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On the Hausdorff dimension of a Julia set with a rationally indifferent periodic point

by

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Abstract. Suppose that $T: \mathbf{C} \to \mathbf{C}$ is a polynomial with a rationally indifferent periodic point ω, such that the corresponding filled-in Julia set $K(T)$ is connected and locally connected. We prove that if A is a connected component of $\operatorname{Int} K(T)$ which is contained in the basin of immediate attraction to ω and the intersection of ∂A and the closure of the trajectory of critical points consists exactly of ω then either ∂A is a real-analytic curve or the Hausdorff dimension of ∂A is strictly greater than 1.

Introduction. Notations. The local topological structure of the behaviour of a rational map $T: \bar{\mathbf{C}} \to \bar{\mathbf{C}}$ around a rationally indifferent periodic point was already described by Leau [L] at the end of the last century. Next results can be found among others in [F1], [F2], [C] and [DH]. Here we want to continue these studies looking at the fractal properties of the Julia set around such a point. A general setting in which our method works can be described as follows. Let $J \subset \bar{\mathbf{C}}$ be a compact nowhere dense set, let $J \subset U \subset \bar{\mathbf{C}}$ be an open neighborhood of J whose complement in $\bar{\mathbf{C}}$ contains at least 3 points and let $T: U \to \bar{\mathbf{C}}$ be a holomorphic map such that the following conditions are satisfied:

(a) $T(J) = J$.

(b) $T'(z) \neq 0$ for every $z \in J$.

(c)

$$\exists_{\omega \in J} \left[\left(\forall_{n \geqslant 0} T^n(z) \in U\right) \Rightarrow \left(z \in J \text{ or } \lim_{n\to\infty} T^n(z) = \omega\right)\right] \text{ and } T(\omega) = \omega.$$

(d) If $z \in U \setminus J$ and $T^n(z) \in U$ for every $n \geqslant 0$, then there exists $r(z) > 0$ such that $T^n\left(B(z, r(z))\right) \subset U$ for every $n \geqslant 0$ and $T^n | B(z, r(z))$ converges uniformly to ω.

(e) There is an open neighbourhood $W \subset U$ of ω such that if $T^n(z) \in W \cap J$ for every $n \geqslant 0$, then $z = \omega$.

In order to prove more advanced results we will also assume (see Sections 4 and 5) that J is the boundary of an open topological disc $A \subset \bar{\mathbf{C}}$ and the

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following condition is satisfied:

$$T(U \cap A) \subset A. \tag{f}$$

Our main result (see Theorem 5.2) is that if moreover J is a Jordan curve and condition (f) is satisfied for both connected components of $\bar{\mathbf{C}}\setminus J$ then J is either a real-analytic curve or the Hausdorff dimension of J is strictly greater than 1. Results of this kind have been proved under different assumptions. Let us mention here the papers [B], [S1], [P3], [PUZ, I, II], [Z2] where, actually in turn, weaker and weaker requirements were needed. In all of them, however, the open topological disc A was assumed to be an RB-domain (see [PUZ, I], p. 5), that is, the formula

$$\bigcap_{n=0}^{\infty} T^{-n}(U \cap \bar{A}) = \partial A = J \tag{1}$$

was satisfied. Under our assumptions (a)–(f) this need not be any longer true and some new ideas are needed. In particular, the map $F\colon S^1 \to S^1$ constructed in Section 4 is not necessarily expanding and because of that we use the jump transformation F^* which is already a piecewise expanding mapping but with an infinite number of pieces of monotonicity. As the tools needed to prove Theorem 5.2 we also obtain some results which concern Hausdorff and conformal measures on J. We show that if t is the Hausdorff dimension of J then the t-dimensional Hausdorff measure of J is finite and there exists a t-conformal measure for $T\colon J \to J$. However, note that if $|T'(\omega)| > 1$ then (1) is satisfied and we are exactly in the setting of [B]–[Z2]. Theorem 5.2 then follows from these papers. Therefore we are essentially interested in the case when $|T'(\omega)| = 1$.

We would also like to mention the paper [Z1]. It concerns the global situation, i.e. rational maps of the Riemann sphere. One of the main results says that the Hausdorff dimension of the Julia set is greater than the Hausdorff dimension of the maximal measure if and only if the rational map is not critically finite with parabolic orbifold. All maps with parabolic orbifold are classified in [DH], §9 (comp. also [Z1], §1) and their Julia set is either the whole sphere $\bar{\mathbf{C}}$, a real-analytic interval or a real-analytic closed curve. If now the rational map is a polynomial then the Hausdorff dimension of the maximal measure is equal to 1 (see [P1]), the Julia set does not coincide with $\bar{\mathbf{C}}$ and Zdunik's result can be reformulated as follows:

Either the Hausdorff dimension of the Julia set is greater than 1 *or the Julia set is a real-analytic* 1-*dimensional manifold.*

Since, however, the Julia set of a polynomial is the boundary of the basin of attraction to infinity which is an RB-domain, this result also fits to the setting of [B]–[Z2]. Note also that although our Corollary 6.4 requires some additional assumptions its statement is stronger than Zdunik's since it says that if the Julia set is not a geometric circle, then not only is the Hausdorff dimension of the Julia set greater than 1 but so is already the Hausdorff dimension of the boundary of the basin of immediate attraction to the rationally indifferent periodic point under consideration.

Concluding this introduction we would like to notice that a large class of examples, which in fact have motivated our work, for which conditions (a)–(f) are satisfied is described in Section 6.

§ 1. Basic notations and definitions. If $A \subset X$ is a subset of a metric space X and t is a real number then the *t-dimensional outer Hausdorff measure* of A is defined to be

$$H_t(A) = \lim_{\varepsilon \searrow 0} \inf_{\mathscr{U}_\varepsilon} \Big\{ \sum_{W \in \mathscr{U}_\varepsilon} r(W)^t \Big\} \tag{1.1}$$

where $\mathscr{U}_\varepsilon$ ranges over all countable covers of A consisting of open balls of radii less than ε and $r(W)$ denotes the radius of the ball W. The outer measure H_t restricted to the Borel σ-algebra of X becomes a measure. The *Hausdorff dimension* $\mathrm{HD}(A)$ of A is defined to be

$$\sup\{t \in \mathbf{R}\colon\ H_t(A) = \infty\} = \inf\{t \in \mathbf{R}\colon\ H_t(A) = 0\}.$$

If μ is a Borel probability measure on X, then the Hausdorff dimension $\mathrm{HD}(\mu)$ of μ is defined by

$$\mathrm{HD}(\mu) = \inf\{\mathrm{HD}(Y)\colon\ \mu(Y) = 1\}.$$

In the presence of holomorphic dynamics we have a useful formula to express this quantity. More precisely, let $G \subset \bar{\mathbf{C}}$ be an open set and $S\colon G \to \bar{\mathbf{C}}$ a holomorphic map. If $K \subset G$ is an S-invariant compact set: $S(K) \subset K$, and μ is a Borel ergodic probability S-invariant measure of positive entropy (for the definition and basic properties of measure-theoretic entropy of endomorphisms see for example the book [Pa] by W. Parry) on K, then (see for example §3 of [P1] and [Ma])

$$\mathrm{HD}(\mu) = h_\mu(S)/\chi_\mu(S) \tag{1.2}$$

where $h_\mu(S)$ is the measure-theoretic entropy of S with respect to the measure μ and $\chi_\mu(S) = \int \log|S'|\, d\mu$ is the Lyapunov exponent. Note that (1.2) makes sense as by Ruelle's inequality proved in [R] we have $\chi_\mu(S) \geqslant \frac{1}{2} h_\mu(S)$.

