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DRUKARNIA UNIWERSYTETU JAGIELLOŃSKIEGO W KRAKOWIE

Periodic orbits indices

by

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Abstract. In this note we define a periodic orbits index as a mapping which transforms fixed point germs generated by local semiflows on a topological space into elements of a given group in such a way that the conditions of Existence, Additivity and Homotopy Invariance are fulfilled. We use the relative Dold fixed point transfer to construct several indices of periodic orbits. The main of these indices is the index j which can be regarded as an extension of the Fuller index. The index j is defined for local semiflows on an ENR-space, and takes values in the group $\Pi \boldsymbol{Z}_p/\oplus \boldsymbol{Z}_p$ (where p ranges over prime numbers). In the case of a smooth flow on a manifold the nontriviality of the Fuller index is equivalent to the nontriviality of the index j.

0. Introduction. In 1967, F. B. Fuller has presented in [Fu] an index related to periodic orbits of smooth flows. Some authors have discovered other methods of defining this index (see [CM–P] and [DGJM]). The methods they have used are based on the assumption, that the flow is generated by a vector-field. C. C. Fenske (see [Fe1]) and A. J. B. Potter (see [Po1]) have posed a question about the possibility of extending the Fuller index to continuous semiflows on ANR's. The paper [Fe1] defines this index for semiflows having isolated periodic orbits and the paper [Fe2] considers the general case. The index obtained in [Fe2] turns out to be a 1-homology class, and it may vanish in some situations, in which the original Fuller index is non-zero. Another approach to this problem, based on approximation methods, is presented in [Po2].

In this note (which is based on unpublished preprints [Sr1] and [Sr2]) we consider mainly the finite-dimensional case (i.e. the underlying space is an ENR). We present two ways extending of the Fuller index. The first one assumes the existence of a certain 1-cohomology class (called a Fuller class), and the resulting index $\hat{\jmath}$ has values in $\boldsymbol{Q}$. The second index, j, is defined in the general situation, behaves as good as Fuller's, but takes values in a certain more complicated group.

We are using the fixed point transfer of fibre-preserving maps presented in the paper [Do3]. Since [Do3] relates to finite-dimensional ANR's, in order to solve the problem of Fenske and Potter in full generality one ought to use an infinite-dimensional version of the above transfer. At the end of this paper we sketch briefly the method of constructing such a general index. The details of this construction will be published elsewhere.

We begin with the introduction of the fixed point homomorphism (Section 1). We define this homomorphism using the fixed point transfer, and indicate some of its properties. Theorem (1.8) plays a basic role in the constructions presented in this paper. The main definitions concerning local semiflows are introduced in Section 2. In Section 3 we present some facts about the Alexander–Spanier approach to Čech cohomologies. Section 4 presents the definition and properties of so-called q-mappings of the circle. The standard generator of a periodic orbit is defined in Section 5. Section 6 is devoted to isolated periodic orbits of smooth flows. The main result of this section (Theorem (6.1)) states that the fixed point index of a Poincaré mapping is equal to the image of a generator of the first cohomology of the orbit by the fixed point homomorphism. Motivated by this result, in Section 7 we introduce so called Fuller classes, and examine their properties. We make use of these classes to define the index $\hat{\jmath}$ in Section 8. Section 9 presents a construction (originating from Fuller) which leads to the index j. The latter index and the auxiliary indices j_q, $q \in N$, are constructed in Section 10. Theorem (10.3) establishes the connection between the Fuller index and the index j. In Section 10 we also present an idea of extending j onto ANR's.

Since the index j plays the main role in this paper, it should be noted that its definition is based only on the notions introduced in Sections 1, 2, 9 and 10. The results of Sections 3–7 are used for proving that j can be treated as an extension of the Fuller index.

The terminology and results of the algebraic topology used here are quoted from the books [Do1] and [Sp]. In particular, H, H^* and $\check{H}^*$ denote the singular homology, the singular cohomology and the Čech cohomology functor, respectively. If $f\colon (X, A) \to (Y, B)$ is a continuous mapping, then it induces a homomorphism f_* on homologies and f^* on cohomologies. If f is an inclusion and u is a cohomology class on (Y, B), then we write $u|_{(X,A)}$ instead of $f^*(u)$. By $\boldsymbol{R}_+$ (or $\boldsymbol{R}_{++}$) we denote the interval $[0, \infty)$ ($(0, \infty)$, respectively). Analogously we define $\boldsymbol{R}_-$ and $\boldsymbol{R}_{--}$.

1. The fixed point homomorphism. Assume that T and X are fixed topological spaces. Let $f\colon E \to X$, where $E \subseteq T \times X$, be a continuous mapping such that

$$\operatorname{Fix}(f) = \{(t, x) \in E\colon f(t, x) = x\}$$

is compact and E is a neighbourhood of $\operatorname{Fix}(f)$ in $T \times X$. Such a mapping will be said to be *compactly fixed*. A mapping $f'\colon E' \to X$ is *equivalent to* f iff there exists an open set V,

$$\operatorname{Fix}(f) = \operatorname{Fix}(f') \subseteq V \subseteq E \cap E'$$

such that

$$f|_V = f'|_V .$$

The equivalence class of f is denoted by $[f]$. A set φ of continuous mappings with domains contained in $T \times X$ and ranges in X is called a *fixed point germ* (shortly: a *germ*) on (T, X) iff there exists an f such that $\varphi = [f]$. For any germ φ, we denote

$$\operatorname{Fix}(\varphi) = \operatorname{Fix}(f) .$$

Let φ_0 and φ_1 be two germs. We say that φ_0 and φ_1 are *disjoint* iff there exist $f_j\colon E_j \to X$, $\varphi_j = [f_j]$, $j = 0, 1$, such that E_0 and E_1 are disjoint. The germ of their join $f_0 \cup f_1\colon E_0 \cup E_1 \to X$ is denoted by $\varphi_0 \cup \varphi_1$.

By $\emptyset$ we denote the empty germ, i.e. the class of mappings f such that $\operatorname{Fix}(f) = \emptyset$.

We say that the germ φ_0 is *homotopical to* φ_1 ($\varphi_0 \simeq \varphi_1$) iff there exist germs φ_α, $\alpha \in I$ (where I denotes the interval $[0, 1]$), such that for any α there are: an $A \subseteq I$, a compact neighbourhood of α, an open set $E \subseteq T \times X$ and a mapping $h\colon E \times A \to X$ such that $\varphi_\beta = [h(\cdot, \beta)]$ for any $\beta \in A$ and $\bigcup_{\beta \in A} \operatorname{Fix}(\varphi_\beta)$ is compact. In this case we say that φ_α is a *homotopy connecting* φ_0 and φ_1.

Let $p\colon T \times X \to T$ denote the projection. If f is a mapping of the form considered above, then the composition (p, f) can be viewed as a fibre-preserving map over the base T, so we can apply to it the Dold's theory ([Do2], [Do3]).

Suppose that R is a given commutative ring with unit. In this section we will assume that all (co-)homologies have coefficients in R, and we will write $H(X)$ instead of $H(X; R)$, etc. Let T be a k-dimensional manifold orientable over R and let X be an ENR. Fix an orientation $\{o_t\}_{t \in T}$, $o_t \in H_k(T, T \setminus t)$. If $f\colon E \to X$ is compactly fixed, we introduce an R-homomorphism

$$\sigma_f\colon \check{H}^k(\operatorname{Fix}(f)) \to R ,$$

called the *fixed point homomorphism*, as follows:

(1.1) Definition. By the *fixed point homomorphism* σ_f we mean the composition $\langle \cdot, o_P \rangle \circ t_{(p,f)}$, where $\langle \cdot, \cdot \rangle$ is the scalar product, P is the image of $\operatorname{Fix}(f)$ under the projection p, $o_p \in H_k(T, T \setminus P)$ is the fundamental class of P generated by $\{o_t\}$ and

$$t_{(p,f)}\colon \check{H}^k(\operatorname{Fix}(f)) \to H^k(T, T \setminus P)$$

is the relative transfer (see (3.9) in [Do3]). If $\varphi = [f]$, we put $\sigma_\varphi = \sigma_f$.

Using the definition of the transfer we can present a direct description of σ_f. Since X is an ENR, we can assume that $X \subseteq \boldsymbol{R}^n$ for some $n \in N$ and that there exists an open neighbourhood Y of X in $\boldsymbol{R}^n$ and a retraction $r\colon Y \to X$. Put

$$D = \{(t, y) \in T \times Y\colon (t, r(y)) \in E\} ,$$

and define

$$g\colon D \ni (t, y) \to f(t, r(y)) \in \boldsymbol{R}^n ,$$

$$F = \operatorname{Fix}(g) = \operatorname{Fix}(f) .$$

Let o^n be a given orientation of $\boldsymbol{R}^n$, i.e. a generator of $H_n(\boldsymbol{R}^n, \boldsymbol{R}^n\setminus 0)$. By $q^n \in H^n(\boldsymbol{R}^n, \boldsymbol{R}^n\setminus 0)$ we denote its dual, i.e. $\langle q^n, o^n\rangle = 1$. The homological cross-products $o_t \times o^n$, $t \in T$, determine an orientation of $T\times \boldsymbol{R}^n$, so we have a fundamental class

$$o_F \in H_{k+n}(T\times \boldsymbol{R}^n, T\times \boldsymbol{R}^n\setminus F)$$

related to this orientation.

(1.2) PROPOSITION. *The fixed point homomorphism σ_f coincides with the direct limit (with respect to the neighbourhoods V of F) of the composition*:

$$H^k(V) \xrightarrow{\times q^n} H^{k+n}(V\times(\boldsymbol{R}^n, \boldsymbol{R}^n\setminus 0)) \xrightarrow{(\mathrm{id}, j-g)^*} H^{k+n}(V, V\setminus F) \xrightarrow{\langle\cdot, o_F\rangle} \boldsymbol{R},$$

where $\times$ denotes the cohomological cross-product (defined in [Sp], *p.* 249), *j is the projection $T\times \boldsymbol{R}^n \to \boldsymbol{R}^n$ and o_F is treated (by excision) as an element of $H_{k+n}(V, V\setminus F)$.*

(1.3) Remark. In the first version of the present paper (and also in [Sr1], [Sr2]) σ_f was presented as the composition of the Poincaré duality isomorphism $\cap o_F$: $\check{H}^k(F) \to H_n(D, D\setminus F)$ with the homomorphism

$$(j-g)_*\colon H_n(D, D\setminus F) \to H_n(\boldsymbol{R}^n, \boldsymbol{R}^n\setminus 0) \cong \boldsymbol{R}.$$

The referee has pointed out the possibility of expressing it in the present form.

Below we present some properties of the fixed point homomorphism, which are direct consequences of the results of Section 3 in [Do3].

(1.4) PROPOSITION. *If φ_0 and φ_1 are disjoint germs and $u_j \in \check{H}^k(\mathrm{Fix}(\varphi_j))$, $= 0, 1$ then*

$$\sigma_{\varphi_0\cup\varphi_1}(u) = \sigma_{\varphi_0}(u_0)+\sigma_{\varphi_1}(u_1)$$

where u is the pre-image of the pair (u_0, u_1) under the isomorphism induced by inclusion

$$\check{H}^*(\mathrm{Fix}(\varphi_0)\cup \mathrm{Fix}(\varphi_1)) \to \check{H}^*(\mathrm{Fix}(\varphi_0)) \oplus \check{H}^*(\mathrm{Fix}(\varphi_1)).$$

Now we state results concerning homotopical germs. Let φ_0 and φ_1 be germs, and let $u_j \in \check{H}^k(\mathrm{Fix}(\varphi_j))$, $j = 0, 1$.

