Assume the $\{a_i^\alpha: i<n\}$ are disjoint outside a root (that is, assume that there is a finite set Δ such that $\{a_i^\alpha - \Delta: i<n\}$ is a disjoint family) be applying the delta-system lemma for finite sets and reindexing. We can ignore the root since it is finite. We can assume that n is fixed and x is fixed and then we can apply Lemma 3 to find a closed unbounded set C, a string σ, a behavior f and an uncountable $A \subset \omega_1$. Let $\alpha \in C \cap S(\sigma, f, x)$ be such that there is an infinite $Y \subset A$ such that $\{a_0^\alpha: \alpha \in Y\}$ increases to α (this requires intersection with another closed unbounded set). Now $\{a_0^\alpha: \alpha \in Y\}$ has code σ and behavior f hereditarily which implies that $\{a_0^\alpha: \alpha \in Y\} \in F^*(\alpha)$. This means that $\{a_i^\alpha: \alpha \in Y\} = \{a_0^\alpha: \alpha \in Y\}(\sigma \restriction i) \in F_{x(i)}$. By closure under finite unions $\{a_i^\alpha: \alpha \in Y, i \in x^{-1}(j)\} \in F_j$ and so $p^{-1}(1) \cap A_j \in \Pi_j(f)$ for some $f \in F_j$. By the definition of X, $\{[p_\alpha]: \alpha \in Y\}$ has nonempty intersection with X.

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Slender modules, endo-slender abelian groups and large cardinals

by

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Dedicated to Professor Hiroyuki Tachikawa on his 60-th birthday

Abstract. We prove the following theorems introducing some new notions.

THEOREM A. *The following* (1)–(3) *are equivalent*:

(1) *For an arbirary infinite cardinal* μ, *there exists an* $L_{\mu\omega}$-*compact cardinal*;

(2) *For an arbitrary ring* R *and module* M_R, *there exists a cardinal* $\varkappa$ *such that* $R_M = R_M^{[\varkappa]}$, *where* $R_M A = \bigcap\{\mathrm{Ker}(h): h \in \mathrm{Hom}_R(A, M)\}$ *and* $R_M^{[\varkappa]} A = \sum\{R_M X_R: X_R \leqslant A_R$ *and* $|X| < \varkappa\}$;

(3) *For an arbitrary ring* R *and module* M_R, *the torsion class* ${}^\perp M_R$ *is singly generated, i.e.*, ${}^\perp M_R = {}^\perp(A_R^\perp)$ *for some* A_R.

An abelian group A is endo-slender, if A is a slender module over its endomorphism ring.

THEOREM B. *Let* $A(=\Pi_{i\in I} A_i)$ *be a direct product of reduced torsion free groups* A_i *of rank* 1. *Then*, A *is endo-slender iff for any infinite subset* X *of* I *there exists a* $j \in I$ *such that*

$$\{i \in X: t(A_i) \leqslant t(A_j)\}$$

is infinite.

THEOREM C. *Let* B *be an* (ω, ∞)-*distributive complete Boolean algebra and* A *a countable reduced torsionfree abelian group. Then, the Boolean power* $A^{(B)}$ *is endo-slender iff* B *satisfies* $(*\omega_1)$, *i.e. For any family of nonzero elements* $\{b_n: n<\omega\}$ *there exist a nonzero* b, *an infinite subset* I *of* ω *and* h_n $(n \in I)$ *such that* $h_n: [0, b_n] \to [0, b]$ *is a countably complete homomorphism and* $h_n(b_n) = b$ *for each* $n \in I$.

0. Introduction. There had been several studies about slender modules and rings as generalizations of slender abelian groups even before the works of Huber [23] and Mader [26]. However, they found a somewhat new situation, where slenderness occurs through consideration of abelian groups and modules as modules over their endomorphism rings. On the other hand, the fundamental theorem about slender groups due to J. Łoś was generalized to arbitrary cardinalities by the author [8, 10]. This clarified why a measurable cardinal appears concerning abelian groups. Though there possibly exist many measurable cardinals [24], the only least measurable cardinal has concerned slender modules. Considering rings and endomorphism rings of large cardinalities, we shall show that larger measurable cardinals concern slender modules in Section 3. (Caution: Here, $\varkappa$ is a measurable cardinal, if there exists a $\varkappa$-complete nonprincipal ultrafilter on $\varkappa$. This usage is common in set theory, but in abelian group theory or in General Topology it does not seem so.) To investigate endo-slenderness, we shall introduce a new notion "primitive slenderness". Though primitive slenderness is strictly stronger than slenderness, many slender modules are primitively slender. In Section 4 we shall study endo-slenderness of infinite direct products of endo-slender modules and apply it to abelian groups, and especially to direct products of torsion free groups of rank 1. In Section 5 we shall study about endo-slenderness of Boolean powers of modules. There, we use Boolean valued models $V^{(B)}$ as in [7] and so we assume that the reader is familiar with Boolean valued models [2, 30]. Since the topic and the methods of this paper are related to a few different areas of mathematics, we shall use usual notions in each area and give definitions as far as possible.

1. Preliminaries; $\varkappa$-slender modules and R-$\varkappa$-sheaves. In this paper a ring R is an associative ring with 1 and all modules are unital R-modules. Right R-modules are written as M_R and left modules appear as modules over their endomorphism rings and are written as ${}_EM$. Sometimes we omit the subscripts R and E. Since right and left properties are defined symmetrically, we define just one sided ones. In case no confusion will occur, we do not mention right or left as usual.

Definition 1.1. A right module M_R is *$\varkappa$-slender*, if $\mathrm{Hom}_R(R^I, M_R) \simeq \oplus_I M_R$ naturally for an index set I of cardinality less than $\varkappa$. More precisely, for any $h \in \mathrm{Hom}_R(R^I, M_R)$ there exist $i_0, \ldots, i_{n-1} \in I$ and $a_0, \ldots, a_{n-1} \in M_R$ such that $h(x) = \sum_{k=0}^{n-1} a_k \pi_{i_k}(x)$ where π_i is the projection to the ith component.

The usual slenderness is the ω_1-slenderness according to this definition and so we use the word "slender" but never use "ω_1-slender", where ω_1 is the least uncountable ordinal. A standard method shows that M_R is slender if for any $h \in \mathrm{Hom}_R(R^\omega, M_R)$ $h(e_n) = 0$ for almost all $n < \omega$, where $e_m(n) = \delta_{mn}$ and ω is the least infinite ordinal. For a right module M_R, let $E = \mathrm{End}_R(M_R)$, then M naturally becomes a left E-module ${}_EM$. M_R is $\varkappa$-endo-slender if the left E-module ${}_EM$ is $\varkappa$-slender. A ring R is right slender if R is slender as a right module. R is $\varkappa$-slender if R is right and left $\varkappa$-slender. The well-known theorem due to Łoś is the following:

Theorem 1.2 (J. Łoś, cf. [25]). *Let I be an index set of cardinality less than the least measurable cardinal and A_i $(i \in I)$ R-modules. If M_R is slender, then $\mathrm{Hom}_R(\prod_{i\in I} A_i, M_R) \simeq \oplus_{i\in I} \mathrm{Hom}_R(A_i, M_R)$ naturally.*

We shall prove a generalized form of this theorem in this section. For this purpose we introduce an *R-$\varkappa$-sheaf* over a $\varkappa$-complete Boolean algebra. A *quasi sheaf* over a countably complete Boolean algebra in [10] is a $\mathbf{Z}$-ω_1-sheaf over an ω_1-complete Boolean algebra according to the present definition. Therefore this

generalization is essentially same as that of [10]. However, neither in [10] nor in [8] we did not explain the intuition behind the generalization and so let us do it here.

$$\begin{array}{ccccc} \text{Point} & — & \text{Component} & — & \text{Projection}\\ \downarrow & & \downarrow & & \downarrow \\ \varkappa\text{-complete} & — & \text{Ultraproduct} & — & \text{Canonical}\\ \text{ultrafilter} & & & & \text{homomorphism} \end{array}$$

In the figure, "Point" means an element of an index set I, "Component" means A_i and "Projection" means π_i. The conversion from "Point" to "ω_1-complete ultrafilter" is fairly known in General Topology concerning the Stone–Čech compacification. The remaining parts may be somewhat unfamiliar. However, famous Hewitt's theorem [22, or 20, Chapter 8] "Any ring homomorphism from a continuous function ring $C(X, \mathbf{R})$ to the real number field $\mathbf{R}$ corresponds to a point of the realcompactification of X" implicitly uses this conversion. Let p be an element of the realcompactification of X and $I_p = \{f \in C(X, \mathbf{R})\colon \{x\colon f(x) = 0\} \in p\}$. (Recall that p is a maximal filter of zero sets.) Then, I_p is an ideal of the ring $C(X, \mathbf{R})$ and $C(X, \mathbf{R})/I_p$ is isomorphic to $\mathbf{R}$. Since p is maximal, this quotient can be seen as a generalization of ultraproducts [4, Chapter 4]. Therefore, our next theorem can be seen as a kind of Hewitt's theorem and under this point of view we can connect Hewitt's theorem and Łoś's theorem. (See also [13].) Before defining R-$\varkappa$-sheaves, we state some prerequisites about Boolean algebras.

A *Boolean algebra* (abbreviated by Ba) B is a complemented distributive lattice with its largest element 1 and least element 0, where its order and operations are denoted by $\leqslant, \vee, \wedge, \neg$ as usual [24, 29]. If $0 = 1$, B is called *trivial*. B is *$\varkappa$-complete* for a cardinal $\varkappa$, if for any subset X of B of cardinality less than $\varkappa$ there exists a least upper bound $\bigvee X$. B is a complete Boolean algebra (abbreviated by cBa), if B is $\varkappa$-complete for any cardinal $\varkappa$. A family $\{b_\lambda\colon \lambda \in \Lambda\}$ is a partition of b, if $b_\lambda \vee b_{\lambda'} = 0$ for $\lambda \neq \lambda'$ and $\bigvee_{\lambda\in\Lambda} b_\lambda = b$. A *filter* F of B is a subset of B which satisfies the following: $1 \in F$; $b, c \in F$ implies $b \wedge c \in F$; $b \in F$ and $b \leqslant c$ imply $c \in F$. An *ultrafilter* is a filter which is maximal among filters not containing 0. A filter is $\varkappa$-complete, if the following holds: For any $X \subset F$ of cardinality less than $\varkappa$, $b = \bigwedge X$ implies $b \in F$.

A cardinal $\varkappa$ is measurable, if there exists a non principal $\varkappa$-complete ultrafilter on $\varkappa$. For a cardinal $\varkappa$, $\varkappa^+$ is the successor cardinal of $\varkappa$ and $\varkappa^{+\mathrm{MC}}$ is the least measurable cardinal which is strictly greater than $\varkappa$. We identify a cardinal with an initial ordinal. A cardinal $\varkappa$ is regular, if $\sup(X) < \varkappa$ holds for any subset X of $\varkappa$ of cardinality less than $\varkappa$. Otherwise, $\varkappa$ is called singular. Throughout this paper, $\varkappa$ is always an infinite cardinal. The next lemma is well known and the proof of the case $\varkappa = \omega$ can be found in [7].

Lemma 1.3 (folklore). *Every $\varkappa^+$-complete ultrafilter F of a μ-complete Ba B is μ-complete, where $\mu \leqslant \varkappa^{+\mathrm{MC}}$.*

Definition 1.4. Let B be a $\varkappa$-complete Ba. An *R-$\varkappa$-sheaf over B* is a pair $(\mathscr{S}, \varrho)$ satisfying the following:

(1) $\mathscr{S}(b)$ is an R-module for every $b \in B$ and $\mathscr{S}(0) = \{0\}$;

(2) ϱ_b^c: $\mathscr{S}(c) \to \mathscr{S}(b)$ is an R-homomorphism, $\varrho_b^c \cdot \varrho_c^d = \varrho_b^d$ and ϱ_b^b the identity for $b \leqslant c \leqslant d$.

(3) If $b = \bigvee_{\alpha<\lambda} b_\alpha$ $(\lambda < \varkappa)$ and $\varrho_{b_\alpha}^b(x) = 0$ for $\alpha < \lambda$, then $x = 0$.

(4) If $b = \bigvee_{\alpha<\lambda} b_\alpha$ $(\lambda<\varkappa)$ and $\varrho_{b_\alpha \wedge b_\beta}^{b_\alpha}(x_\alpha) = \varrho_{b_\alpha \wedge b_\beta}^{b_\beta}(x_\beta)$ for $\alpha, \beta < \lambda$, then there exists an $x \in \mathscr{S}(b)$ such that $\varrho_{b_\alpha}^b(x) = x_\alpha$ for every $\alpha < \lambda$. We denote $\mathscr{S}(1)$ by $\mathscr{S}^\wedge$. If B is a cBa and $(\mathscr{S}, \varrho)$ is an R-$\varkappa$-sheaf over B for any $\varkappa$, we say $(\mathscr{S}, \varrho)$ is an R-sheaf over B.