If $t \in \mathbf{R}$ and m is a Borel probability measure on K, then m is said to be *t-conformal* for the map $S\colon K \to K$ if and only if

$$m(S(A)) = \int_A |S'|^t\, dm \tag{1.3}$$

for any Borel set $A \subset K$ such that $S|A$ is injective. This is a slight generalization of the notions of conformal measure introduced by S. Patterson [Pat] for Fuchsian groups and D. Sullivan [S2] for rational maps. For an even more general notion of conformal measure and its basic properties the reader is referred to the paper [DU].

Finally, let us formulate a version of the Koebe distortion theorem (see [Hi]).

THE KOEBE DISTORTION THEOREM. *There exists a function* $k\colon [0,1)\to[1,\infty)$ *such that for any* $z\in\mathbf{C}$, *any* $r>0$ *and any univalent holomorphic function* $S\colon B(z,r)\to\mathbf{C}$ *we have*

$$|S'(y)|/|S'(x)|\leqslant k(t)\quad \textit{if only } |y-z|, |x-z|\leqslant tr.$$

We put $K=k(1/2)$.

§ 2. Basic lemmas. In this section we establish some basic analytic and topological properties of a holomorphic mapping $T\colon U\to\bar{\mathbf{C}}$ that satisfies conditions (a)–(e) of Section 1. We fix a Riemannian metric on $\bar{\mathbf{C}}$ and, unless stated otherwise, we consider all distances and derivatives with respect to this metric. Given a connected, simply connected open set $H\subset\bar{\mathbf{C}}$ and an integer $n\geqslant 0$ we say that a holomorphic map $T_v^{-n}\colon H\to U$ is an *inverse branch* of T^n if and only if $T^n(T_v^{-n}(H))$ is well-defined and $T^n\circ T_v^{-n}=\mathrm{Id}|H$. Moreover, we say that *all the inverse branches of* T^n *are well-defined on* H if and only if for every $z\in J$ such that $T^n(z)\in H$ there exists a holomorphic inverse branch $T_v^{-n}\colon H\to U$ of T^n which satisfies $T_v^{-n}(T^n(z))=z$.

LEMMA 2.1. *Let* $V\subset J$ *be an open neighbourhood of* ω. *Then there exists an* $r>0$ *such that for every* $z\in J\backslash V$ *all the inverse branches of* T^n, $n=0,1,2,\ldots$, *are well-defined on* $B(z,2r)$.

Proof. As J is compact, it follows from (b) and (a) that for sufficiently small $\delta>0$, $B(J,\delta)\subset U$ and all the inverse branches of T are well-defined on the balls $B(z,\delta)$, $z\in J$, and therefore also on any connected, simply connected open set contained in $B(J,\delta)$. Take now an open neighborhood $V_1\subset\bar{\mathbf{C}}$ of ω in $\bar{\mathbf{C}}$ such that $\bar{V}_1\cap(J\backslash V)=\emptyset$ and let $0<\eta<\frac{2}{3}\delta$ be so small that

(2.1) $$B(z,\eta)\cap V_1=\emptyset\quad\text{for every } z\in J\backslash V.$$

Fix $y\in J\backslash V$ and suppose that for every $0<\varepsilon\leqslant\eta$ not all the inverse branches T_v^{-n} of T^n are well-defined on $B(y,\varepsilon)$. Therefore for any sequence $\eta\geqslant\varepsilon_n\searrow 0$ there exists an infinite sequence $\{m_n\}_{n=1}^{\infty}$ of positive integers such that

(2.2) $$T_v^{-m_n}(B(y,\varepsilon_n))\not\subset B(J,\eta)$$

for some inverse branch $T_v^{-m_n}$ well-defined on $B(y,\varepsilon_n)$

and

(2.3) $$T^k\big(T_v^{-m_n}(B(y,\varepsilon_n))\big)\subset B(J,\eta)\quad\text{for every } 1\leqslant k\leqslant m_n.$$

Since $T_v^{-m_n}(B(y,\varepsilon_n))$ is connected and $T_v^{-m_n}(y)\in J$, we can find by (2.2) for every integer $n\geqslant 1$ a point $z_n\in T_v^{-m_n}(B(y,\varepsilon_n))$ such that $\mathrm{dist}(z_n,J)=\frac{1}{2}\delta$. By compactness of $\bar{U}\subset\bar{\mathbf{C}}$ we can assume that $z_n\to z$ for some $z\in\bar{U}$. So

(2.4) $$\mathrm{dist}(z,J)=\tfrac{1}{2}\delta.$$

CLAIM. $T^k(z)$ *is well-defined for every integer* $k\geqslant 1$.

For $k=1$ this follows immediately from (2.4) as $B(J,\delta)\subset U$. So, suppose that $T^q(z)$ is well-defined for some $q\geqslant 1$ and take an integer $s\geqslant 1$ so large that $m_s\geqslant q$ and $\mathrm{dist}(T^q(z_s),T^q(z))<\frac{1}{2}\eta$. Therefore, using (2.3) we get

$$\mathrm{dist}(T^q(z),J)\leqslant\mathrm{dist}(T^q(z),T^q(z_s))+\mathrm{dist}(T^k(z_s),J)<\tfrac{1}{2}\eta+\eta=\tfrac{3}{2}\eta<\delta.$$

Hence $T^q(z)\in U$ and therefore $T^{q+1}(z)=T(T^q(z))$ is well-defined. The claim is proved.