(1.5) DEFINITION. We say that (φ_0, u_0) is *homotopical to* (φ_1, u_1), in symbols $(\varphi_0, u_0) \simeq (\varphi_1, u_1)$, iff there is a homotopy φ_α connecting φ_0 and φ_1 and $u_\alpha \in \check{H}^k(\mathrm{Fix}(\varphi_\alpha))$, $\alpha \in I$, such that the following condition holds: For every $\alpha \in I$ there exist an open subset U of $T\times X$, an element $u \in \check{H}^k(U)$ and a neighbourhood A of α in I such that for any $\beta \in A$:

$$\mathrm{Fix}(\varphi_\beta) \subseteq U \quad \text{and} \quad u|_{\mathrm{Fix}(\varphi_\beta)} = u_\beta .$$

(1.6) PROPOSITION. *If $(\varphi_0, u_0) \simeq (\varphi_1, u_1)$ then*

$$\sigma_{\varphi_0}(u_0) = \sigma_{\varphi_1}(u_1).$$

The following result states a kind of continuity property of $\sigma_f(u)$ with respect to f. As usual, we assume that $f\colon E\to X$ is compactly fixed. Let U be an open neighbourhood of $\mathrm{Fix}(f)$, let $\mathrm{cl}(U)$ be compact and contained in E, and let $u \in \check{H}^k(U)$. Proposition (1.6) implies the following:

(1.7) COROLLARY. *There is a neighbourhood N of f in the compact-open topology of mappings $E\to X$ such that for any $g \in N$ with* $\mathrm{Fix}(g) \subseteq U$

$$\sigma_g(u|_{\mathrm{Fix}(g)}) = \sigma_f(u|_{\mathrm{Fix}(f)}).$$

Now we state the main theorem of this section, which is an immediate consequence of the already presented results.

(1.8) THEOREM. *If $u \in \check{H}^k(T\times X)$ and Φ denotes the set of all fixed point germs on (T, X) then the function*

$$i_u\colon \Phi \ni \varphi \to \sigma_\varphi(u|_{\mathrm{Fix}(\varphi)}) \in \boldsymbol{R},$$

has the following properties:

(1.8a) (*Existence*) *if $i_u(\varphi) \neq 0$ then* $\mathrm{Fix}(\varphi) \neq \emptyset$,

(1.8b) (*Additivity*) *if φ_0 and φ_1 are disjoint, then*

$$i_u(\varphi_0\cup\varphi_1) = i_u(\varphi_0)+i_u(\varphi_1),$$

(1.8c) (*Homotopy Invariance*) *if $\varphi_0 \simeq \varphi_1$ then*

$$i_u(\varphi_0) = i_u(\varphi_1).$$

(1.9) Remark. If T is a point, then we do not distinguish between X and $T\times X$. In this case, if $1 \in \check{H}^0(X)$ is the unit, i_1 is the ordinary fixed point index.

2. Local semiflows and periodic orbits. We begin with the definition of a local semiflow. Let X be a topological space and let $\pi\colon D\to X$ be a continuous mapping, where $D \subseteq \boldsymbol{R}_+ \times X$.

(2.1) DEFINITION. π is called a *local semiflow* iff D is open in $\boldsymbol{R}_+\times X$, $\{0\}\times X \subseteq D$, for any $x \in X$ the set

$$I_x = \{t \in \boldsymbol{R}_+\colon (t, x) \in D\}$$

is an interval and the following conditions hold:

(2.1a) $\pi(0, x) = x$ for any $x \in X$,

(2.1b) $s \in I_x$, $t \in I_{\pi(s,x)}$ iff $s+t \in I_x$ and

$$\pi(t, \pi(s, x)) = \pi(s+t, x).$$

Define the set of periodic orbits as

$$\mathrm{Per}(\pi) = \{(t, x) \in D\colon t>0, \pi(t, x) = x, \exists s\colon \pi(s, x) \neq x\} \subseteq \boldsymbol{R}_{++}\times X.$$

We are especially interested in isolated compact subsets of $\mathrm{Per}(\pi)$. Let C be such a subset. By definition, there exists an open subset E of $\boldsymbol{R}_{++}\times X$ such that $C = \mathrm{Fix}(\pi|_E)$ (see Section 1). The fixed point germ $[\pi|_E]$ will be denoted by (π, C), and called a *flow-germ on X*. By $\Pi(X)$ we will denote the set of all flow-germs on X.

Let Π be a subset of $\Pi(X)$.

(2.2) Definition. We say that a flow-germ (π^0, C^0) is *homotopical to* a flow-germ (π^1, C^1) in Π ($(\pi^0, C^0) \underset{\Pi}{\simeq} (\pi^1, C^1)$) iff there are $(\pi^\alpha, C^\alpha) \in \Pi$, $\alpha \in I$, such that for any $\alpha \in I$ there exist: a compact neighbourhood A of α in I, an open subset E of $\boldsymbol{R}_{++} \times X$ and a continuous mapping $\varrho: E \times A \to X$, which satisfy the following conditions for any $\beta \in A$:

(2.2a) $E \subseteq D^\beta$, where D^β is the domain of π^β,

(2.2b) $C^\beta = E \cap \operatorname{Per}(\pi^\beta)$,

(2.2c) $\pi_\beta|_E = \varrho(\cdot, \beta)$,

(2.2d) $\bigcup_{\beta \in A} C^\beta$ is compact.

In the case when $\Pi = \Pi(X)$ we write $\simeq$ instead of $\underset{\Pi(X)}{\simeq}$.

Now we introduce the main object of our considerations. Let G be a given abelian group, and let $\Pi \subseteq \Pi(X)$.

(2.3) Definition. A function

$$i: \Pi \to G$$

is called a *periodic orbits index on* Π iff the following conditions are fulfilled:

(2.3a) (Existence) if $i(\pi, C) \neq 0$, then $C \neq \emptyset$,

(2.3b) (Additivity) if C^0 and C^1 are disjoint, then

$$i(\pi, C^0 \cup C^1) = i(\pi, C^0) + i(\pi, C^1),$$

(2.3c) (Homotopy Invariance) if $(\pi^0, C^0) \underset{\Pi}{\simeq} (\pi^1, C^1)$, then

$$i(\pi^0, C^0) = i(\pi^1, C^1).$$

The following definition will be useful in the sequel.

(2.4) Definition. We say that $o \in H_1(\boldsymbol{R}, \boldsymbol{R}\setminus 0; \boldsymbol{Z})$ is a *standard orientation of* $\boldsymbol{R}$ iff o is the homology class of a singular simplex $\sigma: \Delta_1 \to \boldsymbol{R}$ such that $\sigma(e_1) > 0$ and $\sigma(e_0) < 0$. (For any $q \in N$, Δ_q denotes the set $\operatorname{conv}\{e_0, \ldots, e_q\}$, called the standard q-dimensional simplex, where $\{e_i\}_i$ is the canonical basis of $\boldsymbol{R}^{q+1}$.) The class o induces an orientation of the 1-dimensional manifold $\boldsymbol{R}_{++}$ over an arbitrary ring R, called also the standard orientation.

We can define immediately some indices, using the following way. Assume that X is an ENR. Consider fixed point homomorphisms determined by the standard orientation of $\boldsymbol{R}_{++}$ over R (see (1.1) and (2.4)). Let $u \in \check{H}^1(\boldsymbol{R}_+ \times X; R)$. By Theorem (1.8), we have:

(2.5) Proposition. *The function*

$$i_u: \Pi(X) \ni (\pi, C) \to \sigma_{(\pi, C)}(u|_C) \in R$$

is a periodic orbits index on $\Pi(X)$.

However, the indices presented above are trivial on spaces for which the first cohomology vanishes, so we must look for other constructions. To this end we introduce several notions.

Let π be a local semiflow on a topological space X. For any point x such that $(t, x) \in \operatorname{Per}(\pi)$ for some t, we define

$$\omega_x = \inf\{t \in \boldsymbol{R}_{++}: \pi(t, x) = x\}.$$

We have always $\omega_x > 0$. If $(\omega, x) \in \operatorname{Per}(\pi)$ we put $m_{(\omega, x)} = \dfrac{\omega}{\omega_x}$. This number is called the *multiplicity of* (ω, x). We define also

$$\theta_{(\pi;\omega,x)}: S^1 \ni \exp\left(2\pi i \frac{t}{\omega}\right) \to (\omega, \pi(t, x)) \in \boldsymbol{R}_+ \times X,$$

the *periodic trajectory of* (ω, x) in π and

$$\Gamma_{(\pi;\omega,x)} = \theta_{(\pi;\omega,x)}(S^1),$$

the *orbit of* (ω, x) in π. The letter π is dropped in these symbols if no confusion can arise.

A set $\Gamma \subseteq \boldsymbol{R}_{++} \times X$ is called a *periodic orbit* (shortly: an *orbit*) iff $\Gamma = \Gamma_{(\omega,x)}$ for some (ω, x). In this case we write m_Γ instead of $m_{(\omega,x)}$, and the latter number is called the *multiplicity of* Γ.

For any flow-germ (π, C), the mapping

$$\theta_C: S^1 \times C \ni \left(\exp\left(2\pi i \frac{t}{\omega}\right), (\omega, x)\right) \to \theta_{(\omega,x)}(t) \in C$$

is a continuous S^1-group action on C. By M_C we denote the set of multiplicities of C, i.e.

$$M_C = \{m_\Gamma: \Gamma \text{ is an orbit}, \Gamma \subseteq C\}.$$

This set is always finite. The maximum of M_C is denoted by m_C.

In Section 4 we will examine the properties of trajectories.

3. Some properties of the circle. In this section we use the construction of Čech cohomologies based on Alexander–Spanier cochains (see [Sp]). It leads to the same theory as the construction in [Do1]. Now we shall briefly indicate some facts concerning this construction. In the sequel we assume that R is a commutative ring with the unit.

Let (X, A) be a topological pair, and let $(\mathscr{U}, \mathscr{V})$ be an open covering of (X, A), i.e. $\mathscr{U}$ is an open covering of X, $\mathscr{V}$ is a covering of A which consists of open subsets of A in the induced topology, and $\mathscr{V}$ is subordinate to $\mathscr{U}$. We define a complex $C^*(\mathscr{U}, \mathscr{V})$ by

$$C^q(\mathscr{U}, \mathscr{V}) = \{\varphi: \bigcup_{U \in \mathscr{U}} U^{q+1} \to R: \varphi \text{ vanishes on } \bigcup_{V \in \mathscr{V}} V^{q+1}\}$$

and the cobord operator

$$\delta: C^q(\mathscr{U}, \mathscr{V}) \to C^{q+1}(\mathscr{U}, \mathscr{V}),$$

$$\delta\varphi(x_0, \ldots, x_{q+1}) = \sum_{i=0}^{q+1} (-1)^i \varphi(x_0, \ldots, \hat{x}_i, \ldots, x_{q+1}).$$

If $(\mathscr{U}', \mathscr{V}')$ is an open covering subordinate to $(\mathscr{U}, \mathscr{V})$, then the restriction defines a chain mapping $C^*(\mathscr{U}, \mathscr{V}) \to C^*(\mathscr{U}', \mathscr{V}')$, so we have a direct system. We put $\check{C}^*(X, A) = \varinjlim C^*(\mathscr{U}, \mathscr{V})$.