Let $(\mathscr{S}, \varrho)$ be an R-$\varkappa$-sheaf over a $\varkappa$-complete Ba B and F a $\varkappa$-complete filter of B. Then, the R-$\varkappa$-sheaf $(\mathscr{S}/F, \varrho/F)$ over a Ba B/F is defined by: $\mathscr{S}/F([b]) = \mathscr{S}(b)/\{x \in \mathscr{S}(b)\colon \varrho_c^b(x) = 0$ for some $c \in F\}$; $\varrho_{[c]}^{[b]}([x]_F^b) = [\varrho_c^b(x)]_F^c$, where $[\ ]_F$: $B \to B/F$ and $[\ ]_F^b$: $\mathscr{S}(b) \to \mathscr{S}/F([b])$ are the canonical maps. It is straightforward to see that $\mathscr{S}/F$ is well defined. If F is an ultrafilter, $B/F \simeq \{0, 1\}$ and so we denote $\mathscr{S}/F(1)$ by $\mathscr{S}/F$.

Let $(\mathscr{S}, \varrho)$ be an R-$\varkappa$-sheaf over a $\varkappa$-complete Ba B. If $\{b_\alpha\colon \alpha < \lambda\}$ is a partition of 1 and $s_\beta \in \mathscr{S}^\wedge$ $(\beta < \mu)$ satisfy $\varrho_{b_\alpha}^1(s_\beta) = 0$ for almost all β and each α, where $\lambda, \mu < \varkappa$, then there exists a unique $s \in \mathscr{S}^\wedge$ such that $\varrho_{b_\alpha}^1(s) = \sum_{\beta<\mu} \varrho_{b_\alpha}^1(s_\beta)$. We denote this s by $\sum_{\beta<\mu} s_\beta$ and say that s_β $(\beta<\mu)$ is a proper sequence.

THEOREM 1.5 (generalized Łoś theorem). *Let B be a μ-complete* Ba *and M_R a $\varkappa$-slender R-module, where $\omega < \varkappa \leqslant \mu \leqslant \varkappa^{+\mathrm{MC}}$. For an R-μ-sheaf $(\mathscr{S}, \varrho)$ over B, $\mathrm{Hom}_R(\mathscr{S}^\wedge, M_R) = \bigoplus_{F \in \mathscr{F}} \mathrm{Hom}_R(\mathscr{S}/F, M_R)$ naturally, where $\mathscr{F}$ is the set of all μ-complete ultrafilters of B.*

Proof. Since the proof is essentially the same as that of [10, Theorem 1], we only take care of μ-completeness. Clearly $\bigoplus_{F\in\mathscr{F}} \mathrm{Hom}_R(\mathscr{S}/F, M_R) \leqslant \mathrm{Hom}_R(\mathscr{S}^\wedge, M_R)$ and hence we show the other inclusion. Let h: $\mathscr{S}^\wedge \to M_R$ be a nonzero R-homomorphism and $F_h = \{b\colon \varrho_b^1(x) = 0$ implies $h(x) = 0$ for any $x \in \mathscr{S}^\wedge\}$. Then, F_h is a filter of B and $0 \notin F$. To show the $\varkappa$-completeness of F_h, suppose that $b_\alpha \in F_h$ $(\alpha < \lambda)$, $\lambda < \varkappa$, $\bigwedge_{\alpha<\lambda} b_\alpha \notin F_h$ and λ is the least cardinal among such cardinals. Then, there exists a pairwise disjoint family $\{c_\alpha\colon \alpha<\lambda\}$ such that $\neg c_\alpha \in F_h$ and $\bigvee_{\alpha<\lambda} c_\alpha (= c) \in F_h$. Since $h \neq 0$, there exists an $x^* \in \mathscr{S}^\wedge$ such that $\varrho_c^1(x^*) = 0$ and $h(x^*) \neq 0$. Let φ: $R^\lambda \to \mathscr{S}^\wedge$ be the map which is naturally defined by: $\varrho_{c_\alpha}^1(\varphi(e_\alpha)) = \varrho_{c_\alpha}^1(x^*)$ and $\varrho_{\neg c_\alpha}^1(\varphi(e_\alpha)) = 0$ for each $\alpha < \lambda$, where $e_\alpha(\beta) = \delta_{\alpha\beta}$. Then, φ is an R-homomorphism and $h \cdot \varphi(e_\alpha) = 0$ and hence $h \cdot \varphi = 0$ by the $\varkappa$-slenderness of M_R, which contradicts to $h(x^*) = h \cdot \varphi(\sum_{\alpha<\lambda} e_\alpha) \neq 0$.

Suppose that B/F is infinite, then there exist a partition $\{b_n\colon n < \omega\}$ and $x_n \in \mathscr{S}^\wedge$ $(n < \omega)$ such that $\varrho_{\neg b_n}^1(x_n) = 0$ and $h(x_n) = 0$. Again, define an R-homomorphism φ: $R^\omega \to \mathscr{S}^\wedge$ such that $\varphi(e_n) = x_n$ $(n<\omega)$ and get a contradiction. Therefore, there exist finite distinct $\varkappa$-complete ultrafilters $F_0, \ldots, F_{n-1}$ which generate F, i.e. $b \in F$ iff $b \in F_i$ for some $1 \leqslant i \leqslant n-1$. By Lemma 1.3, $F_i (0 \leqslant i \leqslant n-1)$ are μ-complete. Let $b_i \in F_i$ $(0 \leqslant i \leqslant n-1)$, $b_i \wedge b_j \neq 0$ $(i \neq j)$ and $\bigvee_{i=0}^{n-1} b_i = 1$. Since $b \in F_i$ iff $b \in F$ for $b \leqslant b_i$, there exist R-homomorphisms h_i: $\mathscr{S}/F_i \to M_R$ $(0 \leqslant i \leqslant n-1)$ such that $h(x) = h_i([x]_{F_i}^1)$ if $\varrho_{\neg b_i}^1(x) = 0$. Hence, $h = \sum_{i=0}^{n-1} h_i \cdot [\ ]_{F_i}^1$.

COROLLARY 1.6. *If M_R is $\varkappa^+$-slender, then M_R is $\varkappa^{+\mathrm{MC}}$-slender. If M_R is $\varkappa$-slender and $\varkappa$ is a singular cardinal, then M_R is $\varkappa^+$-slender.*

Proof. Let $\varkappa^+ \leqslant \mu < \varkappa^{+\mathrm{MC}}$. The power set $\mathrm{P}(\mu)$ is a cBa and R^μ has a sheaf structure over $\mathrm{P}(\mu)$ naturally. Every μ-complete ultrafilter of a cBa is $\varkappa^{+\mathrm{MC}}$-complete and the cardinality of $\mathrm{P}(\mu)$ is less than $\varkappa^{+\mathrm{MC}}$. Therefore, every $\varkappa^{+\mathrm{MC}}$-complete ultrafilter of $\mathrm{P}(\mu)$ is principal. By Theorem 1.5, $\mathrm{Hom}_R(R^\mu, M_R) = \bigoplus_\mu M_R$ naturally and hence M_R is $\varkappa^{+\mathrm{MC}}$-slender. To prove the next proposition, let $\mu < \varkappa$ and $\{I_\alpha\ \alpha<\mu\}$ a partition of $\varkappa$ so that $|I_\alpha| < \varkappa$. Then, $R^\varkappa = \prod_{\alpha<\mu} R^{I_\alpha}$. By Theorem 1.5 and $\varkappa$-slenderness $\mathrm{Hom}_R(R^\varkappa, M_R) = \bigoplus_{\alpha<\mu} \mathrm{Hom}_R(R^{I_\alpha}, M_R) = \bigoplus_{\alpha<\mu} \bigoplus_{\beta \in I_\alpha} M_R = \bigoplus_\varkappa M_R$.

EXAMPLE 1.7. Let B be a $\varkappa$-complete Ba and M_R and R-module of cardinality less than $\varkappa$. The Boolean power $M^{(B)}$ is the R-module consisting of all f such that f: $M \to B$, $\bigvee_{u \in M} f(u) = 1$ and $f(u) \wedge f(v) = 0$ for $u \neq v$. The operations are defined as follows: $(f+g)(u) = \bigvee_{u=v+w} f(v) \wedge g(w)$: $(f \cdot r)(u) = \bigvee_{u=vr} f(v)$ for $u, v \in M$ and $r \in R$. Let $\mathscr{S}(b) = \{f \in M^{(B)}\colon \neg b \leqslant f(0)\}$, $\varrho_c^b(f)(u) = f(u) \wedge c$ and $\varrho_c^b(f)(0) \geqslant \neg c$. Then, $(\mathscr{S}, \varrho)$ is an R-$\varkappa$-sheaf over B. Another example will appear in Section 5.

2. Primitively slender modules and related concepts. First we define $\varkappa$-approximation property and primitively slender modules.

DEFINITION 2.1. A right R-module M_R *satisfies $\varkappa$-approximation property*, if there exist finitely generated left ideals I_α $(\alpha < \varkappa)$ of R and subsets K_α $(\alpha<\varkappa)$ of M_R which satisfy the following:

(1) $M_R = \bigcup_{\alpha<\varkappa} K_\alpha$, $K_\alpha = -K_\alpha$, $\{0\} \subset K_\alpha \subset K_\beta$, $I_\beta \subset I_\alpha$ and $M_R \cdot I_\alpha \cap K_\alpha = \{0\}$ $(\alpha \leqslant \beta < \varkappa)$;

(2) For each $\alpha < \varkappa$ and $x \in M_R$ there exists a $\beta < \varkappa$ such that $x + K_\alpha \subset K_\beta$;

(3) For any $\alpha < \varkappa$ and $x \in M_R$ $x \cdot I_\alpha = \{0\}$ implies $x = 0$.

If M_R satisfies ω-approximation property, we say M_R is primitively slender. M_R is primitively endo-slender, if the left module ${}_EM$ is primitively slender, where $E = \mathrm{End}_R(M_R)$.

Note. Let $r_0, \ldots, r_{k-1}$ be generators of a left ideal I and M_R a right module. Then, $M \cdot I (= \{\sum_{i=0}^{n-1} m_i s_i\colon m_i \in M, s_i \in I, n < \omega\}) = \{\sum_{i=0}^{k-1} m_i r_i\colon m_i \in M\}$ holds. Therefore, $I^\omega = R^\omega \cdot I$ holds. $M \cdot I$ is a subgroup of M, but not always an R-submodule.

LEMMA 2.2. *If M_R satisfies the $\varkappa$-approximation property for a regular cardinal $\varkappa$, then, for every $h \in \mathrm{Hom}_R(R^\varkappa, M_R)$, there exists an $\alpha < \varkappa$ such that $h(R^{\varkappa-\alpha}) = \{0\}$.*

Proof. First we show (a): If $m \notin K_\alpha$, then there exists β such that $(m + M \cdot I_\beta) \cap K_\alpha = \varnothing$. There exists $\lambda < \varkappa$ such that $m \in K_\gamma$ and therefore $\alpha < \lambda$.

Choose β so that $K_\gamma - m \subset K_\beta$. This is the desired β, for if there is an $m' \in M \cdot I_\beta$ and $m + m' \in K_\alpha$, then $m' = (m+m') - m \in K_\beta$ and hence $m' = 0$. But, this contradicts $m \notin K_\alpha$; hence, (a) holds. Next we show (b): There exist $x \in R^\varkappa$ and $\alpha, \beta, \lambda < \varkappa$

such that $h(x+R^{\varkappa-\alpha}\cdot I_\beta)\subset K_\gamma$. Under the negation of the conclusion, we construct x_α and β_α ($\alpha\leqslant\varkappa$) in the following way. Let $x_0=0$ and $\beta_0=0$.

(Successor case.) Since $h(x_\alpha+R^{\varkappa-\alpha}\cdot I_{\beta_\alpha})\not\subset K_{\alpha+1}$, there exists an $x_{\alpha+1}$ such that $x_{\alpha+1}\in x_\alpha+R^{\varkappa-\alpha}\cdot I_{\beta_\alpha}$ and $h(x_{\alpha+1})\notin K_{\alpha+1}$. By (a) there exists $\beta_{\alpha+1}\geqslant\beta_\alpha$ such that $(h(x_\alpha)+M\cdot I_{\beta_{\alpha+1}})\cap K_{\alpha+1}=\emptyset$.