As T is defined on U, it follows from the claim that $T^k(z)\in U$ for every $k\geqslant 1$. Thus, in view of (2.4) and (c) we obtain

(2.5) $$\lim_{k\to\infty}T^k(z)=\omega.$$

So, as $z\in U\backslash J$, we can find by (d) an integer $l\geqslant 0$ so large that for every $k\geqslant l$ we have

(2.6) $$T^k\big(B(z,r(z))\big)\subset V_1.$$

Since $\lim_{n\to\infty}z_n=z$, we can find n so large that $m_n\geqslant l$ and $z_n\in B(z,r(z))$. Hence by (2.6), $T^{m_n}(z_n)\in V_1$. On the other hand, $T^{m_n}(z_n)\in B(y,\varepsilon_n)\subset B(y,\eta)$. This contradicts formula (2.1).

Consequently for every $y\in J\backslash V$ we have found $0<\varepsilon(y)\leqslant\eta$ such that all the inverse branches of T^n, $n=0,1,2,\ldots$, are well-defined on $B(y,\varepsilon(y))$. Now for r required in the lemma, it is sufficient to take $\frac{1}{4}$ of the Lebesgue number of the open cover $\{B(y,\varepsilon(y))\}_{y\in J\backslash V}$ of $J\backslash V$. ∎

COROLLARY 2.2. *If* $H\subset B(J\backslash V,2r)$ *is a connected simply connected open set then all the inverse branches of* T^n, $n\geqslant 0$, *are well-defined on* H.

LEMMA 2.3. *If* $z\in J\backslash V$ *and* $\mathscr{B}_n(z)$ *indexes all the holomorphic inverse branches of* T^n *defined on* $B(z,2r)$, $n\geqslant 0$, *then the family* $\{T_v^{-n}\colon B(z,2r)\to\bar{\mathbf{C}}\colon v\in\mathscr{B}_n(z),\ n=0,1,\ldots\}$ *is normal in the sense of Montel. Moreover, all its accumulation points are constant functions and consequently for every* $0\leqslant\gamma<1$

(2.7) $$\lim_{n\to\infty}\max\{\mathrm{diam}\,T_v^{-n}(B(z,2\gamma r))\colon v\in\mathscr{B}_n(z)\}=0.$$

Proof. Since $T_v^{-n}(B(z,2r))\subset U$ for every $n\geqslant 0$ and $v\in\mathscr{B}_n(z)$ and the complement of U in $\bar{\mathbf{C}}$ contains at least three points, the family $\{T_v^{-n}\colon v\in\mathscr{B}_n(z),\ n=0,1,\ldots\}$ is normal in view of Montel's theorem.

Suppose now that there is $z\in J\backslash V$, an increasing sequence $\{n_k\}_{k=1}^{\infty}$ and holomorphic inverse branches $T_v^{-n_k}\colon B(z,2r)\to\bar{\mathbf{C}}$ converging almost uniformly to a holomorphic function $H\colon B(z,2r)\to\bar{\mathbf{C}}$ which is not constant. Thus $H(B(z,2r))$ is an open set and for any point $x\in H(B(z,2r))$ there exists $m\geqslant 1$ such that $x\in T^{-n_k}(B(z,2r))$ for every $k\geqslant m$. Hence $T^{n_k}(x)\in B(z,2r)\subset U$ for every $k\geqslant m$. So $T^j(x)\in U$ for every $j\geqslant 0$ and $T^j(x)$ avoids a fixed neighbourhood of ω for infinitely many j (of the form n_k). Therefore in view of (c) and (d), $x\in J$. Consequently $H(B(z,2r))\subset J$, which contradicts the openness of $H(B(z,2r))$ as J is nowhere dense in $\bar{\mathbf{C}}$. The proof is finished. ∎

Corollary 2.4. *$\forall_{\lambda<1}\exists_q\forall_{n\geqslant q}\forall_{z\in J\setminus V}$ if T_v^{-n}: $B(z,2r)\to\bar{\mathbb{C}}$ is an inverse branch of T^n then $|(T_v^{-n})'(x)|<\lambda^{-1}$ for every $x\in B(z,r)$.*

Proof. Since $J\subset V$ is compact, there exists a finite set $E\subset J\setminus V$ such that

$$\bigcup_{x\in E} B(x,r/2)\supset J\setminus V. \tag{2.8}$$

It follows from Lemma 2.3 that

$$\lim_{n\to\infty}\max\{\sup\{|(T_v^{-n})'(z)|:\ z\in B(x,3r/2)\}:\ v\in\mathscr{B}_n(x),\ x\in E\}=0.$$

Therefore the corollary is proved since by (2.8) for every $z\in J\setminus V$ one can find $x\in E$ such that $B(z,r)\subset B(x,\frac{3}{2}r)$. ∎

As an immediate consequence of this corollary and Lemma 2.1 we get

Corollary 2.5. *The set $K(V)=\{z\in J:\ T^n(z)\notin V$ for every $n\geqslant 0\}$ is closed, $T(K(V))\subset K(V)$ and $\forall_{\lambda>1}\exists_{n\geqslant 1}\forall_{x\in K(V)}\ |(T^n)'(z)|\geqslant\lambda$.*

Now we shall prove

Theorem 2.6. *The map T: $J\to J$ is positively expansive.*

Proof. Let $\eta>0$ be so small that $B(\omega,\eta)\subset W$ and

$$T|B(z,\eta)\ \text{is injective for every}\ z\in J. \tag{2.9}$$

Let $Q=B(\omega,\eta)\cap J$ and let $r>0$ be the radius given by Lemma 2.1 with $V=Q$. Since $J\setminus Q$ is compact, there exists a finite set $E\subset J\setminus Q$ such that

$$\bigcup_{z\in E} B(z,r/2)\supset J\setminus Q. \tag{2.10}$$

Since E is finite, it follows from (2.7) (Lemma 2.3) that there exists $m\geqslant 1$ such that for every $z\in E$, $n\geqslant m$ and $v\in\mathscr{B}_n(z)$

$$\operatorname{diam} T_v^{-n}(B(z,r))\leqslant\eta. \tag{2.11}$$

Choose finally $0<\beta<\frac{1}{2}r$ so small that for every $s\in J\setminus V$, every $0\leqslant n<m$ and every holomorphic inverse branch T_v^{-n}: $B(s,2r)\to\bar{\mathbb{C}}$ defined on $B(s,2r)$ we have

$$\operatorname{diam} T_v^{-n}(B(s,\beta))<\eta. \tag{2.12}$$

We shall show that $\delta=\min(\eta,\frac{1}{2}\beta)$ is an expansive constant for T: $J\to J$. So, consider $x,y\in J$ such that

$$\operatorname{dist}(T^n(x),T^n(y))<\delta\quad \text{for every}\ n\geqslant 0. \tag{2.13}$$

We need to prove that $x=y$. There are the following two possibilites. Either

$$T^k(x)=\omega\quad\text{for some}\ k\geqslant 0,\ \text{or} \tag{2.14}$$

$$T^n(x)\neq\omega\quad\text{for every}\ n\geqslant 0. \tag{2.15}$$

Let us consider the case (2.14). Then by the definition of δ, $T^n(y)\in B(\omega,\eta)\subset W$ for every $n\geqslant k$. Hence by (e), $T^k(y)=\omega=T^k(x)$ and let $q\geqslant 0$ be the smallest integer for which $T^q(x)=T^q(y)$. As $x\neq y$ we deduce that $q\geqslant 1$ and $T^{q-1}(x)\neq T^{q-1}(y)$. But this contradicts the fact that $\operatorname{dist}(T^{q-1}(x), T^{q-1}(y))<\delta\leqslant\eta$ and (2.9).