The homologies of this complex are identical with the Čech cohomologies, so they are denoted by $\check{H}^*(X, A; R)$. We will drop the letter R if it does not lead to confusion. If φ is an element of $C^q(\mathscr{U}, \mathscr{V})$, then we denote by $\check{\varphi}$ its image in $\check{C}^q(X, A)$. If φ is a cocycle, then $\check{\varphi}$ is also a cocycle, and $[\check{\varphi}]$ denotes its cohomology class in $\check{H}^q(X, A)$. If $A = \emptyset$, then $\mathscr{V} = \{\emptyset\}$, and we write $C^*(\mathscr{U})$, $\check{C}^*(X)$, $\check{H}^*(X)$ instead of $C^*(\mathscr{U}, \{\emptyset\})$, $\check{C}^*(X, \emptyset)$, $\check{H}^*(X, \emptyset)$ respectively. Since $C^q(\mathscr{U}, \mathscr{V}) \subseteq C^q(\mathscr{U})$, we can regard $\check{C}^*(X, A)$ as a subcomplex of $\check{C}^*(X)$.

Let $\varphi \in C^q(\mathscr{U})$. The *support* $|\varphi|$ of φ is defined as follows:

$$|\varphi| = \{x \in X\colon \forall W \text{ open},\ x \in W\ \exists (x_0, \dots, x_q) \in W^{q+1}\colon \varphi(x_0, \dots, x_q) \neq 0\}.$$

If $c \in \check{C}^*(X, A)$, $c = \check{\varphi}$, then we define the support of c as $|c| = |\varphi|$, and the latter set does not depend on the choice of φ. If A is open, then $c \in \check{C}^*(X, A)$ iff $c \in \check{C}^*(X)$, $|c| \subseteq X \setminus A$. In order to define an element of $\check{C}^*(X)$, it suffices to define it in a neighbourhood of its support.

Let $f\colon (X, A) \to (X', A')$ be a continuous mapping, $(\mathscr{U}', \mathscr{V}')$ an open covering of (X', A') and $(\mathscr{U}, \mathscr{V})$ an open covering of (X, A) subordinate to $(f^{-1}(\mathscr{U}'), f^{-1}(\mathscr{V}'))$. Define

$$C^*(f)\colon C^*(\mathscr{U}', \mathscr{V}') \to C^*(\mathscr{U}, \mathscr{V})$$

by $C^*(f)\varphi(x_0, \dots, \varphi_q) = \varphi(f(x_0), \dots, f(x_q))$. The limit mapping is denoted by $\check{C}^*(f)$. It induces a homomorphism

$$f^* = \check{H}^*(f)\colon \check{H}^*(X', A') \to \check{H}^*(X, A).$$

If $i\colon (X, A) \hookrightarrow (X', A')$ is the inclusion and $c \in \check{C}^*(X', A')$, we will write $c|_{(X,A)}$ instead of $\check{C}^*(i)(c)$; if $u \in \check{H}^*(X', A')$, then $u|_{(X,A)}$ denotes $i^*(u)$.

Analogously, we define $S^q(\mathscr{U}, \mathscr{V})$ as the group of all mappings from the set of all q-dimensional singular simplexes $\sigma\colon \Delta_q \to X$ (see (2.4)) such that $\sigma(\Delta_q) \subseteq U$ for some $U \in \mathscr{U}$, to R and equal to zero for any σ with $\sigma(\Delta_q) \subseteq V$ for some $V \in \mathscr{V}$.

In the same manner as above we can define the cobord operator on $S^*(\mathscr{U}, \mathscr{V})$, etc., and finally obtain a complex $\varinjlim S^*(\mathscr{U}, \mathscr{V})$. The homologies of this complex are naturally isomorphic to singular cohomologies, so they are denoted by $H^*(X, A; R)$ (or $H^*(X, A)$).

If $\varphi \in S^q(\mathscr{U})$, then we can define $|\varphi|$, the support of φ by $|\varphi| = \{x \in X\colon \forall W \text{ open},\ x \in W\ \exists$ a singular simplex $\sigma\colon \Delta_q \to X$ such that $\sigma(\Delta_q) \subseteq W$ and $\varphi(\sigma) \neq 0\}$.

There is a natural chain mapping

$$\mu\colon C^*(\mathscr{U}, \mathscr{V}) \to S^*(\mathscr{U}, \mathscr{V})$$

$\mu(\varphi)(\sigma) = \varphi(\sigma(e_0), \dots, \sigma(e_q))$ which induces a natural homomorphism (also denoted by μ)

$$\mu\colon \check{H}^*(X, A) \to H^*(X, A).$$

If (X, A) is an ANR-pair, then μ is an isomorphism.

Now we shall construct a generator of $\check{H}^1(S^1)$. Let $\mathscr{U}$ be an open covering of S^1, with $\mathscr{U} = \{U_0, U_1, U_2\}$, where;

$$U_0 = \{\mathrm{re} > -\tfrac{1}{2}\}, \quad U_1 = \{\mathrm{re} < 0, \mathrm{im} > -\tfrac{1}{2}\}, \quad U_2 = \{\mathrm{re} < 0, \mathrm{im} < \tfrac{1}{2}\}.$$

Define $\varphi \in C^1(\mathscr{U})$ as follows: if $(x_0, x_1) \in U_i^2$ for $i = 1, 2$ then $\varphi(x_0, x_1) = 0$, if $(x_0, x_1) \in U_0^2$, then $\varphi(x_0, x_1) = \alpha(x_1) - \alpha(x_0)$, where $\alpha\colon U_0 \to R$,

$$\alpha(x) = \begin{cases} 1 & \text{if } \mathrm{im}(x) \geqslant 0, \\ 0 & \text{if } \mathrm{im}(x) < 0. \end{cases}$$

(3.1) Lemma. *The cochain φ defined above is a cocycle and $z = [\check{\varphi}]$ is a generator of $\check{H}^1(S^1)$.*

(3.2) Definition and Remark. In the sequel the generator z defined as above will be called the *standard generator*. Since S^1 is an ANR, then z induces $\mu(z)$, which is also a generator of $H^1(S^1)$. We will call $\mu(z)$ also the standard generator. For simplicity, we will write z instead of $\mu(z)$.

Proof of (3.1). We will use the Eilenberg–Steenrod axioms for Čech cohomologies. The mapping

$$1\colon \boldsymbol{R}_{++} \ni x \to 1 \in R$$

is a cocycle in $C^0(\boldsymbol{R}_{++})$, and [1] is a generator of $\check{H}^0(\boldsymbol{R}_{++})$. Hence the class $[\psi|_{\boldsymbol{R}\setminus 0}]$, where $\psi\colon \boldsymbol{R} \to R$, $\psi(x) = 1$ if $x \geqslant 0$ and 0 if $x < 0$, defines a generator of $\check{H}^0(\boldsymbol{R}\setminus 0, \boldsymbol{R}_{--})$. Since the connecting homomorphism in the exact sequence of the triple $(\boldsymbol{R}, \boldsymbol{R}\setminus 0, \boldsymbol{R}_{--})$ is an isomorphism, $[\check{\chi}]$ is a generator of $\check{H}^1(\boldsymbol{R}, \boldsymbol{R}\setminus 0)$, where $\chi \in C^1(\boldsymbol{R}, \{\boldsymbol{R}_{++}, \boldsymbol{R}_{--}\})$, $\chi(x_0, x_1) = \delta\psi(x_0, x_1)$. One can verify that z is the image of $[\check{\chi}]$ under the following composition of isomorphisms:

$$\check{H}^1(\boldsymbol{R}, \boldsymbol{R}\setminus 0) \xrightarrow{\cong} \check{H}^1(U_0, U_0\setminus 1) \xrightarrow{\cong} \check{H}^1(S^1, S^1\setminus 1) \xrightarrow{\cong} \check{H}^1(S^1),$$

where the first arrow is induced by excision and the exponential mapping, the second is induced by excision and the third is induced by the inclusion. This ends the proof.

It is easy to verify that a class member of z can be also defined as follows. Let $g\colon U_0 \to \boldsymbol{R}$ be a mapping such that:

(3.3a) $g(x) > 0$ if $\mathrm{re}(x) \leqslant 0$, $\mathrm{im}(x) > 0$,

(3.3b) $g(x) < 0$ if $\mathrm{re}(x) \leqslant 0$, $\mathrm{im}(x) < 0$.

Define $\beta\colon U_0 \to R$, as follows:

$$\beta(x) = \begin{cases} 1 & \text{if } g(x) \geqslant 0, \\ 0 & \text{if } g(x) < 0. \end{cases}$$

Let $\psi \in C^1(\mathscr{U})$, $\psi(x_0, x_1) = 0$ if $(x_0, x_1) \in U_i^2$ for $i = 1, 2$, and $\psi(x_0, x_1) = \beta(x_1) - \beta(x_0)$ if $(x_0, x_1) \in U_0^2$. Then $z = [\check{\psi}]$. Indeed, one can verify that $\varphi - \psi = \delta\chi$, where $\chi\colon S^1 \to R$ and

$$\chi(t) = \begin{cases} \alpha(t) - \beta(t) & \text{if } \mathrm{re}(t) > 0, \\ 0 & \text{if } \mathrm{re}(t) \leqslant 0. \end{cases}$$

Let us consider a more general situation. Assume that r is a positive integer and let $\{1, w, \ldots, w^{r-1}\}$ be the set $\sqrt[r]{1}$. We put

$$U_{0,0} = \exp\left(\pi i\left(-\frac{2}{3r}, \frac{2}{3r}\right)\right),$$

$$U_{1,0} = \exp\left(\pi i\left(\frac{1}{2r}, \frac{7}{6r}\right)\right),$$

$$U_{2,0} = \exp\left(\pi i\left(-\frac{7}{6r}, -\frac{1}{2r}\right)\right).$$

Then $U_{i,0}$ is the image of U_i under a suitably chosen branch of $\sqrt[r]{\ }$. Denote by h the branch with $U_{0,0} = h(U_0)$. Fix $j = 0, \ldots, r-1$. We put $U_{i,j} = \tau_{w^j}(U_{i,0})$, where

$$\tau_s\colon S^1 \ni t \to st \in S^1 .$$

Let $g_j\colon U_{0,j} \to \boldsymbol{R}$ be a mapping such that the composition

$$g_j \circ \tau_{w^j} \circ h\colon U_0 \to \boldsymbol{R}$$

has the properties (3.3a) and (3.3b). For $x \in U_{0,j}$, put $\beta_j(x) = 1$ if $g_j(x) \geqslant 0$, and $\beta_j(x) = 0$ if $g_j(x) < 0$. Define $\psi_r \in C^1(\mathcal{U}_r)$, where

$$\mathcal{U}_r = \{U_{i,j}\}_{\substack{i=0,1,2\\ j=0,\ldots,r-1}},$$

as follows: $\psi_r(x_0, x_1) = 0$ if $(x_0, x_1) \in U_{i,j}^2$, $i = 1, 2$ and $\psi_r(x_0, x_1) = \beta_j(x_1) - \beta_j(x_0)$ if $(x_0, x_1) \in U_{0,j}^2$. Repeating the argument presented above and using the fact that the mapping

$$\sigma^r\colon S^1 \ni t \to t^r \in S^1$$

induces the homomorphism

$$\sigma^{r*}\colon \check{H}^1(S^1) \ni u \to ru \in \check{H}^1(S^1),$$

one can prove the following result:

(3.4) Proposition. *ψ_r is a cocycle in $C^1(\mathcal{U}_r)$ and $[\psi_r] = rz$, where z is the standard generator of $\check{H}^1(S^1)$.*

Now we shall present a certain fact concerning singular (co-)homologies of the circle. If $z \in H^1(S^1; \boldsymbol{Z})$ is the standard generator, then an element ζ is called the standard generator of $H_1(S^1; \boldsymbol{Z})$ iff $\langle z, \zeta\rangle = 1$ (where $\langle\cdot,\cdot\rangle$ denotes the scalar product).