(Limit case.) Let $\beta^*=\sup_{\gamma<\alpha}\beta_\gamma$, $x^*(\gamma)=x_\gamma(\gamma)$ for $\gamma<\alpha$ and $x^*(\gamma)=0$ for $\gamma\geqslant\alpha$. Then, according to the construction $x^*\in x_\gamma+R^{\varkappa-\gamma}\cdot I_{\beta_\gamma}$ ($\gamma<\alpha$) and

$$h(x^*+R^{\varkappa-\alpha}\cdot I_{\beta^*})\not\subset K_\alpha$$

by the assumption. There exists an $x_\alpha\in x^*+R^{\varkappa-\alpha}\cdot I_{\beta^*}$ such that $h(x_\alpha)\notin K_\alpha$. By (a) there exists $\beta_\alpha\geqslant\beta^*$ such that $(h(x_\alpha)+M\cdot I_{\beta_\alpha})\cap K_\alpha=\emptyset$. Finally, let $x_\varkappa(\alpha)=x_\alpha(\alpha)$ for $\alpha<\varkappa$; as before, $x_\varkappa\in x_\alpha+R^{\varkappa-\alpha}\cdot I_{\beta_\alpha}$ ($\alpha<\varkappa$). Then, $h(x_\varkappa)\in K_\alpha$ for some α and so $h(x_\varkappa)\in(h(x_\alpha)+M\cdot I_{\beta_\alpha})\cap K_\alpha$, which is a contradiction

The statement (b) implies the existence of x and α such that $h(x)+h(R^{\varkappa-\alpha}\cdot I_\alpha)\subset K_\alpha$ and $h(x)\in K_\alpha$. On the other hand, there exists $\alpha^*\geqslant\alpha$ such that $K_\alpha-h(x)\subset K_{\alpha^*}$. Consequently, $h(R^{\varkappa-\alpha^*}\cdot I_{\alpha^*})\subset K_{\alpha^*}$, which implies $h(R^{\varkappa-\alpha^*})\cdot I_{\alpha^*}=h(R^{\varkappa-\alpha^*}\cdot I_{\alpha^*})=0$ and $h(R^{\varkappa-\alpha^*})=0$.

THEOREM 2.3. *If an R-module M_R is primitively slender, then M_R is slender. In addition if M_R satisfies the $\varkappa$-approximation property for every measurable cardinal or regular limit of measurable cardinals $\varkappa\leqslant\mu$, then M_R is $\mu^{+\mathrm{MC}}$-slender.*

Proof. The first proposition follows from the case $\varkappa=\omega$ of Lemma 2.2. We prove λ^+-slenderness for $\lambda<\mu^{+\mathrm{MC}}$ by induction. If λ is neither measurable nor regular limit of measurable cardinals, λ^+-slenderness follows from Corollary 1.6. The remaining case follows from Lemma 2.2 by the assumption.

EXAMPLES and Remarks 2.4. (1) The group of integers $\boldsymbol{Z}$ is primitively slender and moreover every countable reduced torsionfree group A is primitively slender. To show the latter, we define finite subsets K_n and $n_k<\omega$ by induction. Let $A=\{a_n\colon n<\omega\}$, $K_0=\{0\}$ and $n_0=0$. In the $(k+1)$th step, let $K_{n_{k+1}}$ be the minimal finite subset of A such that $a_k\in K_{n_{k+1}}$, $K_{n_k}+K_{n_k}\subset K_{n_{k+1}}$ and $K_{n_{k+1}}=-K_{n_{k+1}}$ and n_{k+1} be the least number such that $n_{k+1}!A\cap K_{n_{k+1}}=\{0\}$. Let $K_i=K_{n_k}$ for $n_k\leqslant i<n_{k+1}$. Then, $K_n, I_n=n!\boldsymbol{Z}$ ($n<\omega$) are the desired entities.

(2) Lady [25] showed that a countable commutative domain which is not a field is slender. It is enough that the cardinality of the domain is less than $2^{\aleph_0}$. Here, we show that a countable left Ore domain which is not a division ring is primitively slender as a right module. First recall that a ring R is a left Ore domain, if the following hold: (a) $a\cdot b=0$ implies that $a=0$ or $b=0$; (b) $Ra\cap Rb=\{0\}$ implies that $a=0$ or $b=0$. Observe that for nonzero a and b there exists a nonzero c such that $Rc\leqslant Ra\cap Rb$. Let r^* be a nonzero element which has no right inverse, then $1\notin Rr^*$. Since R is a domain, $x\notin Rr^*x$ for all nonzero x. Now, we construct K_n and principal left ideals $I_n=R\cdot x_n$ ($n<\omega$) as in Example (1). Let $R=\{r_n\colon n<\omega\}$, $K_0=\{0\}$ and $x_0=1$. In the $(n+1)$th step, let K_{n+1} be the minimal finite subset

of R such that $r_n\in K_{n+1}$, $K_n+K_n\subset K_{n+1}$ and $K_{n+1}=-K_{n+1}$. Next, let x_{n+1} be a nonzero element such that $R\cdot x_{n+1}\leqslant\bigcap\{Rr^*x\colon 0\neq x\in K_{n+1}\}$. Then, K_n and $I_n=R\cdot x_n$ ($n<\omega$) have the desired properties.

(3) A submodule of a primitively slender module is clearly primitively slender. If $(M_\lambda)_R$ ($\lambda\in\Lambda$) are primitively slender and if, in addition, we can take $M_\lambda=\bigcup_{n<\omega}K_{\lambda n}$ and I_n ($n<\omega$) uniformly (i.e. I_n are independent of λ), then the direct sum $\bigoplus_{\lambda\in\Lambda}M_\lambda$ is primitively slender. Consequently, subgroups of the direct sum of countable torsion free reduced groups are primitively slender. To show the former statement, let $K_n=\sum_\lambda K_{\lambda_n}=\{x_{\lambda_1}+\ldots+x_{\lambda_k}\colon x_{\lambda_i}\in K_{\lambda_i n}$ and $\lambda_i\neq\lambda_j$ $(i\neq j)\}$. For $m<\omega$ and $x=x_{\lambda_1}+\ldots+x_{\lambda_k}\in\bigoplus_{\lambda\in\Lambda}M_\lambda$, where $x_{\lambda_i}\in K_{\lambda_i m}$, there exists $n<\omega$ such that $x_{\lambda_i}+K_{\lambda_i m}\subset K_{\lambda_i n}$ for $1\leqslant i\leqslant k$. Since $K_{\lambda m}\subset K_{\lambda n}$ ($\lambda\in\Lambda$), $x+K_m\subset K_n$ holds. Other properties clearly hold.

(4) O'Neil [27] has shown that the polynomial ring $R[X]$ over any ring R is right and left slender. One can easily see that $R[X]$ is primitively slender. To show the existence of $\varkappa$-slender ring for an arbitrary cardinal $\varkappa$, we prove $\varkappa$-approximation property of modified semigroup rings, which also implies that any polynomial ring is primitively slender. Let R be a ring and G a semigroup with 1. Then, the semigroup ring $R[G]$ is an R-free module $\oplus\langle g\colon g\in G\rangle$ generated by G and the multiplication is defined by the operations on R and G naturally where elements of R commute with those of G. In case G has 0, the cyclic R-submodule $\langle 0\rangle$ forms a two-sided ideal. We denote the factor ring $R[G]/\langle 0\rangle$ by $R[G]^*$ and hence we may identify $R[G]^*$ with $\oplus\langle g\colon g\neq 0\rangle$.

PROPOSITION 2.4 (4). *Let G be a semigroup with* 0, 1 *and let $\{u_\alpha\colon \alpha<\varkappa\}$ be its subset such that*:

(a) *If $\alpha\leqslant\beta$, then $g\cdot u_\alpha=u_\beta$ for some g*;

(b) $\bigcap_{\alpha<\varkappa}G\cdot u_\alpha=\{0\}$;

(c) *$g\cdot G\cdot u_\alpha=\{0\}$ implies $g=0$ for each $\alpha<\varkappa$.*

Then, the ring $R[G]^*$ satisfies $\varkappa$-approximation property as a right $R[G]^*$-module.

Proof. Let $I_\alpha=R[G]^*\cdot u_\alpha$ and $K_\alpha=\oplus\langle g\colon x\cdot u_\alpha\neq g$ for any $x\rangle$, then it is easy to check the $\varkappa$-approximation property.

Now, let $M_S=\oplus_\varkappa A_S$ for a nonzero S-module A_S and $E=\mathrm{End}_S(M_S)$. For any cardinal $\lambda\leqslant\varkappa$, there exists a decomposition $M=\oplus_{\alpha<\lambda}M_\alpha$ such that $M_\alpha\neq\{0\}$. Let $p_\alpha^\lambda\colon M\to\oplus_{\beta\geqslant\alpha}M_\beta$ be the projection. Then, E as a semigroup satisfies the property of the proposition and moreover with the same $\{p_\alpha^\lambda\colon\alpha<\lambda\}$ E also satisfies the symmetric one. Therefore, $R[E]^*$ satisfies λ-approximation property as a right and left module for each $\lambda\leqslant\varkappa$ by Proposition 2.4 (4). Consequently, $R[E]^*$ is a $\varkappa^+$-slender ring. Moreover, taking a subring E' of E containing p_α^λ ($\alpha<\lambda\leqslant\varkappa$), we get a $\varkappa^+$-slender ring $R[E']^*$ of cardinality $\varkappa$.

(5) M. Huber [23, Theorem 3.3] proved that any module of the form $\oplus_\omega M_R$ is endo-slender. It is also primitively endo-slender. Let $E=\mathrm{End}_R(\oplus_\varkappa M_R)$. For any $\lambda\leqslant\varkappa$, there exists a decomposition $\oplus_\varkappa M_R=\oplus_{\alpha<\lambda}(A_\alpha)_R$, where $A_\alpha\simeq A_\beta$ for all α and β. Let $p_\alpha\colon\oplus_\varkappa M_R\to\oplus_{\beta\geqslant\alpha}A_\beta$, $I_\alpha=p_\alpha\cdot E$ and $K_\alpha=\oplus_{\beta<\alpha}A_\beta$. Then, it is easy to see

that ${}_E(\oplus_\varkappa M_R)$ satisfies λ-approximation property for any $\lambda \leqslant \varkappa$ and consequently ${}_E(\oplus_\varkappa M_R)$ is a $\varkappa^+$-slender E-module.

(6) We prove Mader's theorem as a corollary of Theorem 2.3 and so modules whose endo-slenderness follows from [26, 2.8 Theorem] are primitively slender. First we state the proposition: Let $M_R = \oplus_{n<\omega}(M_n)_R$ and $E = \mathrm{End}_R(M_R)$. If $\oplus_{k<n}M_k$ contains no nonzero E-submodule of ${}_EM$, then ${}_EM$ is slender [26, 2.8 Theorem].

Let $p_n\colon M_R \to \oplus_{k\geqslant n}M_k$ be the projection, $I_n = p_n\cdot E$ and $K_n = \oplus_{k<n}M_k$. We only check the property (3). Suppose that $I_n\cdot x = \{0\}$ for a nonzero $x \in M_R$. Then, $x \in \oplus_{k<n}M_k$. By the assumption $E\cdot x \not\subset \oplus_{k<n}M_k$ and hence $p_n\cdot E\cdot x \neq \{0\}$, which is a contradiction.

(7) Here, we remark that primitive slenderness is strictly stronger than slenderness. For that purpose we refer the reader to [19, or 21] for T-slenderness for a monotone subgroup T of Z^N. A monotone subgroup T itself is not T-slender, of course, and if T is different from Z^N, then T is slender [19, Theorem 1]. On the other hand, a primitively slender group is T-slender for any monotone subgroup T which is different from the subgroup consisting of all bounded functions. Therefore, there exist slender, but not primitively slender groups. We prove the previous sentence. Let M be a primitively slender group and $M = \bigcup_{n<\omega}K_n$, $I_n = n!Z$ and other properties in Definition 2.1 hold. There exists a monotone increasing unbounded sequence n_k $(k<\omega)$ such that $1 \leqslant n_k$ and $\sum_{k<\omega}n_k!e_k \in T$. Suppose that $h(e_k) \neq 0$ for infinitely many k. We define a strictly monotone increasing sequence k_i $(i<\omega)$ by induction. Let k_0 be the least number so that $h(e_k) \neq 0$ and k_{i+1} be the least number such that $k_i < k_{i+1}$, $h(e_{k_{i+1}}) \neq 0$ and $K_{n_{k_i}} - h(\sum_{j=0}^{i}n_{k_j}!e_{k_j}) \subset K_{n_{k_{i+1}}}$. Then, there exists an i such that $h(\sum_{j<\omega}n_{k_j}!e_{k_j}) \in K_{n_i}$. For $m \geqslant i+1$, $h(\sum_{j\geqslant m}n_{k_j}!e_{k_j}) = h(\sum_{j<\omega}n_{k_j}!e_{k_j}) - h(\sum_{j=0}^{m-1}n_{k_j}!e_j) \in K_{n_{k_m}}$ and hence $h(\sum_{j\geqslant m}n_{k_j}!e_{k_j}) = 0$. Now,

$$h(n_{k_{i+1}}!e_{k_{i+1}}) = h(\textstyle\sum_{j\geqslant i+1}n_{k_j}!e_{k_j}) - h(\sum_{j\geqslant i}n_{k_j}!e_{k_j}) = 0,$$

which is a contradiction.