So, let us consider the case (2.15). Condition (e) implies that $T^n(x)\notin W$ for infinitely many n. Therefore, since $Q\subset W$ and J is compact, we can find a sequence $\{n_j\}_{j=1}^{\infty}$ of positive integers increasing to infinity such that

$$T^{n_j}(x)\notin Q\quad\text{for every}\ j=1,2,\ldots, \tag{2.16}$$

$$\lim_{j\to\infty}T^{n_j}(x)=g\quad\text{for some}\ g\in J, \tag{2.17}$$

$$T^{n_j}(x)\in B(g,\beta/2)\quad\text{for every}\ j=1,2,\ldots \tag{2.18}$$

By (2.16) and (2.17), $g\in J\setminus Q$. Hence it follows from Lemma 2.1 that for every $j\geqslant 1$ there exists a unique holomorphic inverse branch $T_v^{-n_j}$: $B(g,2r)\to\bar{\mathbb{C}}$ of T^{n_j} defined on $B(g,2r)$ and determined by the condition

$$T_v^{-n_j}(T^{n_j}(x))=x, \tag{2.19}$$

which makes sense because of (2.18). Now, since $\operatorname{dist}(T^{n_j}(x),T^{n_j}(y))<\delta\leqslant\frac{1}{2}\beta$, we conclude from (2.18) that

$$T^{n_j}(x),\ T^{n_j}(y)\in B(g,\beta)\quad\text{for every}\ j=1,2,\ldots, \tag{2.20}$$

Fix now $j\geqslant 1$, define $y_j=T_v^{-n_j}(T^{n_j}(y))$ and consider $0\leqslant l\leqslant n_j-1$ such that

$$T^{l+1}(y_j)=T^{l+1}(y). \tag{2.21}$$

Since $T^l\circ T_v^{-n_j}$: $B(g,2r)\to\bar{\mathbb{C}}$ is a holomorphic inverse branch of T^{n_j-l} and since, by (2.10) and the definition of β, $B(g,\beta)\subset B(z,r)$ for some $z\in E$, it follows from (2.11) and (2.12) that

$$\operatorname{diam} T^l\circ T_v^{-n_j}(B(g,\beta))<\eta.$$

Therefore, using (2.19), (2.20) and the definition of y_j we get $\operatorname{dist}(T^l(x),T^l(y_j))<\eta$. Thus, in view of (2.13),

$$T^l(y),\ T^l(y_j)\in B(T^l(x),\eta).$$

Hence, applying (2.9) and (2.21), we conclude that $T^l(y_j)=T^l(y)$. Since (2.21) is true for $l=n_j-1$, we therefore deduce by induction that $y_j=y$. Since, by (2.19), (2.20) and (2.7), $\lim_{j\to\infty}\operatorname{dist}(y_j,x)=0$ we obtain $y=x$. The proof is finished. ∎

Corollary 2.7. *If J is a Jordan curve then there exists an integer $d\geqslant 2$ such that the map T: $J\to J$ is topologically conjugate to the map $S^1\to S^1$, $z\to z^d$, where $S^1=\{z\in\mathbb{C}:\ |z|=1\}$.*

Proof. Since J is a Jordan curve and T: $J\to J$ is the restriction to J of a holomorphic map, the mapping $T\colon J \to J$ is topologically conjugate to a continuous orientation preserving mapping $h\colon S^1 \to S^1$. By Theorem 2.6, the map $h\colon S^1 \to S^1$ is positively expansive and therefore by a result of [HR], h is a covering map with finite degree at least two. Hence $h\colon S^1 \to S^1$ is homotopic to a map $S^1 \to S^1$, $z \mapsto z^d$ with $d \geqslant 2$, and consequently these two maps are topologically conjugate in view of (2), p. 99, [CR]. The proof is finished. ∎

Therefore if J is a Jordan curve then for every $z \in J$ the set $\bigcup_{n=0}^{\infty} T^{-n}(x)$ is dense in J and we obtain

COROLLARY 2.8. *If J is a Jordan curve then every t-conformal measure for $T\colon J \to J$ is positive on nonempty open sets.*

§ 3. Hausdorff and conformal measures on J. In this section we assume that J is a Jordan curve, conditions (a)–(e) are satisfied and under these assumptions we prove our main results concerning Hausdorff and conformal measures on J. We want to emphasize that we do not assume condition (f) to be satisfied.

THEOREM 3.1. *If m is a t-conformal measure for $T\colon J \to J$ and H_t the t-dimensional Hausdorff measure on J, then H_t is absolutely continuous with respect to m with bounded density. Consequently $H_t(J)$ is finite and $t \geqslant \mathrm{HD}(J)$.*

Proof. Let $r > 0$ be the radius given by Lemma 2.1 with $V = W$, where $W \subset \bar{\mathbf{C}}$ is the neighbourhood of ω guaranteed in (e). By Corollary 2.8, m is positive on open sets. Hence

$$M = \inf\{m(B(x,r))\colon x \in J\} > 0. \tag{3.1}$$

Let $B \subset J$ be any Borel set. Put $F = B \setminus \{T^{-n}(\omega)\}_{n=0}^{\infty}$. Since $\{T^{-n}(\omega)\}_{n=0}^{\infty}$ is at most countable, $H_t(B) = H_t(F)$. Fix an $\varepsilon > 0$. As m is a Borel probability measure, for every $z \in F$ we can find $\beta(z) > 0$ such that

$$m\Big(\bigcup_{z \in F} B(z, \beta(z))\Big) \leqslant \varepsilon + m(F). \tag{3.2}$$