(3.5) Proposition. *ζ is the homology class of the singular simplex*

$$\sigma\colon \Delta_1 \ni (1-t)e_0 + te_1 \to \exp(2\pi it) \in S^1 .$$

Proof. One can see that the homology class of σ in $H_1(S^1, S^1\setminus 1)$ is equal to the class of

$$\sigma'\colon \Delta_1 \ni (1-t)e_0 + te_1 \to \exp(\pi i(t - \tfrac{1}{2})) \in S^1 .$$

By (3.1) the cochain φ induces also a generator of $H^1(S^1, S^1\setminus 1)$ and $\langle\varphi, \sigma'\rangle = 1$. The assertion follows from definition and the naturality of the scalar product.

4. q-mappings of the circle. Let X be a topological space. A continuous mapping $\theta\colon S^1 \to X$ is called a *q-mapping on X* (q is a positive integer) if can be represented as a composition of the form

$$\theta\colon S^1 \xrightarrow{\sigma^q} S^1 \xrightarrow{h} \Gamma \hookrightarrow X$$

where $\sigma^q(t) = t^q$ and h is a homeomorphism.

(4.1) Example. For a given local semiflow π and $(\omega, x) \in \mathrm{Per}(\pi)$, the trajectory $\theta_{(\omega,x)}$ defined in Section 2 is a $m_{(\omega,x)}$-mapping on $\boldsymbol{R}_{++} \times X$. Indeed, $\theta_{(\omega,x)}$ coincides with the composition

$$\left\{\exp\left(2\pi i\frac{t}{\omega_x}\right) \to (\omega, \pi(t, x))\right\} \circ \sigma^{m(\omega,x)} .$$

Let θ be a q-mapping, $\theta = (\Gamma \hookrightarrow X) \circ h \circ \sigma^q$. If z is the standard generator of $\check{H}^1(S^1)$, we define

$$z_\theta = h^{-1*}(z) \in \check{H}^1(\Gamma) .$$

One can easily prove the following fact:

(4.2) Lemma. *If τ_s is the multiplication by s, then*

$$z_\theta = z_{\theta \circ \tau_s} .$$

Now we are going to present the main results of this section. We assume that X is an euclidean space with the scalar product $\langle\cdot,\cdot\rangle$ and the induced norm $|\cdot|$. For any two continuous functions θ, $\theta'\colon S^1 \to X$ we put

$$|\theta - \theta'| = \max\{|\theta(t) - \theta'(t)|\colon t \in S^1\} .$$

(4.3) Theorem. *Let $\theta\colon S^1 \to X$ be a q-mapping,*

$$\theta = (\Gamma \hookrightarrow X) \circ h \circ \sigma^q .$$

Then there exist $\varepsilon > 0$, an open neighbourhood U of Γ and $u \in \check{H}^1(U; R)$ such that if $\theta'\colon S^1 \to X$ is a q'-mapping, $\theta'(S^1) = \Gamma'$ and $|\theta - \theta'| < \varepsilon$, then q' divides q and

$$u|_{\Gamma'} = \frac{q}{q'} z_{\theta'} \in \check{H}^1(\Gamma', R) .$$

Proof. Since the transformation between Čech cohomologies with coefficients in $\boldsymbol{Z}$ and R induced by the canonical ring-with-unit homomorphism $\boldsymbol{Z} \to R$ is natural, it suffices to prove the theorem in the case where $R = \boldsymbol{Z}$.

It is easy to find $\varepsilon_0 > 0$ such that if $|\theta - \theta'| < \varepsilon_0$ then q' divides q. To this end one can use a retraction of a neighbourhood of Γ onto Γ itself. Let $y_+ = h(i)$, $y_- = h(-i)$, and let L_1, L_2 be closed arcs given by the formulae

$$L_1 = \exp(\pi i[-\tfrac{1}{2}, \tfrac{1}{2}]), \quad L_2 = \exp(\pi i[\tfrac{1}{2}, \tfrac{3}{2}]) .$$

By translating the origin if necessary, we can assume that there exists a linear form $g\colon X\to \boldsymbol{R}$ such that $g(y_+)>0$ and $g(y_-)<0$. We can assume that $g(x)=\langle x, v\rangle$ for some $v\in X$, $|v|=1$. Define $H=\ker(g)$. The sets $h(L_1)\cap H$ and $h(L_2)\cap H$ are compact and disjoint, so there exists a Urysohn Lemma mapping $f\colon H\to[0,1]$ such that

$$h(L_1)\cap H\subseteq f^{-1}(0), \qquad h(L_2)\cap H\subseteq f^{-1}(1)$$

and the set $f^{-1}([0,1))$ is bounded. Let $c\in(0,1)$ be an arbitrary number. We put $U=X\setminus f^{-1}(c)$. Since $h(L_1)$ is compact and separated from the set $f^{-1}([c,1])$, there exists an ε_1; $0<\varepsilon_1<\varepsilon_0$, such that

$$h(L_1)\cap(f^{-1}([c,1])+[-\varepsilon_1,\varepsilon_1]v)=\emptyset .$$

We define $\mathscr{V}$, an open covering of U, as follows: $\mathscr{V}=\{V_0,V_1,V_2,V_3\}$, where:

$$\begin{aligned}
V_0 &= (f^{-1}([0,c))+\boldsymbol{R}v)\cup g^{-1}((\varepsilon_1,\infty)\cup(-\infty,-\varepsilon_1)),\\
V_1 &= g^{-1}((0,\infty)),\\
V_2 &= g^{-1}((-\infty,0)),\\
V_3 &= f^{-1}((c,1])+(-\varepsilon_1,\varepsilon_1)v .
\end{aligned}$$

Define $\psi\in C^1(\mathscr{V})$ by the formula

$$\psi(x_0,x_1)=\begin{cases}0 & \text{if } (x_0,x_1)\in V_j^2, \text{ for } j=1,2,3;\\ \beta(g(x_1))-\beta(g(x_0)) & \text{if } (x_0,x_1)\in V_0^2,\end{cases}$$

where

$$\beta(x)=\begin{cases}1 & \text{if } x\geqslant 0,\\ 0 & \text{if } x<0.\end{cases}$$

Then ψ is a cocycle. Let $u=[\check{\psi}]\in\check{H}^1(U)$. Choose an ε, $0<\varepsilon<\varepsilon_1$, so small that if $|\theta-\theta'|<\varepsilon$, then for any $j=0,\dots,q-1$:

$$\theta'\left(\exp\left(\pi i\left[\frac{2j}{q}-\frac{1}{2q},\frac{2j}{q}+\frac{1}{2q}\right]\right)\right)\subseteq V_0,$$

$$\theta'\left(\exp\left(\pi i\left(\frac{2j}{q}+\frac{1}{2q}\right)\right)\right)\subseteq V_1,$$

$$\theta'\left(\exp\left(\pi i\left(\frac{2j}{q}-\frac{1}{2q}\right)\right)\right)\subseteq V_2,$$

$$\theta'\left(\exp\left(\pi i\left(\left[\frac{2j}{q}+\frac{1}{2q},\frac{2j}{q}+\frac{3}{2q}\right]\right)\right)\right)\subseteq V_1\cup V_2\cup V_3 .$$

Let $\theta'\colon S^1\to X$ be a q'-mapping with $|\theta-\theta'|<\varepsilon$. Then θ' can be represented as a composition of the form $(\Gamma'\hookrightarrow X)\circ h'\circ\sigma^{q'}$ for some homeomorphism h'. Now

we can use Proposition (3.4). In fact, the cocycle presented in this proposition differs from the cocycle $C^*(h'\circ\sigma^{q'})(\psi|_{\Gamma'})$, but they induce the same limit elements. By (3.4),

$$\check{H}^*(h'\circ\sigma^{q'})(u|_{\Gamma'})=[\check{C}^*(h'\circ\sigma^{q'})(\check{\psi}|_{\Gamma'})]=qz .$$

Since $\sigma^{q'}$ induces the multiplication by q', we have

$$\check{H}^*(h')(u|_{\Gamma'})=\frac{q}{q'}z,$$

and thus $u|_{\Gamma'}=\dfrac{q}{q'}z_\theta$. The proof is finished.

(4.4) Corollary. *Let θ be as in (4.3), let V be an open set, $\Gamma\subseteq V$ and $v\in\check{H}^1(V;R)$, $v|_\Gamma=rz_\theta$ for some $r\in R$. Then there exists a $\delta>0$ such that for any q'-mapping θ' with $|\theta-\theta'|<\delta$ the following holds*:

$$\Gamma'=\theta'(S^1)\subseteq V,\ q' \text{ divides } q \text{ and } v|_{\Gamma'}=r\frac{q}{q'}z_{\theta'}\in\check{H}^1(\Gamma';R).$$

Proof. Choose ε, U and u as in the conclusion of (4.3). Since $ru|_\Gamma=v|_\Gamma$, the tautness of Čech cohomology implies the existence of an open set W such that $\Gamma\subseteq W\subseteq U\cap V$ and $ru|_W=v|_W$. The corollary holds for any $0<\delta<\varepsilon$ such that if $|\theta-\theta'|<\delta$ then $\Gamma'\subseteq W$.

5. Standard generators of periodic orbits. Let X be a topological space, and let π be a local semiflow on X. (4.1) and (4.2) enables us to introduce the following:

(5.1) Definition. If Γ is a periodic orbit of π, then by z_Γ (the *standard generator of* $\check{H}^1(\Gamma;R)$) we mean $z_{\theta(\pi;\omega,x)}$ for any $(\omega,x)\in\Gamma$.

This z_Γ can also be treated as an element of $H^1(\Gamma;R)$, so we may introduce $\zeta_\Gamma\in H_1(\Gamma;R)$ such that $\langle z_\Gamma,\zeta_\Gamma\rangle=1$. By Proposition (3.5), we have the following result:

(5.2) Proposition. *For any $(\omega,x)\in\Gamma$, the element ζ_Γ is the homology class of the singular simplex*

$$\Delta_1\ni a^{-1}(t)\to(\omega,(\pi(t,x))\in\Gamma,$$

where

$$a\colon [0,\omega_x]\ni t\to\left(1-\frac{t}{\omega_x}\right)e_0+\frac{t}{\omega_x}e_1\in\Delta_1 .$$

6. Isolated periodic orbits of smooth flows. In this section we will use the notion of a standard generator in the case of a smooth flow.