(8) Recently, P. Eklof and A. Mekler [16] have proved that a principal ideal domain R is slender iff R is not a complete valuation domain.

3. Large cardinals and cardinality restrictions of radicals. We have shown in [12] that the existence of an $L_{\omega_1\omega}$-compact cardinal is equivalent to the cardinality condition of the radical R_Z. In this section we shall prove a relative property which concerns R-modules for all rings R. To state the theorem some definitions from infinitary logic and theory of radicals are necessary.

For infinite cardinals μ and ν, $L_{\mu\nu}$ is an infinitary language which admits α-sequences of disjunctions and conjunctions and β-sequences of quantifiers for $\alpha<\mu$ and $\beta<\nu$. We refer the reader to [4] for a precise definition. A cardinal $\varkappa$ is λ-$L_{\mu\nu}$-compact, if the following hold for a set of $L_{\mu\nu}$-sentences $\mathcal{T}$ of cardinality λ: If any subset $\mathcal{S}$ of $\mathcal{T}$ of cardinality less than $\varkappa$ has a model, then $\mathcal{T}$ itself has a model. If $\varkappa$ is λ-$L_{\mu\nu}$-compact for every λ, then $\varkappa$ is said to be an *$L_{\mu\nu}$-compact cardinal*. For cardinals $\varkappa$ and λ, $P_\varkappa\lambda = \{x \subset \lambda\colon |x| < \varkappa\}$ and $\lambda^{<\varkappa} = |P_\varkappa\lambda|$, where $|x|$ is the cardi-

nality of x. Let $U_x = \{y\colon x \subset y \in P_\varkappa\lambda\}$ for each $x \in P_\varkappa\lambda$ and $F_{\varkappa\lambda} = \{X\colon U_x \subset X$ for some $x \in P_\varkappa\lambda\}$. Then, $F_{\varkappa\lambda}$ is a $\varkappa$-complete filter on $P_\varkappa\lambda$ and a $\varkappa$-complete Boolean algebra $P(P_\varkappa\lambda)/F_{\varkappa\lambda}$ is denoted by $B_{\varkappa\lambda}$. A $\varkappa$-representable, $\varkappa$-complete Boolean algebra is the quotient Boolean algebra of a $\varkappa$-complete field modulo a $\varkappa$-complete filter [29, § 29].

For an R-module M_R, radicals $\mathbf{R}_M$ and $\mathbf{R}_M^\infty$ are defined by:

$$\mathbf{R}_M X = \bigcap\{\mathrm{Ker}(h)\colon h \in \mathrm{Hom}_R(X, M_R)\}$$

and

$$\mathbf{R}_M^\infty X = \sum\{Y_R\colon Y_R \leqslant X, \mathrm{Hom}_R(Y, M) = \{0\}\}$$

for each R-module X. For a cardinal $\varkappa$ and a preradical $\mathbf{T}$ (i.e. a subfunctor of the identity) for R-modules, $\mathbf{T}^{[\varkappa]}$ is defined by:

$$\mathbf{T}^{[\varkappa]}X = \sum\{\mathbf{T}Y_R\colon Y_R \leqslant X \text{ and } Y_R \text{ is } {}^{<}\varkappa\text{-generated}\},$$

where Y_R is ${}^{<}\varkappa$-generated if Y_R has a set of generators of cardinality less than $\varkappa$. $\mathbf{T}$ satisfies the cardinality condition, if there exists a cardinal $\varkappa$ such that $\mathbf{T} = \mathbf{T}^{[\varkappa]}$. For a class of R-modules $\mathbf{X}$ ${}^\perp\mathbf{X} = \{Y_R\colon \mathrm{Hom}_R(Y_R, X_R) = \{0\}$ for all $X_R \in \mathbf{X}\}$ and $\mathbf{X}^\perp = \{Y_R\colon \mathrm{Hom}_R(X_R, Y_R) = \{0\}$ for all $X_R \in \mathbf{X}\}$. If $\mathbf{X}$ is a singleton $\{X_R\}$, then we write ${}^\perp X_R$ and $X_R^\perp$ instead of ${}^\perp\{X_R\}$ and $\{X_R\}^\perp$ respectively. A torsion class $\mathbf{X}$ is singly generated, if there exists an X_R such that ${}^\perp(X_R^\perp) = \mathbf{X}$. We refer the reader to [31, § 1 and 2 of Chapter VI] for a general theory of radicals and torsion theory and to [5, 6, 12, 14, 15, 17] for related topics.

Now, we can state the main theorem of this section.

THEOREM 3.1. *Let $\varkappa$, μ, λ be infinite cardinals such that $\mu^+ \leqslant \varkappa$, $\lambda = \lambda^{<\varkappa}$ and $\varkappa$ is regular. Then, the following are equivalent:*

(1) *$\varkappa$ is a λ-$L_{\mu^+\mu^+}$-compact cardinal;*

(2) *$\varkappa$ is a λ-$L_{\mu^+\omega}$-compact cardinal;*

(3) *If $|R| \leqslant \mu$, $|M_R| \leqslant \mu$ and $|A_R| \leqslant \lambda$, then $\mathbf{R}_M A = \mathbf{R}_M^{[\varkappa]}A$ holds;*

(4) *If $|R| \leqslant \mu$, $|M_R| \leqslant \mu$ and $|A_R| \leqslant \lambda$, then $\mathbf{R}_M^\infty A = (\mathbf{R}_M^\infty)^{[\varkappa]}A$ holds;*

(5) *If $|R| \leqslant \mu$, $|M_R| \leqslant \mu$, $|A_R| \leqslant \lambda$ and $\mathrm{Hom}_R(A_R, M_R) = \{0\}$, then A belongs to ${}^\perp C^\perp$. (Here, $C = \oplus[X_R\colon |X_R| < \varkappa$ and $\mathrm{Hom}_R(X_R, M_R) = \{0\}]$ and $[X_R\colon |X_R| < \varkappa \ldots]$ means the set of all isomorphism types of such modules.)*

(3′) *If $|R| \leqslant \mu$ and $|A_R| \leqslant \lambda$, then $\mathbf{R}_R A = \mathbf{R}_R^{[\varkappa]}A$ holds;*

(4′) *If $|R| \leqslant \mu$ and $|A_R| \leqslant \lambda$, then $\mathbf{R}_R^\infty A = \mathbf{R}_R^{\infty[\varkappa]}A$ holds;*

(5′) *If $|R| \leqslant \mu$, $|A_R| \leqslant \lambda$ and $\mathrm{Hom}_R(A_R, R_R) = \{0\}$, then A belongs to $({}^\perp C^\perp)$, where $C = \oplus[X_R\colon |X_R| < \varkappa$ and $\mathrm{Hom}_R(X_R, R_R) = \{0\}]$;*

(6) *If $|R| \leqslant \mu$ and $|A_R| \leqslant \lambda$, then $\mathbf{R}_R^{[\varkappa]}A = \{0\}$ implies $\mathbf{R}_R A = \{0\}$;*

(7) *Let $|R| \leqslant \mu$ and B be a $\varkappa$-representable, $\varkappa$-complete* Ba *of cardinality less than or equal to λ, then $\mathbf{R}_R(R^{(B)}) = \{0\}$;*

(8) *If $|R| \leqslant \mu$ and S is a nonzero submodule of $R^{(B_{\varkappa\lambda})}$ with $|S| \leqslant \lambda$, then* $\mathrm{Hom}_R(S, R_R) \neq \{0\}$

LEMMA 3.2 [29, 29.3]. *Let B be a $\varkappa$-representable, $\varkappa$-complete* Ba. *If $b \neq 0$ and $\bigvee_{\beta<\mu} b_{\alpha\beta} = 1$ for $\alpha < \nu$, where $\mu, \nu < \varkappa$, then there exists an $f \in {}^{\nu}\mu$ such that $\{b, b_{\alpha f(\alpha)}: \alpha < \nu\}$ satisfies the finite intersection property.*

Proof of Theorem 3.1. (1)$\to$(2): Clear.

(2)$\to$(3): Suppose that $a^* \notin \mathbf{R}_M^{[\varkappa]}A$. Let $\mathscr{T}$ be the following set of $L_{\mu^+\omega}$-sentences:

(a) $\underline{a} \neq \underline{a}'$ for $a \neq a'$, $a, a' \in A$; $\underline{a}+\underline{b} = \underline{c}$, $\underline{a}\cdot\underline{r} = \underline{b}$ for $a+b = c$, $a\cdot r = b$, $a, b, c \in A$ and $r \in R$;

(b) The axioms of R-modules;

(c) $\forall x \bigvee_{u\in M}(H_u(x) \,\&\, \bigwedge_{v\neq u, v\in M} \neg H_v(x))$;

$\forall x, y \bigvee_{u,v,w\in M, u+v=w}(H_u(x) \,\&\, H_v(y) \,\&\, H_w(x+y))$;

$\forall x \bigvee_{u,v\in M, ur=v}(H_u(x) \,\&\, H_v(x\cdot\underline{r}))$ for all $r \in R$;

$\bigvee_{u\in M, u\neq 0} H_u(\underline{a}^*)$.

For a subset $\mathscr{S}$ of $\mathscr{T}$ of cardinality less than $\varkappa$, there exists an R-submodule X_R of A_R with the following: X_R is of cardinality less than $\varkappa$; a^* belongs to X_R; If $\underline{a}$ appears in $\mathscr{S}$, then a belongs to X_R. By the assumption, there exists an R-homomorphism $h\colon X_R \to M_R$ with $h(a^*) \neq 0$ and hence (X_R, h) is a model of $\mathscr{S}$. By (2), $\mathscr{T}$ has a model A^*. Then, A^* is an R-module and A_R is an R-submodule of A^* and $a^* \notin \mathbf{R}_M A^*$. Hence, $a^* \notin \mathbf{R}_M A$.

(3)$\to$(4): As is well known, $\mathbf{R}_M^{\infty}$ can be obtained by iterated applications of $\mathbf{R}_M$, i.e. $\mathbf{R}_M^0 A = \mathbf{R}_M A$, $\mathbf{R}_M^{\alpha+1} A = \mathbf{R}_M(\mathbf{R}_M^{\alpha} A)$, $\mathbf{R}_M^{\alpha} = \bigcap_{\beta<\alpha} \mathbf{R}_M^{\beta} A$ for a limit α and $\mathbf{R}_M^{\infty} A = \mathbf{R}_M^{\alpha} A = \mathbf{R}_M^{\alpha+1} A$ for some α. Therefore, it is easy to see that $(\mathbf{R}_M^{\infty})^{[\varkappa]} A = (\mathbf{R}_M^{[\varkappa]})^{\infty} A$ by the regularity of $\varkappa$ [14, Theorem 1.2]. If $|A| \leqslant \lambda$, then $(\mathbf{R}_M^{\infty})^{[\varkappa]} A = \mathbf{R}_M^{\infty} A$ by (3).

(4)$\to$(5): By the assumption and (4),

$$A = \mathbf{R}_M^{\infty} A = (\mathbf{R}_M^{\infty})^{[\varkappa]} A = \sum \{X \leqslant_R A\colon \mathrm{Hom}_R(X, M_R) = \{0\}, |X| < \varkappa\}.$$

This implies that A is a homomorphic image of a direct sum of copies of C and hence A belongs to ${}^{\perp}(C^{\perp})$.

(3′)$\to$(4′) and (4′)$\to$(5′): The proofs are special cases of above proofs. (5′) is a special case of (5).

