b_p^{l'(p)} b_q)^{l(q)}$$

and it is easy to define an admissible mappings

$$P_2 \to P' = \{b_0, \ldots, b_q;\ b_0^n b_q^{l(q)}, v_1, \ldots, v_p, [b_i, b_j] \text{ for } 0 \leqslant i < j \leqslant q\}$$

and

$$P' \to P'' = \{b_0, b_q;\ b_0^n b_q^{l(q)}, [b_0, b_q]\}$$

(above, the first mapping is induced by replacing $b_q$ by $b_q(b_1^{l'(1)} \ldots b_p^{l'(p)})^{-1}$ and the second by replacing $b_1, \ldots, b_p$ by the empty word).

Observe that we may assume that $n(j) \neq 0$ for each $j \in \{1, \ldots, k\}$. Otherwise, we apply an argument similar to the one given at the end of the proof of Lemma (3.4).

As a corollary of the above lemmas we obtain the main result of this section.

(3.9) Fundamental Lemma. *Let $(L, L_0)$ be a pair of finite polyhedra such that $\dim L = 2$. Then for any map $p\colon L_0 \to S$ there is a map $f\colon L \to K$ such that*

(3.10) *$K$ is a finite 2-dimensional CW-complex with the presentation $Q$ given by* (3.8), *such that $n(j) \neq 0$ for each $j$, and*

(3.11) *$f$ is admissible with respect to $p$ and the identity map on $S$.*

Checking the proofs of (3.3)–(3.7) one can obtain:

(3.12) Proposition. *Let*

$$P = \{b_0, \ldots, b_p;\ w_0, \ldots, w_r, [b_i, b_j] \text{ for } 0 \leqslant i < j \leqslant p\}.$$

*Then there exist fixed on $b_0$ mappings $P \to Q$ and $Q \to P$, where $Q$ has the form given by* (3.8) *and $n(j) \neq 0$ for each $j$.*

**4. Main results.** For compact pairs $(X, A)$ with $\dim(X \times X) < 4$, the process of simplifying the polyhedral structure of the complex $K$ of (3.9) is concluded by the following

(4.1) THEOREM. *Let* $g: (X, A) \to (D^2, S)$ *be a map of a compact pair* $(X, A)$ *such that* $\dim(X \times X) < 4$. *Then there exists a mapping*

$$h: (X, A) \to (B, S)$$

*such that* $h(x) = g(x)$ *for* $x \in A$ *and* $B$ *is given by the presentation of the form*

$$P = \{b_0, b_1;\ b_0^n b_1^{n^2}, [b_0, b_1]\},$$

*where* $b_0$ *corresponds to* $S$.

The proof of the above theorem is divided into two lemmas:

(4.2) LEMMA. *Under assumptions of* (4.1) *there exists a mapping*

$$h: (X, A) \to (K, S)$$

*such that* $h(x) = g(x)$ *for* $x \in A$ *and* $K$ *is given by the presentation*

$$Q = \{b_0, \ldots, b_k;\ b_0^{n(j)} b_j^{m(j)} \text{ for } 1 \leqslant j \leqslant k,\ [b_i, b_j] \text{ for } 0 \leqslant i < j \leqslant k\},$$

*where*

(i) $m(j)$ *divides* $(n(j))^2$ *and* $n(j) \neq 0$ *for* $1 \leqslant j \leqslant k$.

Proof. Since $\dim(X \times X) < 4$, the homomorphism

$$H^4(g \times g): H^4((D^2, S) \times (D^2, S)) \to H^4((X, A) \times (X, A))$$

is zero and also $\dim X \leqslant 2$. It follows that the homomorphism

$$H^2(g) \otimes H^2(g): H^2(D^2, S) \otimes H^2(D^2, S) \to H^2(X, A) \otimes H^2(X, A)$$

is zero. By a standard argument there exist a pair of connected polyhedra $(L, L_0)$ with $\dim L = 2$ and mappings

$$g'': (X, A) \to (L, L_0) \quad \text{and} \quad g': (L, L_0) \to (D^2, S)$$

such that $g' \circ g''$ and $g$ are homotopic (as the maps of pairs) and $H^2(g')$ has the property $H^2(g') \otimes H^2(g') = 0$. By (3.9), there exists a map $f: (L, L_0) \to (K, S)$ which satisfies conditions (3.10) and (3.11).