We will deal with the ordinary fixed point index, denoted by i_1 (this notation is explained in (1.9)). We use a similar convention as in Section 1; For any mapping $g: U\to X$ (where U is an open subset of X), $i_1(g,x)$ denotes the index of a germ $[g|_D]$, where D is open in U and $\mathrm{Fix}(g|_D)=\{x\}$.

Assume that X is a smooth (i.e. of C^∞) manifold and π: $\boldsymbol{R}\times X\to X$ is a smooth flow on M. Let $\Gamma\subseteq\mathrm{Per}(\pi)$ be an isolated periodic orbit with the multiplicity $m_\Gamma=m$.

By γ we denote the image of Γ under the projection $\boldsymbol{R}\times X\to X$. Such a γ will be called the *underlying orbit of* Γ. (Usually, γ is called a *periodic* or *closed* orbit of π (see [MeP]), but this term has another meaning here.) Let P be a Poincaré mapping associated with γ. P is defined for some section Σ transversal to γ. Let $\gamma\cap\Sigma=\{x_0\}$. We will prove the following result:

(6.1) THEOREM. *Under the assumptions presented above,*

$$i_1(P^m, x_0)=\sigma_{(\pi,\Gamma)}(z_\Gamma),$$

where P^m is the m-th iterate of P, $\sigma_{(\pi,\Gamma)}$ is the fixed point homomorphism determined by the standard orientation of $\boldsymbol{R}_{++}$ and z_Γ is the standard generator of $\check{H}^1(\Gamma,\boldsymbol{Z})$ (see Definitions (1.1), (2.4) *and* (5.1)).

(6.2) Remark. (6.1) is also valid if $\boldsymbol{Z}$ is rplaced by R. In this case, the left side is interpreted as the image of $i_1(P^m, x_0)\in\boldsymbol{Z}$ under the canonical ring-with-unit homomorphism $\boldsymbol{Z}\to R$. The proof is the same as in the case when $R=\boldsymbol{Z}$.

Before we start the proof of (6.1), we introduce a lemma.

(6.3) LEMMA. *Let X and Y be euclidean spaces, let D be an open neighbourhood of $x_0\in X$, and let $\varphi\colon D\to Y$ be a differentiable mapping with $\varphi(x_0)=0$. Let X be an orthogonal direct sum $H\oplus K$, where $K=\ker d_{x_0}\varphi$, and let $m\in\boldsymbol{N}$. Then there exists an $\varepsilon>0$ such that for any δ, $0<\delta<\varepsilon$, if $|x-x_0|<\delta$ and $\inf_{k\in K}|(x-x_0)-k|>\frac{\delta}{m}$, then*

$$|\varphi(x)-d_{x_0}\varphi(x-x_0)|<|d_{x_0}\varphi(x-x_0)|.$$

Proof of (6.3). Let $L>0$ be such that for any $v\in H$, $v\neq 0$

$$L|v|<|d_{x_0}\varphi(v)|.$$

There exists an $\varepsilon>0$ such that if $0<|x-x_0|<\varepsilon$, then

$$|\varphi(x)-d_{x_0}\varphi(x-x_0)|<\frac{1}{m+1}L|x-x_0|.$$

Let $\delta<\varepsilon$ and $0<|x-x_0|<\delta$. Put $x-x_0=v+w$, $v\in H$, $w\in K$. If $|v|>\frac{\delta}{m}$, then $|w|<\delta<m|v|$ by the orthogonality of v and w. Thus

$$|\varphi(x)-d_{x_0}\varphi(x-x_0)|<\frac{L}{m+1}(|v|+|w|)<L|v|<|d_{x_0}\varphi(v+w)|$$

and the lemma is proved.

Proof of (6.1). It is well known that he number $i_1(P^m, x_0)$ and the eigenvalues of $d_{x_0}P^m$ do not depend on the choice of the Poincaré mapping P. First we assume that 1 is not an eigenvalue of $d_{x_0}P^m$. In this case it suffices to prove that

$$\sigma_{(\pi,\Gamma)}(z_\Gamma)=\operatorname{sgn}\det(\mathrm{id}-d_{x_0}P^m).$$

Let n be the dimension of X. Without any loss of generality, we can assume that X is a submanifold of $\boldsymbol{R}^{k+n}$, $x_0=0$, $T_{x_0}X=0\times\boldsymbol{R}^n$ (first k coordinates are equal to 0), $\frac{\partial\pi}{\partial t}(0, x_0)=(0,\ldots,0,\lambda)$ for some $\lambda>0$ and that $(\omega, x_0)\in\Gamma$. Let V be a normal tubular neighbourhood of X (see [H]), and let $r\colon V\to X$ be the induced retracion. We choose an open set D such that $D\cap\operatorname{Per}(\pi)=\Gamma$. Put

$$E=\{(t,y)\in\boldsymbol{R}\times\boldsymbol{R}^{k+n}\colon (t, r(y))\in D\}.$$

Let us assume the following convention. Any point in $\boldsymbol{R}\times\boldsymbol{R}^{k+n}$ will be written as a quadruple (t,y,x,s), where $t,s\in\boldsymbol{R}$, $y\in\boldsymbol{R}^k$ and $x\in\boldsymbol{R}^{n-1}$. We also put

$$c=(\omega,0,0,0).$$

Let W be a normal tubular neighbourhood of Γ contained in E. Denote by r_1 the induced retraction $r_1\colon W\to\Gamma$. We define a singular cocycle ψ as follows: the support $|\psi|$ (see Section 3) is equal to $J=r_1^{-1}(c)$ and there exists a neighbourhood Z of J in W such that for any singular simplex $\sigma\colon\Delta_1\to W$ with $\sigma(\Delta_1)\subseteq Z$ we have

$$\psi(\sigma)=\alpha(\sigma(e_1))-\alpha(\sigma(e_0))$$

where $\alpha(w)=1$ if the last cord inate of $w\in Z$ is greater than 0, and $\alpha(w)=0$ otherwise. Such a ψ determines an element $v\in H^1(W, W\setminus J)$. By results of Sections 3 and 4, the class v restricted to Γ is equal to the class z_Γ defined in (5.1). For any natural number l, let $o^l\in H_l(\boldsymbol{R}^l, \boldsymbol{R}^l\setminus 0)$ be a given orientation (i.e. generator), and let $q^l\in H^l(\boldsymbol{R}^l, \boldsymbol{R}^l\setminus 0)$ be such that $\langle q^l, o^l\rangle=1$. We assume that $o^1=o$ (o is defined in (2.4)) and we put $q=q^1$. Let $\hat{q}$ denote the class $q^k\times q^{n-1}\times q$. The orientation $o\times o^k\times o^{n-1}\times o$ determines a fundamental class $o_\Gamma\in H_{k+n+1}(W, W\setminus\Gamma)$, and by (1.2) our task is to compute

$$\langle(\mathrm{id}_W, f)^*(v|_W\times\hat{q}), o_\Gamma\rangle,$$

where

$$f\colon (W, W\setminus\Gamma)\to(\boldsymbol{R}^{k+n}, \boldsymbol{R}^{k+n}\setminus 0)$$

is defined as

$$f\colon (t,y,x,s)\to(y,x,s)-\pi(t, r(y,x,s)).$$

By the naturality of the considered products, it suffices to compute

$$\langle(\mathrm{id}_W, f)^*(v\times\hat{q}), o_c\rangle$$

where

$$(\mathrm{id}_W, f)^*\colon H^{1+k+n}((W, W\setminus J)\times(\boldsymbol{R}^{k+n}, \boldsymbol{R}^{k+n}\setminus 0))\to H^{1+k+n}(W, W\setminus c).$$

We can assume that the Poincaré mapping P is defined on a section Σ, where Σ is a neighbourhood of 0 in $(\boldsymbol{R}^k\times\boldsymbol{R}^{n-1}\times 0)\cap X$. By standard calculations one can verify that $d_c f$ has the following matrix form in coordinates (t,y,x,s) and (y,x,s):

$$\begin{vmatrix} 0 & \mathrm{id} & 0 & 0\\ 0 & 0 & \mathrm{id}-d_0P^m & 0\\ -\lambda & 0 & b & 0\end{vmatrix},$$

where b is a certain operator $\boldsymbol{R}^{n-1} \to \boldsymbol{R}$ and id is the identity. Let Δ_δ denote the interval $(\omega-\delta, \omega+\delta)$, and let K_δ^l denote the open euclidean ball with 0 as the centre and the radius δ in $\boldsymbol{R}^l$, $l \in N$. Put:

$$\Phi_\delta = \Delta_\delta \times K_\delta^k \times K_\delta^{n-1} \times K_\delta^1$$
$$A_\delta = \mathrm{cl}(\Delta_{\delta/2}) \times \mathrm{cl}(K_{\delta/2}^k) \times \mathrm{cl}(K_{\delta/2}^{n-1}) \times K_\delta^1 .$$

Lemma (6.3) implies that there exists an $\varepsilon > 0$ such that for any δ, $0 < \delta < \varepsilon$ and any $\alpha \in I$ we have

$$f_\alpha \neq 0 \quad \text{on } \Phi_\delta \setminus A_\delta ,$$

where $f_\alpha = \alpha f + (1-\alpha) d_c f \circ \tau_c$ (here $\tau_c(w) = w - c$). Not let $\eta < \varepsilon$ be so small that Φ_η is contained in Z. Denote $\Phi = \Phi_\eta$ and $A = A_\eta$. Observe that the following diagram commutes (the vertical arrows are induced by inclusions):

$$\begin{tikzcd}
H^*((W, W\setminus J) \times (\boldsymbol{R}^{k+n}, \boldsymbol{R}^{k+n}\setminus 0)) \arrow[r, "(\mathrm{id}, f)^*"] \arrow[d] & H^*(W, W\setminus c) \arrow[d] \\
H^*((\Phi, \Phi\setminus J) \times (\boldsymbol{R}^{k+n}, \boldsymbol{R}^{k+n}\setminus 0)) \arrow[r, "(\mathrm{id}, d_c f \circ \tau_c|_\Phi)^*"] & H^*(\Phi, \Phi\setminus (A \cap J)) .
\end{tikzcd}$$

Indeed, if $z \in \Phi \setminus A$, then $(\mathrm{id}, f_\alpha)(z) \in \Phi \times (\boldsymbol{R}^{k+n}\setminus 0)$, and if $z \in A \setminus J$ then $(\mathrm{id}, f_\alpha)(z) \in (\Phi\setminus J) \times \boldsymbol{R}^{k+n}$ for any $\alpha \in I$, so the homotopy axiom implies the commutativity. By this diagram and the naturality of the scalar product our problem reduces to the computation of

$$\langle (\mathrm{id}, d_c f \circ \tau_c|_\Phi)^*(v|_{(\Phi, \Phi\setminus J)} \times \hat{q}), o_c \rangle ,$$

where o_c is the fundamental class. Let p denote the projection $(t, y, x, s) \to s$. Then p induces an isomorphism

$$p^*\colon H^1(\boldsymbol{R}, \boldsymbol{R}\setminus 0) \to H^1(\Phi, \Phi\setminus J) ,$$

and by the proof of (3.1) we have $p^*(q) = v|_{(\Phi, \Phi\setminus J)}$, so by the naturality of products and the definition of $\hat{q}$ we have to compute

$$\langle (p, d_c f)^*(q \times q^k \times q^{n-1} \times q), o \times o^k \times o^{n-1} \times o \rangle .$$

Using the matrix form of $d_c f$, it is easy to verify that the above scalar product is equal to $\operatorname{sgn}\det(\mathrm{id} - d_0 P^m)$, so the theorem is proved in the considered case.