(5′)$\to$(8): Suppose that $\mathrm{Hom}_R(S, R_R) = \{0\}$, then S belongs to ${}^{\perp}(C^{\perp})$ by (5′). Let X be a submodule of $R^{(B_{\varkappa\lambda})}$ of cardinality less than $\varkappa$. We show that $\mathbf{R}_R X = \{0\}$. Let x^* be a nonzero element of X, then there exists $r^* \neq 0$ such that $x^*(r^*) \neq 0$. By Lemma 3.2, there exists an $f\colon X \to R$ such that $\{x^*(r^*), x(f(x))\colon x \in X\}$ satisfies the finite intersection property. By the finite intersection property we get: $f(x^*) = r^* \neq 0$; $x\cdot r(f(x)\cdot r) \geqslant x(f(x)) \neq 0$, and hence $f(x)\cdot r = f(x\cdot r)$;

$$(x+y)(f(x)+f(y)) \geqslant x(f(x)) \wedge y(f(y)) \neq 0;$$
$$(x+y)(f(x)+f(y)) \wedge (x+y)(f(x+y)) \neq 0,$$

and hence $f(x+y) = f(x)+f(y)$. Now, we get a desired homomorphism and so $\mathbf{R}_R X = \{0\}$. Therefore, $\mathrm{Hom}_R(Y, M_R) = \{0\}$ and $|Y| < \varkappa$ imply $\mathrm{Hom}_R(Y, S) = \{0\}$ and so $\mathrm{Hom}_R(C, S) = \{0\}$, which contradicts $S \in {}^{\perp}(C^{\perp})$.

(3′)$\to$(6): Clear.

(6)$\to$(7): Since $|R^{(B)}| \leqslant \lambda^{\mu} \leqslant \lambda^{<\varkappa} = \lambda$, what we must show is: $\mathbf{R}_R^{[\varkappa]} R^{(B)} = \{0\}$. For a submodule S of $R^{(B)}$ of cardinality less than $\varkappa$ and a nonzero x, we get a desired homomorphism as in the proof of (5′)$\to$(8).

(7)$\to$(8): Let C be the $\varkappa$-complete subalgebra of $B_{\varkappa\lambda}$ generated by $\{s(r)\colon s \in S, r \in R\}$, then $|C| \leqslant \lambda$ holds. Since C is also $\varkappa$-representable [29, 29.3], $\mathbf{R}_R S \leqslant \mathbf{R}_R R^{(C)} \leqslant \{0\}$ and hence (8) holds.

(8)$\to$(1): In the proof of the corresponding part in [12], we referred to [28, or 3] and did not write down the whole proof and so here we present the whole proof of (8)$\to$(1). We use usual notions in model theory without precise definitions. Let $\mathscr{T}$ be a set of $L_{\mu^+\mu^+}$-sentences of cardinality λ whose any subset of cardinality less than $\varkappa$ has a model. Introducing infinitary Skolem functions, we may suppose that $\mathscr{T}$ consists of universal sentences by the cardinality assumptions, i.e. $\lambda = \lambda^{<\varkappa}$, $\mu^+ \leqslant \varkappa$. Let $\mathfrak{A}_x$ be a model of x for $x \in P_\varkappa\mathscr{T}$ and $\mathfrak{B}$ be a substructure of the direct product $\prod_{x\in P_\varkappa\mathscr{T}} \mathfrak{A}_x$. We can get $\mathfrak{B}$ so that $|\mathfrak{B}| \leqslant \lambda$. For each open formula $\varphi(v_\xi\colon \xi < \alpha)$, $\alpha < \varkappa$, of $L_{\mu^+\mu^+}$ and each sequence $f = (f_\xi\colon \xi < \alpha)$ from $\mathfrak{B}$. let $X_{\varphi,f} = \{x \in P_\varkappa\mathscr{T}\colon \mathfrak{A}_x \models \varphi(f_\xi(x)\colon \xi < \alpha)\}$. The $\varkappa$-complete subfield of $P(P_\varkappa\mathscr{T})$ generated by all $X_{\varphi,f}$ and U_x has the cardinality λ. We denote the subfield by $\mathscr{F}$ and the quotient map from $P(P_\varkappa\mathscr{T})$ to $B_{\varkappa\lambda}$ by σ. Then, $\sigma(\mathscr{F})$ is a $\varkappa$-complete subalgebra of $B_{\varkappa\lambda}$. By Example 2.4 (4), there exists a μ^+-slender ring R of cardinality μ. The Boolean power $R^{(\sigma(\mathscr{F}))}$ is an R-submodule of $R^{(B_{\varkappa\lambda})}$ of cardinality equal to or less than λ. By (8), $\mathrm{Hom}_R(R^{(\sigma(\mathscr{F}))}, R_R) \neq \{0\}$. Theorem 1.5 implies the existence of μ^+-complete ultrafilter of $\sigma(\mathscr{F})$. Since σ is $\varkappa$-complete, $\mathscr{F}$ has a μ^+-complete ultrafilter F which contains all U_x ($x \in P_\varkappa\mathscr{T}$). Define a modified ultraproduct $\mathfrak{B}/F$ by: $f \sim_F g$ if $\{x\colon f(x) = g(x)\} \in F$, and the equivalence class is denoted by $[f]_F$; $\mathfrak{B}/F \models \underline{P}([f_\xi]\colon \xi < \alpha)$ if $\{x\colon \mathfrak{A}_x \models \underline{P}(f_\xi(x)\colon \xi < \alpha)\} \in F$ for each predicate $\underline{P}$ of $L_{\mu^+\mu^+}$ and each sequence $(f_\xi\colon \xi < \alpha)$ from $\mathfrak{B}$ with $\alpha < \varkappa$. Since F is μ^+-complete, this definition is correct and moreover $\mathfrak{B}/F \models \varphi([f_\xi]\colon \xi < \alpha)$ iff

$$\{x\colon \mathfrak{A}_x \models \varphi(f_\xi(x)\colon \xi < \alpha)\} \in F$$

for each open formula of $L_{\mu^+\mu^+}$. If φ is a sentence in $\mathscr{T}$, then there exists an open formula $\psi(v_\xi\colon \xi < \alpha)$ such that φ is the universal closure of $\psi(v_\xi\colon \xi < \alpha)$ with respect to variables v_ξ ($\xi < \alpha$). If φ belongs to x for an $x \in P_\varkappa\mathscr{T}$, then $\mathfrak{A}_x \models \varphi$ and hence $\mathfrak{B}/F \models \psi([f_\xi]_F\colon \xi < \alpha)$ for every sequence $(f_\xi\colon \xi < \alpha)$ from $\mathfrak{B}$, i.e. $\mathfrak{B}/F \models \varphi$ for every $\varphi \in \mathscr{T}$.

Using Boolean powers $M_R^{(B)}$ instead of $R^{(B)}$ in the proof, we get

COROLLARY 3.3. *Let $\varkappa, \mu, \lambda$ be infinite cardinals such that $\mu^+ \leqslant \varkappa$, $\lambda = \lambda^{<\varkappa}$ and $\varkappa$ is regular and let R be a ring with $|R| \leqslant \mu$ and M_R a slender R-module with $|M_R| \leqslant \mu$. Then, the following are equivalent:*

(1) *$\varkappa$ is a λ-$L_{\mu^+\mu^+}$-compact cardinal;*

(2) *$\varkappa$ is a λ-$L_{\mu^+\omega}$-compact cardinal;*

(3) *For an R-module A_R with $|A_R| \leqslant \lambda$, $\mathbf{R}_M A = \mathbf{R}_M^{[\varkappa]} A$ holds;*

(4) *For an* R*-module* A_R *with* $|A_R| \leqslant \lambda$, $\mathbf{R}_M^\infty A = \mathbf{R}_M^{\infty[\varkappa]} A$ *holds*;

(5) *If an* R*-module* A_R *satisfies* $|A_R| \leqslant \lambda$ *and* $\mathrm{Hom}_R(A_R, M_R) = \{0\}$, *then* A_R *belongs to* ${}^{\perp}(C^{\perp})$, *where* $C = \oplus[X_R\colon |X_R| < \varkappa \,\&\, \mathrm{Hom}_R(X_R, M_R) = \{0\}]$;

(6) *For an* R*-module* A_R *with* $|A_R| \leqslant \lambda$, $\mathbf{R}_M^{[\varkappa]} A = \{0\}$ *implies* $\mathbf{R}_M A = \{0\}$.

In the following corollary the equivalence of (1) and (2) is well known and can be proved by a standard method.

COROLLARY 3.4. *The following propositions are equivalent*:

(1) *For an arbitrary infinite cardinal* μ, *there exist an* $L_{\mu\omega}$*-compact cardinal*;

(2) *For an arbitrary infinite cardinal* μ, *there exists a cardinal* $\varkappa$ *such that any* $\varkappa$*-complete filter on an arbitrary set can be extended to a* μ*-complete ultrafilter*;

(3) *For any ring* R *and* R*-module* M_R, *the radical* $\mathbf{R}_M$ $(\mathbf{R}_M^\infty)$ *satisfies the cardinality condition*;

(4) *For any ring* R, *the radical* $\mathbf{R}_R$ $(\mathbf{R}_M^\infty)$ *satisfies the cardinality condition:*

(5) *For any ring* R *and* R*-module* M_R, *the torsion class* ${}^{\perp}M_R$ *is singly generated*;

(6) *For any ring* R, *the torsion class* ${}^{\perp}R_R$ *is singly generated.*

The equivalence of (2) and (5) of Corollary 3.3 in the case that $R = \mathbf{Z}$, $M_R = \mathbf{Z}$, $\varkappa = \omega_1$ and $\mu = \omega$ implies that the existence of $L_{\omega_1\omega}$-compact cardinal is equivalent to the singly generatedness of the torsion class ${}^{\perp}\mathbf{Z}$, which answers a question in [5].

4. Endo-slenderness of direct products. First we state some preliminary facts about endo-slenderness.

PROPOSITION 4.1. *If* R *is a commutative ring and an* R*-module* M_R *is slender, then* M_R *is endo-slender.*

The proposition is clear, since there exists a ring homomorphism from R to $\mathrm{End}_R(M_R)$ which preserves 1. We remark that in a noncommutative case slenderness does not always imply endo-slenderness.

PROPOSITION 4.2 [26, 3.2 Lemma]. *If* A_R *is a fully invariant submodule of* M_R, *i.e.* $h(A_R) \leqslant A_R$ *for all* $h \in \mathrm{End}_R(M_R)$, *and* A_R *is not endo-slender, then* M_R *is also not endo-slender.*

PROPOSITION 4.3. *Let* A *be a nonzero divisible abelian group of finite rank. If* A *is either a* p*-group or a torsionfree group, then* A *is not endo-slender.*

The former case is [26, 3.7 Proposition]. (A correct statement there should be "If A is a nonreduced p-group and the divisible part of A is of finite rank, then A is not endo-slender.") The latter case holds, because A is a finite dimensional vector space over Q.

PROPOSITION 4.4. *Let* A_R *be endo-slender. If* M_R *is isomorphic to a submodule of* A_R^I *for some* I, *i.e.* $\mathbf{R}_A M = \{0\}$, *and* M_R *contains a summand which is isomorphic to* A_R, *then* M_R *is endo-slender.*

Proof. Suppose that $\varphi\colon E^\omega \to_E M$ is an E-homomorphism with $\varphi(e_n) \neq 0$ for any $n < \omega$. Let $M_R = A_R \oplus B_R$ and $p_A\colon M_R \to A_R$ and $p_B\colon M_R \to B_R$ be the projections. Then, $\varphi | p_A E^\omega\colon p_A E^\omega \to p_A M_R$ $(= A_R)$ is an $\mathrm{End}_R(A_R)$-homomorphism. By the assumption, there exists an $m < \omega$ such that $\varphi(p_A E^{\omega - m}) = \{0\}$. Since $p_A\varphi(e_m) = 0$, $p_B\varphi(e_m) \neq 0$. There exists an $h \in \mathrm{Hom}_R(B_R, A_R)$ such that $h \cdot p_B \varphi(e_m) \neq 0$. Now, $\varphi(p_A \cdot h \cdot p_B \cdot e_m) = p_A \cdot h \cdot p_B \varphi(e_m) = h \cdot p_B \varphi(e_m) \neq 0$, which is a contradiction.