Now fix $\theta > 0$ and consider $z \in F$. By (e), $T^n(z) \notin W$ for infinitely many positive integers n. Therefore by Corollary 2.4 we can find an integer $n = n(z) \geqslant 1$ for which $T^n(z) \notin W$ and so large that $|(T^n)'(z)| \geqslant \max\big(2rK\theta^{-1}, 2rK(\beta(z))^{-1}\big)$ where $K = k(1/2) > 0$ is the constant defined in the Koebe distortion theorem stated in the introduction. Putting $r(z) = 2rK|(T^n)'(z)|^{-1}$ we thus get

$$r(z) \leqslant \min(\theta, \beta(z)). \tag{3.3}$$

By Lemma 2.1 there exists an inverse branch $T_v^{-n}\colon B(T^n(z), 2r) \to \bar{\mathbf{C}}$ of T^n determined by the condition $T_v^{-n}(T^n(z)) = z$. Let $A(z) = T_v^{-n}\big(B(T^n(z), r)\big)$. In view of (1.3), the Koebe distortion theorem and (3.1) we have

$$\operatorname{diam} A(z) \leqslant 2rK|(T^n)'(z)|^{-1} = r(z), \tag{3.4}$$

$$m(A(z)) \geqslant m\big(B(T^n(z), r)\big)K^{-t}|(T^n)'(z)|^{-t} \geqslant M(2rK^2)^{-t} r(z)^t. \tag{3.5}$$

Now, by the Besicovitch covering theorem (see for example [G]) we can find a sequence $\{z_j\}_{j=1}^{\infty}$ in F such that $\bigcup_{j=1}^{\infty} B(z_j, r(z_j)) \supset F$ and the cover $\{B(z_j, r(z_j))\}_{j=1}^{\infty}$ is of multiplicity bounded by a universal constant $c \geqslant 1$. In view of (3.2)–(3.5) we can estimate

$$\sum_{j=1}^{\infty} r(z_j)^t \leqslant \sum_{j=1}^{\infty} M^{-1}(2rK^2)^t m(A(z_j)) \leqslant M^{-1}(2rK^2)^t \sum_{j=1}^{\infty} m\big(B(z_j, r(z_j))\big)$$
$$\leqslant cM^{-1}(2rK^2)^t m\Big(\bigcup_{j=1}^{\infty} B(z_j, r(z_j))\Big) \leqslant cM^{-1}(2rK^2)^t(\varepsilon + m(F)).$$

So, letting $\theta \searrow 0$ and then $\varepsilon \searrow 0$ we obtain $H_t(B) = H_t(F) \leqslant cM^{-1}(2rK^2)^t m(B)$. The proof is finished. ∎

Remark 3.2. Note that in the proof of Theorem 3.1, the assumption for J to be a Jordan curve was needed only to have formula (3.1).

Our next aim is to prove the existence of a $\mathrm{HD}(J)$-conformal measure for $T\colon J \to J$. To this end, improving the techniques worked out in [U3] (comp. also [M], [U1], [U2]), we shall study some special sets of the form $K(V)$. As was proved in Corollary 2.5 they are all hyperbolic but we will also need more particular properties. For this note that in view of Corollary 2.7 for every $k \geqslant 1$ there exists an open arc $V = V_k \subset J$ containing ω and such that

$$\bar{V} \cap T^{-k}(\omega) = \{\omega\} \cup \partial V. \tag{3.6}$$

Then $\bar{V} \cap K(V) = \emptyset$ and therefore we can find an open neighbourhood $H \subset U$ (for instance $(\bar{\mathbf{C}} \setminus \bar{V}) \cap U$) of $K(V)$ such that $H \cap V = \emptyset$. Hence

$$T^{-1}(K(V)) \cap H = K(V) = T(K(V)). \tag{3.7}$$

LEMMA 3.3. *For every nonempty open set $G \subset K(V)$ there exists an integer $n \geqslant 0$ such that $G \cup T(G) \cup \ldots \cup T^{n-1}(G) \cup T^n(G) = K(V)$.*

Proof. In view of Corollary 2.7 we can find an integer $n \geqslant 1$ and an open arc $Q \subset J$ such that

$$\emptyset \neq Q \cap K(V) \subset G, \tag{3.8}$$

$$T^n|Q \text{ is injective, and} \tag{3.9}$$

$$T^n(Q) = J \setminus \{\omega\}. \tag{3.10}$$

Assume first that $V \cap (Q \cup T(Q) \cup \ldots \cup T^{n-1}(Q)) = \emptyset$ and let $y \in K(V)$. By (3.9) there exists $x \in Q$ such that $y = T^n(x)$. As $x, T(x), \ldots, T^{n-1}(x) \notin V$ and $y \in K(V)$, we conclude that $x \in K(V)$ and we are done in this case.

So, suppose now that $V \cap (Q \cup T(Q) \cup \ldots \cup T^{n-1}(Q)) \neq \emptyset$ and let $m \in \{0, \ldots, n-1\}$ be the smallest number such that

$$V \cap T^m(Q) \neq \emptyset. \tag{3.11}$$

By (3.8), $T^m(Q) \cap (S^1 \setminus V) \neq \emptyset$. Therefore, as V and $T^m(Q)$ are open arcs, it follows from (3.11) that $T^m(Q)\cap\partial V\neq\emptyset$. Hence, using (3.6), we can find $x\in Q$ such that $T^{m+k}(x)=\omega$. In view of (3.10) this gives

$$(3.12)\qquad m+k>n.$$

Suppose now also that $\omega\notin T^m(\partial Q)$. Then, again as $T^m(Q)$ and V are open arcs, it follows from (3.11) that $T^m(\partial Q)\cap(V\setminus\{\omega\})\neq\emptyset$. In view of (3.6) and (3.10) this implies that $n-m>k$. But this contradicts (3.12) and shows that

$$(3.13)\qquad \omega\in T^m(\partial Q).$$

Let $\partial Q=\{q_1,q_2\}$ and $\partial V=\{v_1,v_2\}$. According to (3.13) and (3.11), without loosing generality, we can assume that $\omega=T^m(q_1)$ and $v_1\in T^m(Q)$ $=(\omega,T^m(q_2))$. The open arc $(\omega,T^m(q_2))$ is uniquely determined by the property $v_1\in(\omega,T^m(q_2))$. All the other arcs, written in the form (a,b), which will appear in this proof are oriented according to the triple $(\omega,v_1,T^m(q_2))$.