Now let us consider the general situation, when $d_{x_0} P^m$ is arbitrary. By a variant of the Kupka–Smale Theorem (see [MeP]), for any relatively compact neighbourhood V of Γ and any neighbourhood N of π in the C^1-topology on the space of smooth flows on X, there is a $\pi' \in N$ such that if a periodic orbit Γ' of π' is contained in V, then the underlying orbit γ' of Γ' is hyperbolic (i.e. the differential of a Poincaré mapping associated to γ' has no eigenvalues in the unit circle in $\boldsymbol{C}$). Fix an open relatively compact neighbourhood V of Γ, where $\Gamma = \mathrm{Per}(\pi) \cap \cap\, \mathrm{cl}(V)$. Consider a section Σ transversal to γ. If a neighbourhood N of π is sufficiently small, then Σ is a section for any $\pi' \in N$. By a shrinking of N if necessary, for any $\pi' \in N$ such that $\mathrm{Per}(\pi') \cap V$ consists of finite number of periodic orbits (denoted by $\Gamma_1, \ldots, \Gamma_k$, where γ_i is the underlying orbit of Γ_i, $i = 1, \ldots, k$), we have: $m_i = m_{\Gamma_i}$ is a divisor of m, γ_i intersects Σ in exactly $\frac{m}{m_i}$ points (denoted by $x_{i,j}$, $j = 1, \ldots, \frac{m}{m_i}$), and the following conditions are satisfied:

$$i_1(P^m, x_0) = \sum_{i=1}^{k} \sum_{j=1}^{m/m_i} i_1((P_{i,j})^{m_i}, x_{i,j}) , \tag{6.4a}$$

where $P_{i,j}$ is the Poincaré mapping associated to π', Σ and γ_i at the point $x_{i,j}$;

(6.4b) There exists an open neighbourhood U of Γ, $U \subseteq V$, such that for any $i = 1, \ldots, k$, $\Gamma_i \subseteq U$ and there exists $u \in \check{H}^1(U)$ with

$$z_\Gamma = u|_\Gamma, \qquad u|_{\Gamma_i} = \frac{m}{m_i} z_{\Gamma_i} ;$$

$$\sigma_{(\pi,\Gamma)}(z_\Gamma) = \sigma_{(\pi', \cup\Gamma_i)}(u|_{\cup\Gamma_i}) . \tag{6.4c}$$

The property (6.4a) can be obtained as follows. The index i_1 does not change if we replace the germ (P^m, x_0) by the fixed point germ of a suitable mapping induced by π' for π' sufficiently close to π. The latter mapping is of the form

$$W \ni x \to \pi'(t(x), x) \in \Sigma ,$$

where $t(x)$ is close to ω and W is open in Σ. The fixed point set of this mapping is $\left\{x_{i,j}\colon i = 1, \ldots, k,\ j = 1, \ldots, \frac{m}{m_i}\right\}$, so the additivity of i_1 implies the required property. (6.4b) is a consequence of Theorem (4.3) and (6.4c) follows from Proposition (1.7). By (6.4c), Proposition (1.4) and (6.4b), we have

$$\sigma_{(\pi,\Gamma)}(z_\Gamma) = \sum_i \sigma_{(\pi',\Gamma_i)}(u|_{\Gamma_i}) = \sum_i \frac{m}{m_i} \sigma_{(\pi',\Gamma_i)}(z_{\Gamma_i}) . \tag{6.5}$$

Now the result follows easily from the previous case. Indeed, we can assume that all the γ_i are hyperbolic, so no iterate of $P_{i,j}$ has 1 as an eigenvalue. By the previous case,

$$i_1((P_{i,j})^{m_i}, x_{i,j}) = \sigma_{(\pi',\Gamma_i)}(z_{\Gamma_i})$$

for any $j = 1, \ldots, \frac{m}{m_i}$. Connecting this fact with (6.4a) and (6.5), we obtain $\sigma_{(\pi,\Gamma)}(z_\Gamma) = i_1(P^m, x_0)$. The proof is finished.

7. Fuller classes. In this section we will deal with a (commutative) field K. The image of $n \in \boldsymbol{Z}$ under the canonical ring-with-unit homomorphism $\boldsymbol{Z} \to K$ will be denoted by $[n]$. If $K = \boldsymbol{Q}$, the square brackets will be omitted.

Assume that π is a given local semiflow on a topological space X. Let $C \subseteq \mathrm{Per}(\pi)$ be compact and isolated.

(7.1) DEFINITION. An element $u \in \check{H}^1(C; K)$ is called a *Fuller class for the germ* (π, C) over the field K iff for any periodic orbit Γ contained in C

$$u|_\Gamma = [m_\Gamma]^{-1} z_\Gamma ,$$

where $z_\Gamma \in \check{H}^1(\Gamma; K)$ is the standard generator (see Definition (5.1)).

Of course, we must have $[m] \neq 0$ for any $m \in M_C$, where M_C is the set of multiplicities of C.

If C consists of a finite number of periodic orbits and the above condition is fulfilled, then such a class exists and is unique. In general neither existence nor uniqueness is guaranteed.

Assume now that X can be embedded into an euclidean space. Let C be isolated and $u \in \check{H}^1(C; K)$ be a Fuller class. By the tautness of the Čech cohomology there exists V, a neighbourhood of C and $v \in \check{H}^1(V; K)$ such that $v|_C = u$.

(7.2) PROPOSITION. *Assume V and v to be as above. Moreover, assume that $\mathrm{cl}(V)$ is compact and $\mathrm{Per}(\pi) \cap \mathrm{cl}(V) = C$. Then there exists a neighbourhood N of π in the compact-open topology such that for any $\pi' \in N$*

$$C' = \mathrm{Per}(\pi') \cap \mathrm{cl}(V) \subseteq V$$

and $v|_{C'}$ is a Fuller class for (π', C').

Proof. If π' is sufficiently close to π, then, for any $(\omega', x') \in C'$, the mapping $\theta_{(\pi'; \omega', x')}$ is close to $\theta_{(\pi; \omega, x)}$ for some $(\omega, x) \in C$ (see Section 2 for the definition), so Corollary (4.4) can be applied. Denote by Γ the orbit of (ω, x) in π, and by Γ' the orbit of (ω', x') in π'. By (4.4) we have

$$v|_{\Gamma'} = [m_\Gamma]^{-1}[m_\Gamma/m_{\Gamma'}] z_{\Gamma'} = [m_{\Gamma'}]^{-1} z_{\Gamma'}$$

so the proposition holds.

Now we present the results concerning Fuller classes of smooth flows on a smooth manifold X. Recall that the Fuller index i^F is defined as follows. If Γ_i, $i = 1, \ldots, k$, are isolated periodic orbits of a smooth flow π, we define

$$i^F(\pi, \bigcup_i \Gamma_i) = \sum_i \frac{1}{m_i} i_1(P_i^{m_i}, x_i) \in \boldsymbol{Q} ,$$

where P_i is a Poincaré mapping of γ_i, the underlying orbit of Γ_i, m_i is the multiplicity of Γ_i and $\{x_i\} = \gamma_i \cap \Sigma_i$, where Σ_i is a section which determines P_i. Let (π, C) be an arbitrary germ such that π is smooth. Since (π, C) is homotopical to some germ (π', C') such that C' consists of a finite number of periodic orbits, we put

$$i^F(\pi, C) = i^F(\pi', C') .$$

Fuller in [Fu] has proved that such a definition is correct and i^F is a periodic orbits index for the set of all smooth flows on X.

The main result of this section is the following:

(7.3) THEOREM. *If C is an isolated set of periodic orbits of a smooth flow π, $u \in \check{H}^1(C; K)$ is a Fuller class for (π, C), and n is a common multiplicity of the set M_C and $[n] \neq 0$, then*

$$\sigma_{(\pi, C)}(u) = [n]^{-1}[n i^F(\pi, C)] ,$$

where $\sigma_{(\pi, C)}$ is the fixed point homomorphism over K determined by the standard orientation of $\boldsymbol{R}_{++}$.

Proof. By the same argument as in the proof of (6.1) we conclude that for any relatively compact isolating neighbourhood V of C and for any neighbourhood N of π in the space of flows there is a $\pi' \in N$ such that the set $C' = \mathrm{Per}(\pi') \cap \mathrm{cl}(V)$ is contained in V and consists of finite number of periodic orbits. Moreover, if N is sufficiently small, we can assume that $(\pi, C) \simeq (\pi', C')$, since π and π' are generated by vector-fields, n is a common multiplicity of M_C, (this follows from Theorem (4.3)) and

$$\sigma_{(\pi', C')}(u') = \sigma_{(\pi, C)}(u) ,$$

where u' is the unique Fuller class for (π', C') (this is a consequence of (7.1) and Proposition (1.7)). By the definition of i^F, it suffices to prove that

$$\sigma_{(\pi' C')}(u') = [n]^{-1}[n i^F(\pi', C')] .$$

Let $C' = \bigcup_{i=1}^{k} \Gamma_i$, Γ_i are isolated, and let m_i be the multiplicity of Γ_i. By the additivity of i^F and Proposition (1.4) it suffices to show that

$$\sigma_{(\pi', \Gamma_i)}([m_i]^{-1} z_{\Gamma_i}) = [n]^{-1}\left[\frac{n}{m_i} i_1(P_i^{m_i}, x_i)\right] \quad \text{for any } i = 1, \ldots, k .$$

(P_i and x_i are described in the definition of i^F.) The latter equation is a trivial consequence of Theorem (6.1) and Remark (6.2).

(7.4) COROLLARY. *If $K = \boldsymbol{Q}$, then under assumptions of (7.3) we have*

$$\sigma_{(\pi, C)}(u) = i^F(\pi, C) \in \boldsymbol{Q} .$$

This corollary indicates a problem of determining which germs have Fuller classes. The following result presents a sufficient condition for their existence.