PROPOSITION 4.5. *If* $(A_i)_R$ $(i \in I)$ *are endo-slender* R*-modules, then* $\oplus_{i\in I} A_i$ *is also endo-slender.*

Proof. Let $E = \mathrm{End}_R(\oplus_{i\in I} A_i)$ and suppose that the conclusion is false. Then, there exists an E-homomorphism $h\colon E^\omega \to \oplus_{i\in I} A_i$ such that $h(e_n) \neq 0$ for $n < \omega$. Let $p_i\colon \oplus_{i\in I} A_i \to A_i$ be the projection and for $x \in \oplus_{i\in I} A_i$ $\operatorname{supp} x = \{i\colon p_i(x) \neq 0\}$. According to the endo-slenderness of A_i's, we get natural numbers $i_n < i_{n+1}$ and finite subsets $I_n \subset I_{n+1} \subset I$ $(n < \omega)$ by induction so that $\{\operatorname{supp}\varphi(e_{i_k})\colon 0 \leqslant k \leqslant n) \subset I_n$ and $p_i\varphi(E^{\omega - i_{n+1}}) = \{0\}$ for every $i \in I_n$. Now, there exists $m < \omega$ such that $\operatorname{supp}\varphi(\sum_{n<\omega} e_{i_n}) \cap \bigcup_{n<\omega} I_n \subset I_m$. Let $p^*\colon \oplus_{i\in I} A_i \to \oplus_{i \in \cup_{n<\omega} I_n} A_i$ be the projection. By the property of the above construction, $p^*\varphi(\sum_{n<\omega}' e_n) = \sum_{i\in I_m} p_i\varphi(\sum_{n<\omega} e_{i_n}) = \varphi(\sum_{k=0}^m e_{i_k})$, i.e. $p^*\varphi(\sum_{k\geqslant m+1} e_{i_k}) = 0$. By the same reasoning, $p^*\varphi(\sum_{k\geqslant m+2} e_{i_k}) = 0$ and hence $\varphi(e_{i_{m+1}}) = 0$, which is a contradiction.

By Propositions 4.2, 4.3, 4.4 and 4.5, we can see that a problem of endo-slenderness of nonreduced torsionfree or torsion groups is splitted to those of reduced groups and divisible groups, where endo-slenderness of divisible groups can be easily determined. Proposition 4.4 has some corollaries, which we next state.

COROLLARY 4.6. *Let* R *be a right principal ideal domain. If* R *is left slender, then every torsionless right* R*-module (i.e. a submodule of* R^I *for some* I*) is endo-slender. Consequently, all torsionless abelian groups and torsionless* $D[X]$*-modules for division rings* D *are endo-slender.*

Proof. Let M_R be a nonzero torsionless R-module, then there exists a nonzero homomorphism $h\colon M_R \to R$. Since R is a right principal domain, there exist an $m \in M$ and $r \in R$ such that $h(m) = r$ and $h(M) = rR$ and so the cyclic right module $\langle m \rangle$ is isomorphic to R_R and is a summand of M_R. Now, the first proposition follows from Proposition 4.4. Slenderness of the ring $\mathbf{Z}$ and $D[X]$ is well known and follows from Example 2.4 (1) and (4) and hence the second claim holds.

COROLLARY 4.7. *If* A *is a torsion endo-slender group, then* $t(A^I)$, *i.e. the torsion subgroup of* A^I, *is endo-slender for every* I.

The proof follows directly from Proposition 4.4. Next we prove a theorem about endo-slenderness of submodules of direct products, by which we will easily determine endo-slenderness of direct products of torsion free groups of rank 1.

THEOREM 4.8. *Let* $(A_i)_R$ *be an endo-slender* R*-module for each* $i \in I$ *and* M_R *an* R*-module such that* $\oplus_{i\in I} A_i \subset M_R \subset \prod_{i\in I} A_i$, *where* $\oplus_{i\in I} A_i = \{x \in \prod_{i\in I} A_i\colon \operatorname{supp} x$ *is finite*$\}$. *If the following condition* (1) *holds, then* M_R *is endo-slender. In case that* $(A_i)_R$ *are slender and the condition* (2) *below holds, then* (1) *holds iff* M_R *is endo-slender.*

(1) *For any infinite* $X \subset I$ *and nonzero* $x_i \in A_i$ $(i \in X)$, *there exists a* $j \in I$ *such that* $x_i \notin \bigcap \mathrm{Ker}(h)\colon h \in \mathrm{Hom}_R(A_i, A_j)\}$ *for infinitely many* $i \in X$.

(2) *There exists a countably complete subfield* B *of* $P(I)$ *satisfying the following*:

(a) *The singleton* $\{i\}$ *belongs to* B *for each* $i \in I$;

(b) $\mathrm{supp}\, u (= \{i\colon u(i) \neq 0\})$ *belongs to* B *for each* $u \in M_R$;

(c) *For any pairwise disjoint subfamily* $\{X_n\colon n < \omega\}$ *of* B *and* $u_n \in M_R$, *there exists a* $u \in M_R$ *such that* $u|X_n = u_n|X_n$ *for every* $n < \omega$.

Proof. By the assumption there exist projections $p_i \in \mathrm{End}_R(M_R)$ $(= E)$ $(i \in I)$ such that $p_i(M) = A_i$. Suppose that there exists an E-homomorphism $\varphi\colon E^\omega \to_E M$ such that $\varphi(e_n) \neq 0$ for any $n < \omega$. Then, there exist i_n $(n \in N)$ and j such that $p_{i_n}\varphi(e_n) \neq 0$ and $p_{i_n}(e_n) \notin \bigcap \{\mathrm{Ker}(h)\colon h \in \mathrm{Hom}_R(A_{i_n}, A_j)\}$ for infinitely many $n < \omega$. In that case, $hp_{i_n}\varphi(e_n) \neq 0$ and so $\varphi(p_j hp_{i_n}) = p_j hp_{i_n}\varphi(e_n) = hp_{i_n}\varphi(e_n) \neq 0$ for some $h \in \mathrm{Hom}(A_{i_n}, A_j)$. However, $\varphi|p_j E^\omega$ is an $\mathrm{End}_R(A_j)$-homomorphism to A_j and hence $\varphi|p_j E^{\omega - m} = \{0\}$ for some m. This is a contradiction.

Next we prove the converse under the additional assumptions. Suppose that (1) does not hold. Then, there exists an infinite subfamily $\{i_n\colon n < \omega\}$ of I (where $i_m \neq i_n$ for $m \neq n$) and $x_n \in A_{i_n}$ $(n < \omega)$ such that if $j \in I$ and $h_n \in \mathrm{Hom}_R(A_{i_n}, A_j)$ $(i < \omega)$ then $h_n(x_n) = 0$ for almost all $n < \omega$. Let $\bar{x}_n(i_n) = x_n$ and $\bar{x}_n(i) = 0$ for $i \neq i_n$ for each $n < \omega$, then $\bar{x}_n$ belongs to M_R by the property (2). Also by (2), there exists an R-ω_1-sheaf $(\mathscr{S}, \varrho)$ over B such that $\mathscr{S}^\wedge \simeq M_R$. (We identify $\mathscr{S}^\wedge$ with M_R.) For $f \in E^\omega$ and $i \in I$, $p_i f(n) \in \bigoplus_{F \in \mathscr{F}} \langle h \cdot \pi_F\colon h \in \mathrm{Hom}_R(\mathscr{S}/F, A_i)\rangle$ by Theorem 1.5. (Here $\mathscr{F}$ is the set of all countably complete ultrafilters of B. $\pi_F(\bar{x}_n) \neq 0$ iff F is the principal ultrafilter generated by the singleton $\{i_n\}$. Hence, $p_i f(n)(\bar{x}_n) = h_n p_{i_n}(\bar{x}_n)$ for some $h_n \in \mathrm{Hom}_R(A_{i_n}, A_i)$ and so $p_i f(n)(\bar{x}_n) = 0$ for almost all n, by the slenderness of A_i. Since $\mathrm{supp}\, f(n)(x_n) \in B$ $(n < \omega)$ and B is a countably complete subalgebra of $P(I)$, there exists a partition $\{b_m\colon m < \omega\}$ of 1 such that $\varrho^1_{b_m}(f(n)(\bar{x}_n)) = 0$ for almost all $n < \omega$. Therefore, $\sum_{n<\omega} f(n)(\bar{x}_n) \in \mathscr{S}^\wedge (= M_R)$. Now, we can define $\varphi\colon E^\omega \to {}_E M$ by: $\varphi(f) = \sum_{n<\omega} f(n)(\bar{x}_n)$. Since $\varphi(e_n) \neq 0$ for all n, it is enough to show that φ is an E-homomorphism for getting the conclusion. $\varphi(f+g) = \varphi(f) + \varphi(g)$ clearly holds. For a $\sigma \in E$, let $p_i\sigma = \sum_{k=0}^{n-1} h_k \pi_{F_k}$ where $h_k \in \mathrm{Hom}_R(\mathscr{S}/F_k, A_i)$ and $F_k \neq F_j$ for $k \neq j$. For an $f \in E^\omega$, let $\{b_m\colon m < \omega\}$ be a partition as above. There exists a unique m such that $b_m \in F_k$ for each $0 \leqslant k \leqslant n-1$. Hence, $p_i\sigma(\sum_{n<\omega} f(n)(\bar{x}_n)) = \sum_{n<\omega} p_i\sigma(f(n)(\bar{x}_n)) = p_i \sum_{n<\omega}\sigma(f(n)(\bar{x}_n))$ for every i. This implies $\sigma(\sum_{n<\omega} f(n)(\bar{x}_n)) = \sum_{n<\omega}\sigma(f(n)(\bar{x}_n)) = \sum_{n<\omega}\sigma \cdot f(n)(\bar{x}_n)$, i.e. $\sigma \cdot \varphi(f) = \varphi(\sigma \cdot f)$.

COROLLARY 4.9. *Let* A_i $(i \in I)$ *be slender groups and*

$$\tilde{\prod}_{i \in I} A_i = \{x \in \prod_{i \in I} A_i\colon \mathrm{supp}\, x \text{ is countable}\}.$$

Then, $\prod_{i \in I} A_i$ *is endo-slender iff* $\tilde{\prod}_{i \in I} A_i$ *is endo-slender iff for any infinite* $X \subset I$ *and nonzero* $x_i \in A_i$ $(i \in X)$ *there exists a* $j \in I$ *such that* $x_i \notin \bigcap \{\mathrm{Ker}(h)\colon h \in \mathrm{Hom}_Z(A_i, A_j)\}$ *for infinitely many* $i \in X$.

Let $B = \{X \subset I\colon X \text{ or } I - X \text{ is countable}\}$, then $\tilde{\prod}_{i \in I} A_i$ and B satisfy the conditions of Theorem 4.8. Now, the corollary follows from Theorem 4.8.

A torsion free group A of rank 1 is isomorphic to a subgroup of Q and identified by its type $\boldsymbol{t}(A)$. We refer the reader to [18, § 85] for the definition of type. We only remark that, for torsion free groups of rank 1 A and A', $\boldsymbol{t}(A) \leqslant \boldsymbol{t}(A')$ iff $\mathrm{Hom}_Z(A, A') \neq \{0\}$ iff A is isomorphic to a subgroup of A'.

COROLLARY 4.10. *Let* A *be a direct product* $\prod_{i \in I} A_i$ *of torsion free groups* A_i $(i \in I)$ *of rank* 1. *Then*:

(1) *If the set* $\{i\colon A_i$ *is isomorphic to* $Q\}$ *is nonempty and finite, then* A *is not endo-slender.*

(2) *If the set* $\{i\colon A_i$ *is isomorphic to* $Q\}$ *is nonempty and infinite, then* A *is endo-slender.*

(3) *Suppose that* A *is reduced, i.e. the set* $\{i\colon A_i$ *is isomorphic to* $Q\}$ *is empty. Then,* A *is endo-slender iff for any infinite subset* X *of* I *there exists a* $j \in I$ *such that* $\{i \in X\colon \boldsymbol{t}(A_i) \leqslant \boldsymbol{t}(A_j)\}$ *is infinite.*

Proof. Propositions 4.2, 4.3 and 4.4 imply (1) and (2). Since a reduced torsion free group of rank 1 is slender and endo-slender by Proposition 1, (3) follows from Theorem 4.8.

Let $R_{\boldsymbol{t}}$ be a torsion free group of rank 1 of type $\boldsymbol{t}$, $T_0 = \{\boldsymbol{t}\colon R_{\boldsymbol{t}}$ is reduced$\}$ and $T_1 = \{\boldsymbol{t}\colon R_{\boldsymbol{t}}$ is p-reduced for every prime $p\}$, where A is p-reduced if $\bigcap_{n<\omega} p^n A = \{0\}$ Considering characteristics of the form $(\infty, \ldots, \infty, 0, 0 \ldots)$, we can see that $\prod_{\boldsymbol{t} \in T_0} R_{\boldsymbol{t}}$ is not endo-slender. On the other hand, T_1 consists of all types which contain the characteristics $f \in {}^\omega\omega$, where $p_0 = 2$, $p_1 = 3$ and so on. Denote the type containing f by $[f]$ for $f \in {}^\omega\omega$. Then, for $f, g \in {}^\omega\omega$, $[f] \leqslant [g]$ holds iff $f(i) \leqslant g(i)$ for almost all i. Since for any countable family $\{f_n \in {}^\omega\omega\colon n < \omega\}$ there exists an $f \in {}^\omega\omega$ such that $f_n(i) \leqslant f(i)$ for almost all i $(n < \omega)$ (see [24, p. 260]), $\prod_{\boldsymbol{t} \in T_1} R_{\boldsymbol{t}}$ is endo-slender.