By Corollary 2.7 there exists $c\in(\omega,T^m(q_2))$ such that $T(c)=T^m(q_2)$. Suppose that $c\in(\omega,v_1)$. Since $T^{n-m+1}(c)=T^{n-m}(T^m(q_2))=\omega$, it follows from (3.6) that $n-m+1>k$. Hence $n-m\geqslant k$. But this contradicts (3.12) and shows that $c\in[v_1,T^m(q_2))$. Consequently $T^m(v_1)\in(v_1,T^m(q_2)]$ and therefore for every $l\in\{0,1,\ldots,n-m\}$ we have

$$(3.14)\qquad [v_1,T^m(q_2))\cup T([v_1,T^m(q_2)))\cup\ldots\cup T^l([v_1,T^m(q_2)))=[v_1,T^{m+l}(q_2)).$$

As $[v_1,T^{m+(n-m)}(q_2))=[v_1,\omega)\supset[v_1,v_2]$, there exists the smallest $s\in\{0,1,\ldots,n-m\}$ such that $v_2\in[v_1,T^{m+s}(q_2))$. Then

$$(3.15)\qquad [v_1,T^m(q_2))\cup T([v_1,T^m(q_2)))\cup\ldots\cup T^s([v_1,T^m(q_2)))\supset[v_1,v_2],$$

$$(3.16)\qquad \big([v_1,T^m(q_2))\cup T([v_1,T^m(q_2)))\cup\ldots\cup T^{s-1}([v_1,T^m(q_2)))\big)\cap V=\emptyset.$$

Consider now $y\in K(V)$. Then $y\in[v_1,v_2]$ and in view of (3.15) and (3.16) we can find $p\in\{0,\ldots,s\}$ and $z\in[v_1,T^m(q_2))$ such that $T^p(z)=y$. Since $[v_1,T^m(q_2))\subset(\omega,T^m(q_2))=T^m(Q)$, there exists $x\in Q$ such that $z=T^m(x)$. Thus $y=T^{m+p}(x)$ and in view of the definition of m and (3.16), x, $T(x),\ldots,T^m(x),T^{m+1}(x),\ldots,T^{m+p-1}(x)\notin V$. As $y=T^{m+p}(x)\in K(V)$, we conclude that $x\in K(V)$. Consequently $K(V)\subset(Q\cap K(V))\cup\ldots\cup T^n(Q\cap K(V))$, which, because of (3.8), completes the proof. ∎

A continuous mapping with the property stated in the lemma is called *locally eventually onto*. As an immediate consequence of Lemma 3.3 we get

COROLLARY 3.4. *The set* $\bigcup_{n=0}^{\infty}(T|K(V))^{-n}(x)$ *is dense in* $K(V)$ *for every* $x\in K(V)$.

In view of (3.7), Corollary 2.5 and Corollary 3.4 we can apply the Corollary on p. 59 in [S1] to obtain

LEMMA 3.5. *There exists an* $\mathrm{HD}(K(V))$*-conformal measure for* $T|K(V)$: $K(V)\to K(V)$. *The* $\mathrm{HD}(K(V))$*-dimensional Hausdorff measure on* $K(V)$ *is finite and equivalent to the* $\mathrm{HD}(K(V))$*-conformal measure.*

In the sequel we will only use the first part of this lemma. Note that it also follows from the Bowen–Manning–McCluskey formula $P(T|K(V),-\mathrm{HD}(K(V))\log|T'|)=0$ (see [B], [McCM]), where P denotes the topological pressure, and Theorem 3.12 of [DU]. One only needs to observe here that the existence of an open neighbourhood H of $K(V)$ such that $H\cap V=\emptyset$ implies that $T|K(V)$ is an open map.

Exactly as Lemma 3.1 of [U3] we can prove the following.

LEMMA 3.6. *Let* $|T'(\omega)|=1$ *and let* $\{G_n\}_{n=1}^{\infty}$ *be a decreasing sequence of open connected neighbourhoods of* ω *such that* $\bigcap_{n=1}^{\infty}G_n=\{\omega\}$. *Let* m_n *be* t_n*-conformal measures for the maps* $T|K(G_n)$, $n\geqslant 1$, *respectively. If the sequence* $\{t_n\}_{n=1}^{\infty}$ *converges,* $t=\lim_{n\to\infty}t_n$ *and the sequence* $\{m_n\}_{n=1}^{\infty}$ *converges in the weak topology of measures on* J, *say to* m, *then* m *is a* t*-conformal measure for* $T\colon J\to J$.

We shall finish this section with the following

THEOREM 3.7. *There is an* $\mathrm{HD}(J)$*-conformal measure for* $T\colon J\to J$. *The* $\mathrm{HD}(J)$*-dimensional Hausdorff measure of* J *is finite* (*possibly* 0).

Proof. If $|T'(\omega)|>1$ the theorem can be proved exactly as in Bowen's paper [B], and for a more detailed discussion see the introduction. So, suppose that $|T'(\omega)|=1$. In view of Lemma 3.5 for every $k\geqslant 1$ there exists a t_k-conformal measure m_k for $T|K(V_k)$, where $t_k=\mathrm{HD}(K(V_k))$. Passing, if necessary, to subsequences we may assume that $\{t_k\}_{k=1}^{\infty}$ and $\{m_k\}_{k=1}^{\infty}$ converge. Let $t=\lim t_k$ and $m=\lim m_k$. In view of Lemma 3.6, m is a t-conformal measure for $T\colon J\to J$. Since $t_k\leqslant\mathrm{HD}(J)$ for every $k\geqslant 1$, also $t\leqslant\mathrm{HD}(J)$. As $t\geqslant HD(J)$ by Theorem 3.1, the first part of Theorem 3.7 is proved. The second part now also follows immediately from Theorem 3.1. The proof is finished. ∎

Remark 3.8. In a forthcoming paper basically devoted to the fractal properties of a Julia set without critical points we shall describe another way of proving the existence of t-conformal measures with $t\leqslant\mathrm{HD}(J)$. Instead of looking at the sets of the form $K(V)$ it uses the Manning–McCluskey picture (see [McCM]) and is based on Theorems 2.5 and 3.12 of [DU].

§ 4. **The jump transformation.** In this section, like in Section 3, we assume that J is a Jordan curve and conditions (a)–(e) are satisfied. Moreover, condition (f) is assumed to be fulfilled for at least one connected component A of $\bar{\mathbf{C}}\setminus J$. We then define a special lifting of $T\colon J\to J$ to an analytic endomorphism F of the circle S^1 and we associate to F the so-called jump transformation $F^*\colon S^1\to S^1$. The next part of the section is devoted to the study of absolutely continuous F^*-invariant measures. So let $R\colon D$ $=\{z\colon |z|<1\}\to A$ be a conformal homeomorphism (Riemann mapping). Since $\partial A=J$ is a Jordan curve, the map R extends homeomorphically to $\bar{D}=\{z\colon |z|\leqslant 1\}$. We leave for this extension as well as for its restriction to the circle $S^1 = \{z\colon |z| = 1\}$ the same symbol R. It will always be clear from the context which set of arguments we consider.