(7.5) PROPOSITION. *Let π be a local semiflow on a topological space. Let (π, C) be a flow-germ. If the orbit space C/S^1 of the action θ_C: $S^1 \times C \to C$ defined in Section 2 is triangulable and $H^2(C/S^1; \boldsymbol{Z}) = 0$, then there exists a Fuller class for (π, C) over $\boldsymbol{Q}$.*

Proof. Let n be a common multiplicity of M_C. Put $\boldsymbol{Z}_n = \sqrt[n]{1} \subseteq S^1$. For $x, y \in C$ introduce a relation

$$x \sim y \quad \text{iff} \quad y \in \boldsymbol{Z}_n x .$$

Denote $B = C/_\sim$. The mapping

$$S^1 \times B \ni (t, [x]) \to t[x] = [wx] \in B$$

where $w \in \sqrt[n]{t}$, is a free S^1-action on B. In order to prove the proposition it suffices to determine a class $v \in \check{H}^1(B; Z)$ such that, for any $b \in B$, $v|_{\gamma(b)}$ is equal to the class z_θ for the mapping $\theta\colon t \to tb$ (where $\gamma(b)$ denotes the orbit of b, i.e. the set $S^1 b$). Indeed, in this case we put $u = \frac{1}{n} q^*(v)$, where q is the quotient mapping $C \to B$. Such a class v can be obtained as follows. Let γ be an orbit of the action on B. Choose a point $b \in \gamma$. Then there exists a slice S_b of the action in b (see [Br], Ch. II. 4) such that the mappings

$$f_b\colon S_b \ni y \to \gamma(y) \in B/S^1$$

and

$$S^1 \times S_b \ni (t, y) \to ty \in B$$

are homeomorphisms onto images under them, and these images are open. Put $W_\gamma = f_b(S_b)$. The set $\mathscr{W} = \{W_\gamma\colon \gamma \in B/S^1\}$ forms an open covering of B/S^1 so we can find a triangulation such that for all vertices γ_i, $i = 1, \ldots, k$, of this triangulation, $\{\mathrm{St}(\gamma_i)\}_i$ is a covering subordinate to $\mathscr{W}$, where St means the star of a vertex. Let $b_i \in B$ be a point such that $\mathrm{St}(\gamma_i) \subseteq W_{\gamma(b_i)}$. Put

$$\Sigma_i = f_{b_i}^{-1}(\mathrm{St}(\gamma_i)), \qquad U_i = S^1 \Sigma_i.$$

Let $g_i\colon \mathrm{St}(\gamma_i) \to \Sigma_i$ be the restriction of $f_{b_i}^{-1}$, and let

$$h_i\colon S^1 \times \mathrm{St}(\gamma_i) \ni (s, \gamma) \to s g_i(\gamma) \in U_i.$$

Define $v_i = (\mathrm{pr}_1 \circ h_i^{-1})^*(z) \in H^1(U_i)$, where z is the standard generator and pr_1 is the projection onto the first factor. Thus for any b such that $\gamma = \gamma(b) \in \mathrm{St}(\gamma_i)$ we have $v_i|_\gamma = z_\theta$ (where θ is the mapping $t \to tb$). We prove that $v_i|_{U_i \cap U_j} = v_j|_{U_i \cap U_j}$. Assume that $\gamma \in U_i \cap U_j$. Since the set $\mathrm{St}(\gamma_i) \cap \mathrm{St}(\gamma_j)$ is contractible, the restriction mapping $H^*(U_i \cap U_j) \to H^*(\gamma)$ is an isomorphism. As it maps $v_i|_{U_i \cap U_j}$ and $v_j|_{U_i \cap U_j}$ onto the same element, so these restrictions must be identical. Now we can use the methods presented in [BT], p. 116–119 and 189–191. To this end one must replace the Čech–De Rham complex by the Čech-singular complex. The free S^1-action determines on B the structure of an S^1-bundle over B/S^1. The set $\{v_i\colon i = 1, \ldots, k\}$ forms an orientation of this bundle. Since $B/S^1 = C/S^1$, the second Čech cohomology of the good covering $\{\mathrm{St}(\gamma_i)\}_i$ of B/S^1 is equal to zero. Thus the Euler class vanishes, and the result follows from the remark on page 119 of [BT].

A sufficient condition under which C/S^1 is triangulable is presented in the paper [Y].

(7.6) Example. The germ $(\pi, \{1\} \times S^3)$ for the Seifert flow π on S^3 (see [Se] or [Fu]) provides an example of a case in which the assumptions of (7.5) are not fulfilled. Indeed, for the S^1-action on S^3 determined by π we have $S^3/S^1 = S^2$. Since $H^1(S^3) = 0$, the above germ does not have any Fuller class.

8. The index $\jmath$. The results of the previous section suggest the possibility of introducing a periodic orbit index in the following way. Let $\Pi^F(X)$ be a set of flow-germs on a given ENR-space X such that, for any $(\pi, C) \in \Pi^F(X)$, the following conditions are satisfied:

(8.1 a) There exists a Fuller class for (π, C) over Q.

(8.1 b) (π, C) is approximable by germs (π', C') having a unique Fuller class (approximability means that the condition presented at the beginning of the proof of (7.3) holds).

A condition which guarantees (8.1 a) is presented in (7.5). The condition (8.1 b) is fulfilled if π is a smooth flow on a manifold.

(8.2) Theorem. *The function*

$$\jmath\colon \Pi^F(X) \ni (\pi, C) \to \sigma_{(\pi, C)}(u) \in Q,$$

where u is an arbitrary Fuller class on (π, C) over Q, and the fixed point homomorphism σ is determined by the standard orientation in (2.4), *is a periodic orbit index.*

Proof. We must prove that $\sigma_{(\pi, C)}(u)$ does not depend on the choice of a Fuller class u. Let u_1 and u_2 be two Fuller classes on (π, C). By the tautness of the Čech cohomology, there exist open neighbourhoods U_1 and U_2 of C and classes $v_i \in \check{H}^1(U_i)$ such that $u_i = v_i|_C$. If (π', C') is a flow-germ from (8.1 b) sufficiently close to (π, C), by the same argument as in the proof of (7.3) we conclude that

$$\sigma_{(\pi, C)}(u_i) = \sigma_{(\pi', C')}(v_i|_{C'})$$

for $i = 1, 2$. By Proposition (7.2), $v_i|_{C'}$ can be regarded as Fuller classes. Since such a class on (π', C') is unique, $v_1|_{C'} = v_2|_{C'}$. Thus we have shown that $\sigma_{(\pi, C)}(u_1) = \sigma_{(\pi, C)}(u_2)$. The additivity and the homotopy invariance of $\jmath$ follow from (1.4) and (1.6), so the result is proved.

(8.3) Remark. If π is a smooth flow on a manifold X and $(\pi, C) \in \Pi^F(X)$, then $\jmath(\pi, C) = i^F(\pi, C)$ — which is stated in Corollary (7.4).

(8.4) Remark. The index $\jmath$ is an extension of the index constructed in [Fe1] in the finite-dimensional case.

Now we compute $\jmath$ in a simple situation.

(8.5) Proposition. *Assume that Y is a compact polytope, and $X = S^1 \times Y$. Then for the flow*

$$\pi\colon R \times X \ni (t, (s, y)) \to (\exp(2\pi i t)s, y) \in X$$

we have $\jmath(\pi, \{1\} \times X) = \chi(Y)$, where χ is the Euler characteristic.

Proof. Obviously, there exists a Fuller class on $\{1\} \times X$, but such a class need not be unique. We prove that π is approximable by flows with the discrete set of periodic orbits. Let Δ_q be the standard q-dimensional simplex (see (2.4)). One can construct a flow ϱ^q on Δ_q such that all the simplexes of the first barycentric subdivision of Δ_q are invariant with respect to ϱ^q (thus all the vertices are stationary), and inside any simplex of this subdivision ϱ^q "flows" into the direction of the vertex which has the greatest number of coordinates equal to zero. Moreover, ϱ^q can be constructed in such a way that if $\Delta_r \to \Delta_q$ is the canonical face operator, then the image of ϱ^r under this operator coincides with the restriction of ϱ^q. By the cartesian multiplication of the flow

$$\mathbf{R}\times S^1 \ni (t, s) \to \exp(2\pi i t)s \in S^1$$

by ϱ^q we obtain a flow on $S^1 \times \Delta_q$. Choose an $\varepsilon > 0$. We can find a triangulation of Y such that any of its simplexes has the diameter less than ε. Glueing together the flows obtained in the above way for each simplex of this triangulation, we obtain a flow on X ε-close to π and having isolated periodic orbits.

For convenience, assume that $Y \subseteq \mathbf{R}^n$. We can find an embedding $e\colon S^1 \to \mathbf{R}^2$ such that $e(\exp(2\pi i t)) = (t, 0)$ for $t \in (-\frac{1}{4}, \frac{1}{4})$, and a neighbourhood retraction $s\colon V \to e(S^1)$ such that $s(x, y) = (x, 0)$ if (x, y) is sufficiently close to $(-\frac{1}{4}, \frac{1}{4}) \times 0$. Let $r\colon W \to Y$ be also a neighbourhood retraction, where W is open in $\mathbf{R}^n$. Put $i\colon Y \hookrightarrow W$. In order to find $\jmath(\pi, \{1\}\times X)$, it suffices to consider the following mapping:

$$\mathbf{R}_+ \times V \times W \ni (t, (x, y), w) \to \big((x, y), w\big) - \Big(e\big(\pi(t, e^{-1}(s(x, y)), r(w)\big)\Big) \in \mathbf{R}^2 \times \mathbf{R}^n .$$

In a certain neighbourhood of $\{1\}\times\{0\}\times Y$ this mapping is of the form:

$$(t, (x, y), w) \to ((1-t, y), w - r(w)) .$$

By similar arguments as in the proof of (6.1), the above fact implies that $\jmath(\pi, \{1\}\times X)$ is equal to the fixed point index of $i \circ r\colon W \to W$. By the Lefschetz Fixed Point Theorem this index is equal to $\chi(Y)$, so the result is proved.

In the following corollary to this proposition we consider also mappings which need not be local semiflows.

(8.6) Corollary. *Let $X = S^1 \times Y$ and π be as above. Assume that $\chi(Y) \neq 0$. Then for any $\varepsilon > 0$ there exists a neighbourhood N of π in the space of all continuous mappings $\mathbf{R}\times X \to X$ with the compact-open topology such that for any $f \in N$ there exists $(t, x) \in (1-\varepsilon, 1+\varepsilon)\times X$ for which $f(t, x) = x$.*

Proof. By the definition of $\jmath$ and (8.5) we have $\sigma_{(\pi, 1\times X)}(u) \neq 0$, where σ is the fixed point homomorphism and $u \in \check{H}^1(1\times X)$ is a Fuller class. By the tautness of the Čech cohomology we can extend u to a certain neighbourhood of $1 \times X$. Now the result follows easily from Proposition (1.7).

9. Fuller's construction. The index $\jmath$ presented in the previous section cannot be regarded as an extension of the Fuller index, since it is not defined for all flows. In order to find a proper extension we will adapt to our considerations some arguments used in the paper [Fu].

Let X be a Hausdorff space, and let p be a given positive integer. Denote

$$\tilde{X} = \{(x_1, \ldots, x_p) \in X^p\colon x_i \neq x_j \text{ for } i \neq j\} .$$

The group $\mathbf{Z}_p$ acts freely on $\tilde{X}$; the corresponding action is generated by

$$(1, (x_1, \ldots, x_p)) \to (x_2, \ldots, x_p, x_1) .$$

Define $X_p = \tilde{X}/\mathbf{Z}_p$. The quotient mapping $\tilde{X} \to X_p$ is a p-sheet covering; the image of $(x_1, \ldots, x_p)$ under this mapping is denoted by $[x_1, \ldots, x_p]$.