5. Endo-slenderness of Boolean powers. Since the content of this section is related to properties of Boolean algebras (abbreviated by Ba), we first introduce some notions concerning Ba's.

A map $h\colon B \to B'$ is a homomorphism for Ba's B and B', if $h(0) = 0$, $h(b \wedge c) = h(b) \wedge h(c)$ and $h(b \vee c) = h(b) \vee h(c)$ for $b, c \in B$. (Note that we do not require $h(1) = 1$.) In addition if $\bigvee X = b$ implies $\bigvee h(X) = h(b)$ for each $b \in B$ and $X \subset B$ of cardinality less than $\varkappa$, h is said to be $\varkappa$-complete. (We use the word "countably complete" instead of "ω_1-complete".) h is complete, if h is $\varkappa$-complete for any $\varkappa$. It should be noted that there exists a complete homomorphism from a noncomplete Ba. For a nonzero $b \in B$, $[0, b] = \{c \in B\colon c \leqslant b\}$ is also a nontrivial Ba. B is homogeneous, if $[0, b]$ is isomorphic to B for every nonzero $b \in B$. B is weakly homogeneous, if 0 and 1 are the only elements which are fixed under all automorphisms of B. Ba's B and C are totally different, if for any nonzero $b \in B$ and $c \in C$ $[0, b]$ and $[0, c]$ are not isomorphic. B is rigid, if the identity is the only automorphism of B. In other words, B is rigid, if $[0, b]$ and $[0, c]$ are totally different for any pairwise disjoint nonzero elements b and c. B satisfies (μ, ν)-distributivity, if $\bigvee_{f \in {}^\mu\nu} \bigwedge_{\alpha < \varkappa} b_{\alpha f(\alpha)} = 1$ holds for any systems $\{b_{\alpha\beta}\colon \alpha < \mu, \beta < \nu\}$ so that $\bigvee_{\beta<\nu} b_{\alpha\beta} = 1$

$(\alpha<\mu)$. The canonical completion $\bar{B}$ of a Ba B is a cBa such that $\bar{B}$ includes B as a subalgebra and for any nonzero $c\in\bar{B}$ there exists a nonzero $b\in B$ with $b\leqslant c$.

Let M_R be an R-module and $\varkappa$ a cardinal greater than both the cardinalities of M_R and R. For a $\varkappa$-complete Ba B, the Boolean power $M_R^{(B)}$ consists of all f's such that $f\colon M_R\to B$, $\bigvee_{u\in M}f(u)=1$ and $f(u)\wedge f(v)=0$ for $u\neq v$, where $(f+g)(u)=\bigvee_{u=v+w}f(v)\wedge g(w)$ and $(f\cdot r)(u)=\bigvee_{v\cdot r=u}f(v)$ for $r\in R$. We also denote f by $\sum_{u\in M}u\cdot 1_{f(u)}$. It is easy to see that there exists an R-$\varkappa$-sheaf $(\mathscr{S},\varrho)$ over B such that $\mathscr{S}^\wedge\simeq M_R^{(B)}$. Let $\{f_\alpha\colon \alpha<\lambda\}\subset M_R^{(B)}$ be a family such that $\lambda<\varkappa$ and there exists a partition $\{b_\beta\colon \beta<\mu\}$ of 1 with the following: $\mu<\varkappa$ and $b_\beta\leqslant f_\alpha(0)$ for almost all α and each β. Then, the given condition corresponds to that after Definition 1.4 and hence we get an element $\sum_{\alpha<\lambda}f_\alpha$ of $M_R^{(B)}$.

The first theorem shows that endo-slenderness occurs without any relationship to either slenderness of abelian groups or infinite direct sums.

THEOREM 5.1. *Let A be a reduced torsion free abelian group and B a homogeneous* Ba *which is not $(\omega,2)$-distributive. Let S be a subgroup of $A^{(\bar{B})}$ that satisfies the following: For any complete homomorphism $h\colon B\to B$ and $\sum_{\lambda\in\Lambda}a_\lambda 1_{b_\lambda}\in S$, $\sum_{\lambda\in\Lambda}a_\lambda 1_{h(b_\lambda)}\in S$ holds, where $\bar{B}$ is the canonical completion of B and $\{b_\lambda\colon \lambda\in\Lambda\}(\subset B)$ is a partition of* 1. *Then, S is endo-slender. Consequently, if B is $\varkappa$-complete and $|A|<\varkappa$, then $A^{(B)}$ is endo-slender.*

Proof. Since B does not satisfy $(\omega,2)$-distributivity, there exists a system $\{b_{mn}\colon m<\omega,\ n<2\}$ such that $b_{m0}\vee b_{m1}=1$, $b_{m0}\wedge b_{m1}=0$ and $\bigwedge_{m<\omega}b_{mf(m)}=0$ for all $f\in{}^\omega 2$ by the homogeneity of B. Suppose that on the contrary an E-homomorphism $\varphi\colon E^\omega\to S$ satisfies $\varphi(e_n)\neq 0$ for any $n<\omega$. Choose $j_m<\omega$ so that $j_m!\,|\,\varphi(e_m)$ but $(j_m+1)!\nmid\varphi(e_m)$. Next pick a $v_m\in A$ so that $b\leqslant\varphi(e_m)(v_m)$ for some nonzero b and $j_m!\,|\,v_m$ but $(j_m+1)\nmid v_m$. By the homogeneity of B there exist $k_m<\omega$ and $\sigma_m\in E$ such that $\sigma_m\varphi(e_m)(0)=b_{m0}$, $\sigma_m\varphi(e_m)(v_m)=b_{m1}$, $2k_m+1\leqslant k_{m+1}$ and $j_m<k_m$. Now,

$$\begin{aligned}\varphi\left(\textstyle\sum_{m<\omega}k_m!\,\sigma_m e_m\right)&=\varphi\left(\textstyle\sum_{m=0}^{n-1}k_m!\,\sigma_m e_m+\sum_{m\geqslant n}k_m!\,\sigma_m e_m\right)\\&=\textstyle\sum_{m=0}^{n-1}k_m!\,\varphi(\sigma_m e_m)+k_n!\,\varphi(\sigma_n e_n)+k_{n+1}!\,\varphi\left(\sum_{m\geqslant n}k_m!/k_n!\cdot\sigma_m e_m\right)\end{aligned}$$

Let $a\in S$ and $f\in{}^n 2(={}^{\{0,1,\dots,n-1\}}\{0,1\})$ such that

$$\varphi\left(\textstyle\sum_{m<\omega}k_m!\,\sigma_m e_m\right)(a)\wedge\bigwedge_{m=0}^{n-1}b_{mf(m)}\neq 0.$$

We claim

(†): $k_n!\,|\,a-\sum_{m=0}^{n-1}f(m)k_m!\,v_m$ but $k_n!\nmid a-\sum_{m=0}^{n-1}g(m)k_m!\,v_m$ for any $g\notin{}^n 2$ with $g\neq f$.

In case $n=0$, $k_0!\,|\,a$, hence the claim holds. Assume the claim is true for n. $k_{n+1}!\,|\,a-\sum_{m=0}^{n}f(m)k_m!\,v_m$ holds. Let $g(i)\neq f(i)$ for some $i<n$. By the inductive hypothesis, $k_n!\nmid a-\sum_{m=0}^{n-1}g(m)k_m!\,v_m$ and $k_n!\,|\,g(n)k_n!\,v_n$ and hence

$$k_{n+1}!\nmid a-\textstyle\sum_{m=0}^{n}g(m)k_m!\,v_m.$$

On the other hand, if $g(n)\neq f(n)$ and $g(i)=f(i)$ for every $i<n$, then $k_{n+1}!\nmid a-\sum_{m=0}^{n}g(m)k_m!\,v_m$ for $k_{n+1}!\nmid k_n!\,v_n$.

The claim (†) implies $\varphi\left(\sum_{m<\omega}k_m!\,\sigma_m e_m\right)(a)\leqslant\bigwedge_{m=0}^{n-1}b_{mf(m)}$. Since this holds for every $n<\omega$, $\varphi\left(\sum_{m<\omega}k_m!\,\sigma_m e_m\right)(a)=0$ for any $a\in S$, which contradicts $\varphi\left(\sum_{m<\omega}k_m!\,\sigma_m e_m\right)\in S\subset A^{(\bar{B})}$.

COROLLARY 5.2. *Let J_p be the group of p-adic integers and B a homogeneous $2^{(\aleph_0)^+}$-complete* Ba. *Then, the Boolean power $J_p^{(B)}$ is endo-slender, iff B satisfies $(\omega,2)$-distributivity.*

Proof. Let $E=\mathrm{End}_Z(J_p^{(B)})$. If B satisfies $(\omega,2)$-distributivity, then $J_p^{(B)}$ is algebraically compact [1, or 10, Proposition 4]. $J_p^{(B)}$ is torsionfree and reduced and hence any Cauchy sequence converges in $\mathbf{Z}$-adic topology. As is well known, $J_p^{(B)}$ is not endo-slender, i.e. $\varphi(f)=\sum_{n<\omega}n!\,f(n)(u)$ (where $u(1)=1$) defines an E-homomorphism from E^ω to $J_p^{(B)}$ and $\varphi(e_n)\neq 0$ $(n<\omega)$, where the infinite sum is taken in the $\mathbf{Z}$-adic topology. The converse immediately follows from Theorem 5.1.

THEOREM 5.3. *Let B be a $\varkappa$-complete* Ba *with the property $(*\varkappa)$ below and M_R a primitively endo-slender module of cardinality less than $\varkappa$. Then, $M_R^{(B)}$ is also endo-slender.*

$(*\varkappa)$ *For any family of nonzero elements $\{b_n\colon n<\omega\}$ there exist $b\neq 0$, an infinite subset I of ω and h_n $(n\in I)$ such that $h_n\colon [0,b_n]\to[0,b]$ is a $\varkappa$-complete homomorphism and $h_n(b_n)=b$.*

LEMMA 5.4. *Let B be a $\varkappa$-complete* Ba, *M_R a primitively slender module of cardinality less than $\varkappa$ and $h\colon R^\omega\to M_R^{(B)}$ an R-homomorphism. Then, for any non-zero $b\in B$ the set $\{n\colon h(e_n)$ is a non-zero constant on $b\}$ is finite, where $x\in M_R^{(B)}$ is said to be a non-zero constant on B if there exists a nonzero $u\in M_R$ such that $b\leqslant x(u)$.*

Proof. Suppose the negation of the conclusion. Without any loss of generality we may assume $h(e_n)(u_n)=1$ and $u_n\neq 0$ for all $n<\omega$. We define $k_n<\omega$ and $s_n\in I_{k_n}$ by induction. Let $k_0=0$ and $s_0\in I_{k_0}$ so that $u_0s_0\neq 0$. There exists k_{n+1} such that $k_n<k_{n+1}$ and $K_{k_n}-\sum_{i=0}^{n}u_is_i\subset K_{k_{n+1}}$. Next, choose $s_{n+1}\in I_{k_{n+1}}$ so that $u_{n+1}s_{n+1}\neq 0$. There exist $0\neq c\in B$ and $u\in K_{k_{m-1}}$ such that $h\left(\sum_{n<\omega}e_ns_n\right)(u)=c\neq 0$. For $n\geqslant m$, $c\leqslant h\left(\sum_{i\geqslant n}e_is_i\right)\left(u-\sum_{i=0}^{n-1}u_is_i\right)$ holds, because $h\left(\sum_{i\geqslant n}e_is_i\right)=h\left(\sum_{i<\omega}e_is_i\right)-$ $-\sum_{i=0}^{n-1}h(e_is_i)$. On the other hand, $h\left(\sum_{i\geqslant n}e_is_i\right)\in M^{(B)}I_{k_n}$, i.e. $u-\sum_{i=0}^{n-1}u_is_i\in M\cdot I_{k_n}$. Since $u-\sum_{i=0}^{n-1}u_is_i\in K_{k_n}$, $u-\sum_{i=0}^{n-1}u_is_i=0$. The same proposition holds for n and hence $u_ns_n=0$, which is a contradiction.