It follows from Proposition 4 of [P2] that $\bar{F} = R^{-1} \circ T \circ R\colon R^{-1}(U) \to D$ extends holomorphically to F on a neighbourhood $\tilde{U}$ of S^1. Moreover, F has no critical points in S^1. We can obviously assume that

$$\bar{\tilde{U}} \subset \{z\colon \tfrac{1}{2} < |z| < \tfrac{3}{2}\} \quad \text{and} \quad \bar{\tilde{U}} \cap D \subset R^{-1}(U). \tag{4.1}$$

Since it follows from the construction of F that F commutes with the map $z \to 1/\bar{z}$ (the Schwarz reflection principle) and $T \circ (R|\overline{\tilde{U} \cap D}) = R \circ (F|\overline{\tilde{U} \cap D})$, we conclude that the Jordan curve S^1, the connected component D of $\bar{\mathbf{C}} \setminus S_1$ and the holomorphic map $F\colon \tilde{U} \to \mathbf{C}$ also satisfy assumptions (a)–(f). In particular, the results proved in Section 2 also apply to F. The role of ω is now played by the point $\Phi = R^{-1}(\omega)$ and by $\tilde{W}$ we denote the neighbourhood of Φ guaranteed in (e). Given two points $x, y \in S^1$, $[x, y]$ denotes one of the two closed intervals which join x and y. It will always be clear from the context which of them we mean (usually the shorter one). The arcs (x, y), $[x, y]$ and so on are understood similarly.

We start the study of the holomorphic map $F\colon S^1 \to S^1$ with the remark that if $|F'(\Phi)| > 1$, then as was argued in the introduction, $F\colon S^1 \to S^1$ is an expanding map. More precisely:

Lemma 4.1. *If $|F'(\Phi)| > 1$, then there is $n_0 \geqslant 1$ such that $|(F^{n_0})'(z)| > 1$ for every $z \in S^1$.*

So, from now on, unless stated otherwise, we will assume that $|F'(\Phi)| = 1$. The neighbourhood $\tilde{W} \subset \bar{\mathbf{C}}$ of Φ given by (e) is assumed to be so small that $F|\tilde{W}$ is injective and all the analytic inverse branches of F are well-defined on $\tilde{W}$. We shall prove the following result which in terminology of [T] says that Φ is a regular source.

Lemma 4.2. *There exist two points $x_0, y_0 \in \tilde{W} \cap S^1$ such that Φ separates them and the functions $|F'|\colon [\Phi, x_0] \to \mathbf{R}$, $|F'|\colon [\Phi, y_0] \to \mathbf{R}$ are strictly increasing.*

Proof. Suppose first that $|F'|\colon [\Phi, x] \to \mathbf{R}$ is not strictly monotone for any $x > \Phi$. Thus F'' vanishes at infinitely many points on S^1. As F is holomorphic on an open neighbourhood of S^1 and $F(\Phi) = \Phi$, $F'(\Phi) = 1$, it follows that F is the identity map. But this contradicts condition (e). Since this argument works as well for $x < \Phi$, we conclude that there are $y_0 < 0 < x_0$ such that $|F'|\colon [\Phi, x_0] \to \mathbf{R}$ and $|F'|\colon [\Phi, y_0] \to \mathbf{R}$ are strictly monotone. If one of these functions, say $|F'|\colon [\Phi, x_0] \to \mathbf{R}$, were decreasing, then $|F'(x)| < 1$ for every $x \in (\Phi, x_0]$, which would again contradict (e). The proof is finished. ∎

In view of Corollary 2.7 there exists a homeomorphism $\psi\colon S^1 \to S^1$ which conjugates the map $S^1 \to S^1$, $z \mapsto z^d$, $d \geqslant 2$, and the map $F\colon S^1 \to S^1$. For $j = 0, 1, \ldots, d-1$ let $a_j = \psi(e^{2\pi i j/d})$, $B_j = [a_j, a_{j+1}]$ where $a_d = \psi(e^{2\pi i d/d}) = a_0$ $= \Phi$. Then for every $j \in \{0, 1, \ldots, d-1\}$

$$F|B_j \text{ is injective,} \tag{4.2}$$

$$F(B_j) = S^1 \tag{4.3}$$

and the partition $\mathscr{B} = \{B_j\}_{j=0}^{d-1}$ is a generator. Since T has no critical points in S^1, it follows from this and Lemma 4.1 that

Lemma 4.3. *The map $F\colon S^1 \to S^1$ satisfies conditions* (T1)–(T4) *of the paper* [T].

Therefore we can apply all the results proved in [T], particularly those concerning the jump transformation F^* associated to F. Following [T] we recall that F^* is defined on $B = S^1 \setminus \bigcup_{n=0}^{\infty} F^{-n}(\Phi)$ as follows.

If $z \notin B_0 \cup B_{d-1}$ we set $n(z) = 0$. For $z \in B_j$, $j \in \{0, d-1\}$, let $n(z) \geqslant 1$ be the smallest integer such that $F^{n(z)}(z) \notin B_j$. We then put $F^*(z) = F^{n(z)+1}(z)$.

Consider now an open neighbourhood $H \subset \tilde{W} \cap [B_0 \cup B_{d-1}]$ of Φ in S^1 such that

$$F(H \cap B_j) \subset B_j \quad \text{for } j \in \{0, d-1\}. \tag{4.4}$$

It follows from this and the definition of the number $n(z)$ that

$$\text{If} \quad n(z) \geqslant 1 \quad \text{then} \quad F^{n(z)-1}(z) \notin H. \tag{4.5}$$

We shall give a short proof of the following.

Lemma 4.4. *There exists $m \geqslant 1$ such that $|((F^*)^m)'| \geqslant 2$ for every $z \in B$.*

Proof. Let $q \geqslant 1$ be the integer claimed in Corollary 2.4 to exist for $V = H$, T replaced by F and $\lambda = \max\left(2(\inf\{|F'(z)|\colon z \in S^1\})^{-1}, 2(\inf\{|(F^2)'(z)|\colon z \in S^1\})^{-1}\right)$. Put $m = q+1$. For $z \in B$ let $x = (F^*)^{m-1}(z)$. By the definition of F^* we have $x = (F^*)^{m-1}(z) = F^n(z)$ with $n \geqslant m-1 \geqslant q$. If $n(x) = 0$ then $x \notin H$ and in view of Corollary 2.4 and the choice of λ we have

$$\begin{aligned} |((F^*)^m)'(z)| &= |((F^*)^{m-1})'(z)| \cdot |(F^*)'(x)| = |(F^n)'(z)| \cdot |F'(x)| \\ &\geqslant \lambda \cdot \inf\{|F'(y)|\colon y \in S^1\} \geqslant 2. \end{aligned} \tag{4.6}$$

If $n(x) \geqslant 1$ then by (4.5), $y = F^{n+n(x)-1}(z) = F^{n(x)-1}(x) \notin H$ and as $F^*(x) = F^{n(x)+1}(x)$ again in view of Corollary 2.4, the choice of λ and $n + n(x) - 1 \geqslant n \geqslant q$ we get

$$|((F^*)^m)'(z)| = |(F^{n(x)+1} \circ F^n)'(z)| = |(F^{n+n(x)-1})'(z)| \cdot |(F^2)'(y)| \geqslant 2.$$

This and (4.6) complete the proof. ∎

Now note that as F is an analytic function without critical points, the number $\sup\{|F''(z)|/|F'(z)|\colon z \in S^1\}$ is finite and therefore the assumptions of Theorem 2 of [T] are fulfilled for $F\colon S^1 \to S^1$. It follows from this and

Lemma 4.4 that the assumptions of Adler's theorem ([A]) are fulfilled for F^*. And applying this result we obtain the following.