Assume that $\pi\colon D \to X$ is a local semiflow on X. Let $\tilde{D} \subseteq \mathbf{R}_+ \times \tilde{X}$ be the set

$$\{(t, x_1, \ldots, x_p)\colon (t, x_i) \in D \text{ and } \forall \tau \in [0, t]: (\pi(\tau, x_1), \ldots, \pi(\tau, x_p)) \in \tilde{X}\} .$$

Since $\tilde{D}$ is open and invariant with respect to the relation

$$(s, (x_1, \ldots, x_p)) \sim (t, (y_1, \ldots, y_p)) \text{ iff } s = t \text{ and there exists } q \in \mathbf{Z}_p\colon$$
$$(y_1, \ldots, y_p) = q(x_1, \ldots, x_p) ,$$

the set $\tilde{D}/_\sim$ is open in $\mathbf{R}_+ \times \tilde{X}/_\sim$, and by (0.4) in [Sw] the set

$$D_p = \{(t, [x_1, \ldots, x_p]) \in \mathbf{R}_+ \times X_p\colon (t, (x_1, \ldots, x_p)) \in \tilde{D}\}$$

is also open. The mapping

$$\pi_p\colon D_p \ni (t, [x_1, \ldots, x_p]) \to [\pi(t, x_1), \ldots, \pi(t, x_p)] \in X_p$$

is a local semiflow on X_p.

Consider the continuous mapping

$$\Phi_p\colon E \ni (t, x) \to \left(\frac{t}{p}, \left[x, \pi\left(\frac{1}{p}t, x\right), \ldots, \pi\left(\frac{p-1}{p}t, x\right)\right]\right) \in D_p$$

where E is the maximal subset of D such that for any $(t, x) \in E$ the element on the right is defined. Analogously as in the smooth case (see [Fu], Lemma 4.3), we have

(9.1) Lemma. *Let p be a prime number. Assume that C is a compact isolated set of periodic orbits for π. If $p > m_C$ (where m_C is the maximum of the set of multiplicities M_C), then $C \subseteq E$ and $\Phi_p(C)$ is a compact isolated set of periodic orbits for π_p. In this case the transformation $\Gamma \to \Phi_p(\Gamma)$ is a bijection between orbits of the germs (π, C) and $(\pi_p, \Phi_p(C))$; moreover, the multiplicities of Γ and $\Phi_p(\Gamma)$ are equal. If $(\pi^0, C^0) \simeq (\pi^1, C^1)$ and (π^α, C^α), $\alpha \in I$, is a homotopy such that $p > m_{C^\alpha}$ for any α, then*

$$(\pi_p^0, \Phi_p(C^0)) \simeq (\pi_p^1, \Phi_p(C^1)) .$$

Now we present some (co-)homological properties of this construction. As at the beginning of this section, we assume that p is a positive integer, and X_p is attached to X and p. If Y is a path component of X_p and $y_0 \in Y$, then we have a natural homomorphism

$$\pi(Y, y_0) \ni [\omega] \to [\sigma_\omega] \in H_1(Y; \mathbf{Z}) ,$$

where π denotes the fundamental group functor, $\omega\colon [0, 1] \to Y$ is a loop in y_0, $[\omega]$ is the class of ω and

$$\sigma_\omega\colon \Delta_1 \ni (1-t)e_0 + te_1 \to \omega(t) \in Y$$

is the singular simplex determined by ω. The monodromy homomorphism $\pi(Y, y_0)\to \boldsymbol{Z}_p$ induces a homomorphism $H_1(Y; \boldsymbol{Z})\to \boldsymbol{Z}_p$ such that the following diagram commutes

$$\begin{tikzcd} \pi(Y, y_0) \arrow[d] \arrow[dr] & \\ H_1(Y; \boldsymbol{Z}) \arrow[r] & \boldsymbol{Z}_p \end{tikzcd}$$

As the latter homomorphism does not depend on the choice of y_0, then we have another homomorphism

$$\Omega\colon H_1(X_p; \boldsymbol{Z}) = \bigoplus_Y H_1(Y; \boldsymbol{Z})\to \boldsymbol{Z}_p .$$

The Universal Coefficients Theorem implies that there exists a unique element $v_p\in H^1(X_p; \boldsymbol{Z}_p)$ such that for any $\xi\in H_1(X_p; \boldsymbol{Z})$

$$\Omega(\xi) = \langle v_p, \xi\rangle \in \boldsymbol{Z}_p .$$

Assume now that X is an ANR. By the Hanner's Theorems X_p is also an ANR. The singular cohomology class v_p defined above induces naturally a Čech cohomology class $u_p\in \check{H}^1(\boldsymbol{R}_+\times X_p; \boldsymbol{Z}_p)$. We have the following result:

(9.2) Theorem. *Let p be a prime number. If (π, C) is a flow-germ on X such that $p > m_C$, then $u_p|_{\Phi_p(C)}$ is a Fuller class for the germ $(\pi_p, \Phi_p(C))$ over $\boldsymbol{Z}_p$.*

Proof. By (9.1), any orbit in $\Phi_p(C)$ is of the form $\Phi_p(\Gamma)$ for some orbit $\Gamma\subseteq C$. For convenience, we will write Γ_p instead of $\Phi_p(\Gamma)$. Let Γ be an orbit in C, let $(\omega, x)\in\Gamma$ and let m be the multiplicity of Γ. The underlying orbit of Γ_p is contained in some path component Y of X_p, and the monodromy homomorphism maps the loop

$$\left[0, \frac{\omega}{p}\right]\ni t\to \pi_p\left(t, \left[x, \pi\left(\frac{1}{p}\omega, x\right), \ldots, \pi\left(\frac{p-1}{p}\omega, x\right)\right]\right)\in Y$$

onto $1\in \boldsymbol{Z}_p$. The multiplicities of Γ and Γ_p are equal by (9.1). Hence if ω_x is the minimal period of x, then the loop $\left[0, \frac{\omega_x}{p}\right]\to Y$ (defined as the restriction of the previous one) is mapped onto $[m]^{-1}\in \boldsymbol{Z}_p$. As the number $\frac{\omega_x}{p}$ is the minimal period of points of the underlying orbit of Γ_p, the latter loop defines the standard generator $\zeta_{\Gamma_p}\in H_1(\Gamma_p; \boldsymbol{Z})$ (see Proposition (5.2)). We have

$$\langle u_p|_{\Gamma_p}, \zeta_{\Gamma_p}\rangle = [m]^{-1} = \langle [m]^{-1} z_{\Gamma_p}, \zeta_{\Gamma_p}\rangle ,$$

which implies that

$$u_p|_{\Gamma_p} = [m]^{-1} z_{\Gamma_p} ,$$

so the result is proved.

10. The indices j_q and j. In this section we assume that X is an ENR-space. Since ENR-space coincide with finite-dimensional, separable and locally compact ANR's (see [Do1]), the results of the previous section imply that the space X_p is also ENR for any p.

Let q be a given positive integer. Denote by $\Pi_q(X)$ the set of all flow-germs on X such that $(\pi, C)\in \Pi_q(X)$ iff $m_C < q$. We have the following result:

(10.1) Proposition. *The function*

$$j_q\colon \Pi_q(X)\ni(\pi, C)\to \{\sigma_{(\pi_p, \Phi_p(C))}(u_p|_{\Phi_p(C)})\}_p\in \prod_{p\ \mathrm{prime}\geq q} \boldsymbol{Z}_p$$

(where we use notation introduced in Section 9 *and the fixed point homomorphisms are determined by o from* (2.4)) *is a periodic orbits index.*

Proof. This follows immediately from (2.5) and (9.1).

Now we introduce the main object of this note. We have the canonical mapping

$$\delta\colon \prod_{p\ \mathrm{prime}\geq q} \boldsymbol{Z}_p\to \prod_{p\ \mathrm{prime}} \boldsymbol{Z}_p/\bigoplus_{p\ \mathrm{prime}} \boldsymbol{Z}_p .$$

Since for any homotopy (π^α, C^α), $\alpha\in I$, of flow-germs on X, the set of its multiplicities $\bigcup_{\alpha\in I} M_{C^\alpha}$ is finite, (10.1) implies

(10.2) Theorem. *The function*

$$j\colon \Pi(X)\ni(\pi, C)\to\delta\big(j_q(\pi, C)\big)\in \prod_{p\ \mathrm{prime}} \boldsymbol{Z}_p/\bigoplus_{p\ \mathrm{prime}} \boldsymbol{Z}_p$$

where q is a sufficiently large number depending on (π, C), is the periodic orbits index defined on the set $\Pi(X)$ of all flow-germs on X.

Now we explain why the index j may regarded as an extension of the Fuller index. To this end we need the following result.

(10.3) Theorem. *If π is a smooth flow on a manifold, (π, C) is a flow-germ, n is a common multiplicity of M_C and i^F denotes the Fuller index, then*

$$j(\pi, C) = \delta(\{[n]_p^{-1}[n i^F(\pi, C)]_p\}_p)$$

where p is prime, $p > n$ and $[\cdot]_p$ is the mod$p$ class.

Proof. Since the fundamental result of [Fu] states that for any $p > m_C$

$$i^F(\pi, C) = i^F(\pi_p, \Phi_p(C))$$

(see [Fu], Lemma 4.5), the assertion follows easily from Theorems (7.3), (9.2) and the definition of j.

(10.4) Corollary. *For any germ (π, C) generated by a smooth flow π we have $j(\pi, C) = 0$ if and only if $i^F(\pi, C) = 0$.*

(10.5) Example. Let a compact manifold (without boundary) X be the total space of an orientable smooth S^1-bundle. This structure induces a S^1-action on X, and, consequently, a smooth flow on X such that any point is periodic with the minimal period equal to 1. The same argument as in [Fu], p. 142–143 shows that $i^F(\pi, \{1\}\times X) = \chi(X/S^1)$, where the right side denotes the Euler characteristic of the orbit space of the S^1-action. In particular, if this characteristic is non-zero, then (10.4) implies that $j(\pi, \{1\}\times X) \neq 0$. This fact induces the following generalization of the Seifert Theorem (see [Se]): For any germ (ϱ, C), where ϱ is a (continuous) local semiflow on X, such that (ϱ, C) is homotopical to $(\pi, \{1\}\times X)$, the set C is nonempty. It is interesting to know whether in this statement the term "local semiflow" can be replaced by the term "continuous mapping" as in (8.6).

(10.6) Remark. For spaces with the nonvanishing first cohomology, the composition $(j, \{i_u\}_u)$, where the indices i_u are defined as in (2.5), has better properties than j alone.

Finishing the paper we present an idea how to extend the index j onto arbitrary ANR's. We assume that the considered local semiflow π is compact, i.e. for any compact set K contained in the domain of π there exists a neighbourhood U of K such that $\pi(U)$ is relatively compact. Let (π, C) be a flow-germ. Since the local semiflows π_p are also compact and defined on ANR-spaces, it suffices to apply the argument presented above using an extended version of the fixed point homomorphism from Section 1. In order to obtain this version, one must introduce a generalization of the fixed point transfer in the spirit of the paper [G]. Some results in this direction were obtained by J. Jezierski [Je] and H. Ulrich [U].

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Added in proof. An extension of the transfer to ANR's is presented in the paper:

B. Schäfer, *Fixpunkttransfer für stetige Familien von* ANR-*Räumen*, Dissertation, Heidelberg 1981.

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