By Lemma (1.1), the map $q: (K, S) \to (D^2, S)$, which is an extension of the identity map on $S$, has the property that $H^2(q) \otimes H^2(q) = 0$. Observe that the cohomology group $H^2(K, S)$ is isomorphic to the group $Z_{m(1)} \oplus \ldots \oplus Z_{m(k)} \oplus Z^r$, where $Z_0 = Z$ and $r$ is the number of all commutators $[b_i, b_j]$ in $Q$. Let us denote by $1$ a generator of the groups $H^2(D^2, S)$ and $H^1(S)$ which are identified by the coboundary homomorphism. Observe that

$$H^2(q)(1) = \delta^*(1) = (n(1), \ldots, n(k), 0, \ldots, 0) \in Z_{m(1)} \oplus \ldots \oplus Z_{m(k)} \oplus Z^r$$

Since

$$(n(1), \ldots, n(k),\ 0, \ldots, 0) \otimes (n(), \ldots, n(k), 0, \ldots, 0) = 0$$

in the tensor product of the group $Z_{m(1)} \oplus \ldots \oplus Z_{m(k)} \oplus Z^r$ by itself, it follows that $m(j)$ divides $(n(j))^2$ for each $1 \leqslant j \leqslant k$.



Observe that $g|A$ and $f \circ g''|A$ are homotopic as maps from $A$ to $S$. Thus by the Borsuk homotopy extension theorem the required map $h$ exists.

Remark. In the proof of (4.2) the assumption $\dim(X \times X) < 4$ can be relaxed to $H^2(g) \otimes H^2(g) = 0$ and $\dim X \leqslant 2$. Using this together with an easy computation, which is implicit in the proof of (4.2), completes the proof of Proposition 1 of the Introduction.

(4.3) Remark. Under the assumptions of (4.1), for a prime $p$ there exists a mapping

$$f: (X, A) \to (L, S)$$

such that $f(x) = g(x)$ for $x \in A$ and $L$ is given by the presentation $R = \{b_0; b_0^p\}$.

Proof. Let $h: (X, A) \to (K, S)$ be as asserted by (4.2). By (2.6), we may assume that $p$ divides each $n(j)$. Thus the lemma follows since there exists an obvious fixed on $b_0$ mapping $Q \to R$. (Remark (4.3) is also a direct consequence of the fact that the condition $\dim(X \times X) < 4$ implies $\mathrm{cdim}_{Z_p} X < 2$.)

(4.4) LEMMA. *Under the assumption of* (4.1), *let*

$$h: (X, A) \to (K, S)$$

*be as asserted by* (4.2) *and let* $j_0 \in \{1, \ldots, k\}$ *and a prime* $p$ *be given. Then the assertions of* (4.2) *are satisfied also with a polyhedron* $K'$ *whose presentation is obtained from that of* $K$ *by replacing* $n(j_0)$ *by* $p \cdot n(j_0)$ *and* $m(j_0)$ *by* $p \cdot m(j_0)$.

Proof. Assume for simplicity that $j_0 = 1$ and consider first the case $k = 1$. In the 2-cell $\sigma$ of $K$, corresponding to the word $b_0^{n(1)} b_1^{m(1)}$, consider a disc $D$ intersecting $\partial\sigma$ along $\{*\}$. Applying (4.3) to $h|h^{-1}(D)$ we get a complex $L$ and a map

$$f: (h^{-1}(D), h^{-1}(\partial D)) \to (L, \partial D)$$

such that $f(x) = h(x)$ for $x \in h^{-1}(\partial D)$ and $L$ is given by the presentation $R = \{c_0; c_0^p\}$ with $c_0$ corresponding to $\partial D$. Replace $D$ by $L$ and $h|h^{-1}(D)$ by $f$ to get a map