Proof of Theorem 5.3. Let $\mathbf{E}=\mathrm{End}_R(M_R^{(B)})$ and $h\colon \mathbf{E}^\omega\to{}_{\mathbf{E}}M^{(B)}$ be an $\mathbf{E}$-homomorphism such that $h(e_n^*)\neq 0$ for all $n<\omega$. Then, $h(e_n^*)$ is nonzero constant on some nonzero b_n. Since a $\varkappa$-complete homomorphism from B to B naturally induces an endomorphism of M_R, $(*\varkappa)$ implies the existence of an infinite subset $I\subset\omega$, $b\neq 0$, $\sigma_n\in\mathbf{E}$ $(n\in I)$ such that $\sigma_nh(e_n^*)$ is nonzero constant on b. $\mathrm{End}_R(M_R)(=E)$ is naturally a subring of $\mathbf{E}$. Define $\varphi\colon E^\omega\to\mathbf{E}^\omega$ by: $\varphi(f)=\sum_{n\in I}f(n)\sigma_ne_n^*$ $(n\in I)$, then φ is an E-homomorphism and $h\varphi(e_n)=h(\sigma_ne_n^*)$ $=\sigma_nh(e_n^*)$ $(n\in I)$, which contradicts Lemma 5.4.

THEOREM 5.5. (*Assume the nonexistence of an inner model with a measurable cardinal.*) *Let B be an (ω, ∞)-distributive* cBa *and M_R a primitively slender and primitively endo-slender module. Then, $M_R^{(B)}$ is endo-slender iff B satisfies $(*\varkappa)$ for an arbitrary $\varkappa$ in Theorem* 5.3.

COROLLARY 5.6. *Let B be an (ω, ∞)-distributive* cBa *and A a countable reduced torsionfree abelian group. Then, the abelian group $A^{(B)}$ is endo-slender iff B satisfies $(*\omega_1)$ of Theorem* 5.3.

Since we use Boolean valued models $V^{(B)}$ for cBa's B [2, 24, 30] in the sequel, we state some definitions and preliminaries.

For an element x, $x^{\vee} \in V^{(B)}$ is defined by: $\mathrm{dom}(x^{\vee}) = \{y^{\vee}: y \in x\}$ and $x^{\vee}(y^{\vee}) = 1$ for $y \in x$. We assume $V^{(B)}$ is separated, i.e. $[\![x = y]\!]^{(B)} = 1$ implies $x = y$. We omit the superscript of $[\![\ldots]\!]^{(B)}$ if no confusion will occur. We say "..." holds in $V^{(B)}$, if $[\![$"..."$]\!] = 1$ holds. For an element $x \in V^{(B)}$, $x^{\wedge} = \{y: [\![y \in x]\!] = 1\}$. Suppose that T is an $R^{\vee}$-module in $V^{(B)}$. Let $\mathscr{S}(b) = \{x \in T^{\wedge}: \neg b \leqslant [\![x = 0]\!]\}$ and $\varrho_c^b(x)$ be defined by: $\neg c \leqslant [\![\varrho_c^b(x) = 0]\!]$ and $c \leqslant [\![\varrho_c^b(x) = x]\!]$. Then, $(\mathscr{S}, \varrho)$ is an R-sheaf over B. Conversely, for any R-sheaf over B there exists a $T \in V^{(B)}$ such that T is an $R^{\vee}$-sheaf in $V^{(B)}$ and the corresponding R-sheaf is isomorphic to $(\mathscr{S}, \varrho)$. Therefore, we identify T and $(\mathscr{S}, \varrho)$ and our notation "$^{\wedge}$" is consistent since $T^{\wedge} \simeq \mathscr{S}^{\wedge}$. Especially if $T = M_R^{\vee}$, $T^{\wedge}$ is isomorphic to $M_R^{(B)}$ as an R-module. This identification is important in the following argument. (We have used such an argument in [7, 11, 15] and such a presentation in [8, 9].) Under this identification, we use the notion "proper sequence" of elements of $M_R^{(B)}$. (See the definition before Theorem 1.5.)

LEMMA 5.7. *Assume the nonexistence of an inner model with a measurable cardinal. If B is an (ω, ∞)-distributive* cBa, *then any countably complete homomorphism from B to B is complete.*

Proof. Suppose that $h: B \to B$ is a countably complete homomorphism which is not complete. Let $\varkappa$ be the least cardinal such that h is not $\varkappa$-complete. By the (ω, ∞)-distributiveness, h induces a non-principal countably complete ultrafilter of $(P\varkappa)^{\vee}$ in $V^{(B)}$. Therefore, as in a usual case in [24, pp. 447–450] we get an inner model with a measurable cardinal.

Proof of Theorem 5.5. The if-part follows from Theorem 5.3. To show the contrapositive, we investigate endomorphisms of $M_R^{(B)}$. Let $\sigma \in \mathrm{End}_R(M_R^{(B)}) (= \mathbf{E})$, then there exists $\tilde{\sigma}$ such that $\tilde{\sigma}: (M_R^{(B)})^{\vee} \to M_R^{\vee}$ is an $R^{\vee}$-homomorphism and $\tilde{\sigma}(x^{\vee}) = \sigma(x)$ in $V^{(B)}$ for $x \in M_R^{(B)}$. Since M_R is primitively slender, $M_R^{\vee}$ is also primitively slender in $V^{(B)}$. By the (ω, ∞)-distributivity of B, $B^{\vee}$ is countably complete and $(M_R^{(B)})^{\vee}$ has an $R^{\vee}$-ω_1-sheaf structure $\mathscr{S}$ over $B^{\vee}$ in $V^{(B)}$. Therefore, $\mathrm{Hom}_{R^{\vee}}((M_R^{(B)})^{\vee}, M_R^{\vee}) = \oplus_{F \in \mathscr{F}} \mathrm{Hom}_{R^{\vee}}(\mathscr{S}/F, M_R^{\vee})$ in $V^{(B)}$ by Theorem 1.5. If $c \leqslant [\![F \in \mathscr{F}]\!]$, define $h_F: B \to B$ by: $h_F(b) = c \wedge [\![b^{\vee} \in F]\!]$ for each $b \in B$. Then, h_F is a countably complete homomorphism with $h_F(b) = c$. Let $\{b_n: n < \omega\}$ be a family of nonzero elements which meets the negation of $(*\varkappa)$ for some $\varkappa$. Pick a nonzero $a \in M_R$ and let $a_n \in M_R^{(B)}$ so that $a_n(b_n) = a$ and $a_n(\neg b_n) = 0$. We claim that

$f(n)(a_n)$ $(n < \omega)$ is a proper sequence for any $f \in \mathbf{E}^{\omega}$. We can take $\tilde{f}$ so that $\tilde{f}(n^{\vee}) = (f(n))^{\sim}$ in $V^{(B)}$ for $n < \omega$. (Here, the symbol "$^{\sim}$" is used in a little bit different meaning to the one in the above usage, but confusion is harmless.) What we need to show is that $\{n < \omega: \tilde{f}(x)((a^{\vee})_n) \neq 0\}$ is finite in $V^{(B)}$. Suppose not. By the (ω, ∞)-distributivity of B, there exists an infinite set I such that

$$[\![\tilde{f}((a^{\vee})_n) \neq 0 \text{ for } n \in I^{\vee}]\!] = b \neq 0 .$$

By the above, there exists a partition P_n of b for each n such that

$$p \leqslant [\![\tilde{f}(n^{\vee}) = \textstyle\sum_{i=0}^{m_p} h_{pi} \pi_{F_{pi}}]\!]$$

for $p \in P_n$ with suitable $m_p < \omega$, $h_{pi} \in (\mathrm{Hom}_{R^{\vee}}(\mathscr{S}/F_{pi}, M_R^{\vee}))^{\wedge}$, $F_{pi} \in \mathscr{F}^{\wedge}$. Since $p \leqslant \bigvee_{i=0}^{m_p} [\![b_n^{\vee} \in F_{pi}]\!]$, we get a refinement Q_n of P_n for each $n \in I$ such that there exists a countably complete homomorphism h: $[0, b_n] \to [0, q]$ with $h(b_n) = q$, for each $q \in Q_n$. Now, using (ω, ∞)-distributivity, we get a nonzero b^* so that there exists a countably complete homomorphism h: $[0, b_n] \to [0, b^*]$, for each $n \in I$. Lemma 5.7 implies that h is complete and hence we have a contradiction which proves the claim. Therefore, we can define a map φ: $\mathbf{E}^{\omega} \to M_R^{(B)}$ by: $\varphi(f) = \sum_{n<\omega} f(n)(a_n)$. It is straightforward to see $\varphi(f+g) = \varphi(f) + \varphi(g)$. Since $\varphi(e_n) = a_n \neq 0$ $(n < \omega)$, what we need to show is that $\varphi(\sigma f) = \sigma\varphi(f)$ for $\sigma \in \mathbf{E}$. As we have shown, $\{n: \tilde{f}(n)((a^{\vee})_n) \neq 0\}$ is finite in $V^{(B)}$ and hence $\sum_{n<\omega} \tilde{f}(n)((a^{\vee})_n) = \sum_{n<\omega} f(n)(a_n)$. Since $\sum_{n<\omega} \tilde{f}(n)((a^{\vee})_n)$ is just a finite sum, $\tilde{\sigma}(\sum_{n<\omega} \tilde{f}(n)((a^{\vee})_n)) = \sum_{n<\omega} \tilde{\sigma} \cdot \tilde{f}(n)((a^{\vee})_n)$ in $V^{(B)}$ for $\sigma \in \mathbf{E}$. $\tilde{\sigma} \cdot \tilde{f}(n^{\vee}) = (\sigma \cdot f)^{\sim}(n^{\vee})$ holds and so $\varphi(\sigma f) = \sigma\varphi(f)$ for $\sigma \in \mathbf{E}$. Now, we have shown the existence of an $\mathbf{E}$-homomorphism φ: $\mathbf{E}^{\omega} \to_{\mathbf{E}} M^{(B)}$ with $\varphi(e_n^*) \neq 0$ $(n < \omega)$.

Proof of Corollary 5.6. Since A is primitively slender and primitively endo-slender by Examples and Remarks 2.4 (1) and Proposition 4.1, the if-part follows from Theorem 5.3. The countable completeness can be proved without Lemma 5.7 as in the proof of Theorem 5.5 hence we get the claim.

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On partitioner-representability of Boolean algebras

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Abstract. It is proved that — consistently with the Martin Axiom — the power-set algebra $P(\omega_1)$ may not be partitioner-representable.

0. Baumgartner and Weese in [B–W] introduced the notion of partitioner-representability of Boolean algebras: if E is m.a.d. (a maximal, almost disjoint family of subsets of ω) then a set $A \subseteq \omega$ is called a *partitioner of* E if for each $e \in E$ either $e \subseteq_* A$ or $e \cap A =_* \emptyset$ (i.e. $e \setminus A$ or $e \cap A$, respectively is finite); the union, intersection and difference of partitioners is again a partitioner, and hence the family $\mathrm{Pt}(E)$ of all the partitioners of E is a Boolean subfield of $P(\omega)$. A Boolean algebra is said to be *partitioner-representable* if for some m.a.d. E it is isomorphic to the factor algebra $\mathrm{Pt}(E) \bmod J$ where J is the ideal generated by fin (the finite sets) and E.

The finite sets and finite unions $e_1 \cup \ldots \cup e_n$ (and their complements) are called *trivial partitioners*. Thus, $\mathrm{Pt}(E) \bmod J$ may be called the *algebra of non-trivial partitioners* of E.

The fundamental theorem in [B–W] (see also [F–Z$_1$]) says that — under CH (the continuum hypothesis) — each algebra of cardinality $\leqslant c = 2^\omega$ is partitioner-representable. A question arises if the same is true if CH is replaced by MA (Martin Axiom). In this note we prove that this is not the case:

THEOREM. *There is a generic extension of the constructible universe in which* MA *holds and $c = \varkappa$, for a given regular $\varkappa > \omega_1$, and the algebra $P(\omega_1)$ is not partitioner-representable.*

Originally, we had $c = \omega_2$ in our model.

We are grateful to the referee who pointed out how to get the more general case.

In [F–Z$_1$] it is proved that partitioner-representability of $P(\omega_1)$ implies the existence of Q-sets. From the theorem it follows that the converse is not true.

The idea of proof is, roughly, as follows. Extend $V = L$ (the constructible sets) via a finite support, c.c.c.-iteration of length $\varkappa$ and assume, for contradiction, that $P(\omega_1)$ is representable in $V[G]$. Each (ω_1, ω_1)-chain $C = \langle\{x_\alpha\}; \{y_\beta\}\rangle$, gives then rise to a species of a Hausdorff gap $H = \langle\{a_\alpha\}; \{b_\beta\}\rangle$, which — at a given stage of iteration — cannot be filled. Now, there are two forcing notions Q