$$h_1: (X, A) \to (K_1, S),$$

where $K_1 = (K \setminus D) \cup L$ has the presentation

$$R' = \{b_0, b_1, c_0;\ b_0^{n(1)} b_1^{m(1)} c_0^{-1}, c_0^p, [b_0, b_1]\}.$$

Next, there is a mapping

$$R' \to Q' = \{b_0, b_1;\ b_0^{p \cdot n(1)} b_1^{p \cdot m(1)}, [b_0, b_1]\}$$

which is fixed on $b_0$ and on $b_1$. It can be defined using the following sequence of transformations applied to $R'$. First, adjoin the commutators $[b_0, c_0]$ and $[b_1, c_0]$ to the words in $R'$. Next replace the word $c_0^p$ by the word $b_0^{p \cdot n(1)} b_1^{p \cdot m(1)}$ applying Procedures R and C. Then, applying (T.4), delete $c_0$ from the generators, delete $b_0^{n(1)} b_1^{m(1)} c_0^{-1}$ from the words and replace $c_0$ by $b_0^{n(1)} b_1^{m(1)}$ in the remaining words.

Finally, delete superfluous words applying (T.2) and apply Procedure C. This finishes the proof of the lemma in the case $k = 1$.

The case $k > 1$ follows from the one $k = 1$ and the fact that the mapping $R' \to Q'$ was fixed on $b_0$ and on $b_1$.

Proof of (4.1). Let $f\colon X \to K$ be given by (4.2). Applying (4.4) successively we may alter the numbers $n(j)$ so as to replace all of them by the least common multiple of $n(1), \ldots, n(k)$. In this process property (i) of (4.2) is being kept and thus we may assume without loss of generality that $K$ is given by

$$Q = \{b_0, \ldots, b_k;\ b_0^n b_j^{m(j)} \text{ for } 1 \leqslant j \leqslant k,\ [b_i, b_j] \text{ for } 0 \leqslant i < j \leqslant k\},$$

where $m(j)$ divides $n^2$ for $j \in \{1, \ldots, k\}$.

Obviously there exists a fixed on $b_0$ mapping

$$Q \to P' = \{b_0, b_1;\ b_0^n b_1^m, [b_0, b_1]\},$$

where $m$ is the least common multiple of $m(1), \ldots, m(k)$.

Finally, using the fact that $m$ divides $n^2$, we can replace $b_0^n b_1^m$ by $b_0^n b_1^{n^2}$ applying (2.4), Procedure A and the Tietze transformation (T.2) successively.

Proof of Theorem 3 (2-dimensional case). One of the implications follows from (4.1). The reverse implication follows from (1.5) in [Kr]. For the sake of completeness let us give a sketch of the argument here. Let $f$ be any mapping from $X$ into a 2-dimensional compact polyhedron $P$ with a triangulation $\mathscr{T}$. For a 2-simplex $\sigma \in \mathscr{T}$, we introduce the following notation

$$X_\sigma = f^{-1}(|\sigma|), \quad A_\sigma = f^{-1}(\partial|\sigma|) \quad \text{and}$$

$$f_\sigma = f|X_\sigma\colon (X_\sigma, A_\sigma) \to (|\sigma|, \partial|\sigma|),$$

where $\partial|\sigma|$ denotes the boundary of $|\sigma|$. By the assumption the mapping $f|A_\sigma\colon A_\sigma \to \partial|\sigma|$ admits an extension over $X_\sigma$ into a Boltyanskii bubble. Since the homomorphism $\delta_k^* \otimes \delta_l^*$ is trivial (where $\delta_n^*\colon H^1(S) \to H^2(B_n^2, S)$ denotes the coboundary homomorphism), it follows that the homomorphism $H^4(f_\sigma \times f_\tau) = H^2(f_\sigma) \otimes H^2(f_\tau)$ is also trivial for any 2-simplexes $\sigma, \tau \in \mathscr{T}$. Thus by the Hopf theorem there is a mapping

$$g_{\sigma,\tau}\colon X_\sigma \times X_\tau \to \partial(|\sigma| \times |\tau|)$$

which coincides with $f \times f$ on $(f \times f)^{-1}(\partial(|\sigma| \times |\tau|))$. It follows that there is a mapping

$$F\colon X \times X \to (P \times P)^{(3)},$$

such that $F(z)$ and $(f \times f)(z)$ belong to the same cell of $\mathscr{T} \times \mathscr{T}$ (by $(P \times P)^{(3)}$ we denote the 3-skeleton of $P \times P$). Suppose $\varepsilon > 0$ is given. We may choose $f$ such that $f \times f$ is an $\varepsilon$-mapping. If $\mathscr{T}$ is chosen sufficiently fine then also $F$ will be an $\varepsilon$-mapping. This implies that $\dim(X \times X) < 4$.



Now, we will prove the case $m = 2$ of Theorem 1. Let $P$ be given by (4.1). It is easy to see that there exist mappings

$$P \to P_1 = \{b_0;\ b_0^n\} \quad \text{and} \tag{4.5}$$

$$P \to P_2 = \{b_0, b_1;\ b_0 b_1^n\} \tag{4.6}$$

which are fixed on $b_0$. (Similar mappings were applied in [K–K].)

To get $P \to P_1$ first, replace $b_0^n b_1^{n^2}$ by $b_0 b_1^n$, using Procedure C and (2.3) and then delete $[b_0, b_1]$ applying the Tietze transformation (T.2).

Thus, by (4.1), we obtain the following

(4.7) Corollary. *Let $X$ be a 2-dimensional compactum with $\dim(X \times X) < 4$ and $A$ be a closed subset of $X$. Then for any map $g\colon A \to S$ there is an integer $n$ and there exist maps*

$$h_i\colon (X, A) \to (K(P_i), S), \quad \textit{for } i = 1, 2,$$

*such that $h_i|A = g$, where $K(P_1)$ and $K(P_2)$ are 2-dimensional CW-complexes whose presentations $P_1$ and $P_2$ are given by (4.5) and (4.6), respectively.*

Now, we invoke the following

(4.8) Proposition (see [K–L], [M–R] and also [K–K]). *There exist mappings*

$$F_1\colon K(P_1) \to D^2 \times D^2 \quad \textit{and} \quad F_2\colon K(P_2) \to D^2 \times D^2$$

*with disjoint images such that*

$$F_1(x) = (x, 0) \quad \textit{and} \quad F_2(x) = (0, x)$$

*for each $x \in S$.*

(In the papers [M–R] and [K–K], this result was stated for polyhedra which are homotopically equivalent to $K(P_1)$ and $K(P_2)$ by homotopies fixed on $S$.)

Clearly, the above result and (4.7) imply Corollary 1 and thus also Theorems 1 and 2 stated in Introduction (see the discussion there).

## 5. Remarks

(5.1) Remark. Theorem 3′ with known techniques (see [Br]) shows that an $m$-dimensional compactum $X$ satisfies $\dim(X \times X) < 2m$ if and only if it can be expressed as

$$X = \varprojlim\{K, f_i^j\}$$

so that: (i) each $K_i$ is an $m$-dimensional polyhedron with triangulation $\mathscr{T}_i$, (ii) for every $m$-simplex $\sigma$ of $\mathscr{T}_{2i-1}$ the inverse image $(f_{2i-1}^{2i})^{-1}(\sigma)$ is a copy of some $B_{n(\sigma)}^m$ with $(f_{2i-1}^{2i})^{-1}(\partial\sigma) = \partial\sigma$ as the distinguished circle of that copy, and (iii) for each $i$

$$\lim_{j \to \infty} \operatorname{mesh} f_i^j(\mathscr{T}_j) = 0.$$

Some of the propositions and their proofs in our paper can be reformulated in the language of abelian groups (then proofs become more concise). For example Proposition (3.12) can be stated in the following way

(5.2) Proposition. *Let $a$ be an element of a finitely generated abelian group $G$. Then there exist an abelian group with a presentation of the form given by* (3.8) *and homomorphisms $f\colon G\to H$, $g\colon H\to G$ such that $f(a) = b_0$ and $g(b_0) = a$.*

Using theorem (3.2) in [Sp], one can prove (compare [Kr]) the following

(5.3) Proposition. *Let $X$ and $Y$ be compacta such that* $\dim X \geqslant 3$. *Then the condition*

$$\dim(X\times Y) < n = \dim X + \dim Y$$

*is satisfied if and only if any two maps*

$$f\colon X\to R^n \quad \text{and} \quad g\colon Y\to R^n$$

*can be approximated arbitrarily closely by maps with disjoint images.*

**Acknowledgements.** I am greatly indebted to Henryk Toruńczyk for his suggestions of improvements to an earlier version of this paper. I would like also to thank Z. Karno for a helpful conversation concerning the papers [Bo] and [Ko], and to J. Krasinkiewicz for several helpful suggestions for improving the exposition.

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*Received 12 January 1989;*
*in revised form 6 March 1989*