LEMMA 4.5. *There exists exactly one absolutely continuous (with respect to Lebesgue measure l) F^*-invariant probability measure μ on S^1. Moreover, μ is equivalent to l.*

Let us also prove

LEMMA 4.6. *The Radon–Nikodym derivative $\varphi = d\mu/dl$ of μ with respect to Lebesgue measure l has a representation which is a real-analytic function on $S^1\backslash\{\Phi\}$.*

Proof. Let

$$P\colon L^1(l)\to L^1(l),\quad P(f)(z)=\sum_{x\in(F^*)^{-1}(z)}\frac{1}{|(F^*)'(x)|}f(x),$$

be the Perron–Frobenius operator of the mapping F with respect to Lebesgue measure l. An easy computation shows that for every $n\geqslant 1$

$$P^n(f)(z)=\sum_{x\in((F^*)^n)^{-1}(z)}\frac{f(x)}{|((F^*)^n)'(x)|}. \tag{4.7}$$

It follows from Lemma 4.5 that the measure μ is ergodic. Hence Lebesgue measure l is also ergodic and therefore any P-absorbing set (for the definition see [K], p. 118) is either of measure 0 or 1. Since moreover, also by Lemma 4.5, φ is strictly positive and $P(\varphi)=\varphi$, it follows from Hopf's theorem (Theorem 3.5 of [K]) that

$$\lim_{n\to\infty} n^{-1}\sum_{j=0}^{n-1}P^j(1)=\varphi \quad l\text{-a.e.} \tag{4.8}$$

Note now that in order to complete the proof we only need to show the local version of the lemma, that is, that any point z in $S^1\backslash\{\Phi\}$ admits an open neighbourhood in S^1 on which φ has a real-analytic version. So, fix $z\in S^1\backslash\{\Phi\}$ and choose in S^1 an open arc V containing Φ such that

$$z\notin\bar{V}. \tag{4.9}$$

Let $r>0$ be the radius claimed in Lemma 2.1 to exist for this arc with T replaced by F. Moreover, r is required to be so small that

$$\Phi\notin B(z,3r). \tag{4.10}$$

For every $x\in S^1$ and $m\geqslant 1$ let $n(x,m)=n(x)+1+n(F^*(x))+1+n((F^*)^2(x))+1+\ldots+n((F^*)^{m-1}(x))+1$ (we make the convention $n(\Phi)=\infty$). Then $(F^*)^m(x)=F^{n(x,m)}(x)$. In view of Lemma 2.1 and (4.9) for every $m\geqslant 1$ and $x\in(F^*)^{-m}(z)$ there exists a unique holomorphic inverse branch $F_{\nu(x,m)}^{-n(x,m)}\colon B(z,2r)\to\bar{\mathbb{C}}$ of $F^{n(x,m)}$ determined by the condition $F_{\nu(x,m)}^{-n(x,m)}(z)=x$. Consider now an arbitrary inverse branch $F_\nu^{-n}\colon B(z,2r)\to\bar{\mathbb{C}}$ with $F_\nu^{-n}(z)\in S^1$. Since $F_\nu^{-n}(S^1\cap B(z,2r))\subset S^1$ is an arc, it easily follows from (4.10) that $F_\nu^{-n}(S^1\cap B(z,2r))$ is contained in the interior of an element of the partition $\mathscr{B}\vee\ldots\vee F^{-(n-1)}(\mathscr{B})$. Therefore, looking at (4.7) and the definition of the jump transformation F^*, we can write

$$P^m(f)(y)=\sum_{x\in(F^*)^{-m}(z)}|(F_{\nu(x,m)}^{-n(x,m)})'(y)|f(F_{\nu(x,m)}^{-n(x,m)}(y)) \tag{4.11}$$

for every $m\geqslant 1$ and $y\in S^1\cap B(z,2r)$. Since $F_{\nu(x,m)}^{-n(x,m)}(S^1\cap B(z,2r))\subset S^1$, we have

$$|(F_{\nu(x,m)}^{-n(x,m)})'(y)|=\frac{y\cdot(F_{\nu(x,m)}^{-n(x,m)})'(y)}{F_{\nu(x,m)}^{-n(x,m)}(y)}\quad\text{for } y\in S^1\cap B(z,2r). \tag{4.12}$$

Thus

$$P^m(1)(y)=\sum_{x\in(F^*)^{-m}(y)}\frac{y\cdot(F_{\nu(x,m)}^{-n(x,m)})'(y)}{F_{\nu(x,m)}^{-n(x,m)}(y)}\quad\text{for every } y\in S^1\cap B(z,2r). \tag{4.13}$$

Let $M(z)=2\big(l(S^1\cap B(z,r))\big)^{-1}<\infty$. Since $1\geqslant l\big((F^*)^{-m}(S^1\cap B(z,r))\big)=\int_{S^1\cap B(z,r)}P^m(1)\,dl$, there exists $y_m\in S^1\cap B(z,r)$ such that

$$P^m(1)(y_m)\leqslant M(z). \tag{4.14}$$

In view of (4.1) we have

$$\left|\frac{y}{F_{\nu(x,m)}^{-n(x,m)}(y)}\right|\leqslant 3\quad\text{for every } m\geqslant 1,\ y\in B(z,r)\text{ and } x\in(F^*)^{-m}(y) \tag{4.15}$$

and in view of the Koebe distortion theorem stated in the introduction

$$|(F_{\nu(x,m)}^{-n(x,m)})'(y)|\leqslant K\,|(F_{\nu(x,m)}^{-n(x,m)})'(y_m)|$$
$$\text{for every } m\geqslant 1,\ y\in B(z,r)\text{ and } x\in(F^*)^{-m}(y).$$

Therefore, using (4.15), (4.11) and (4.14), we get

$$\sum_{x\in(F^*)^{-m}(z)}\left|\frac{y\cdot(F_{\nu(x,m)}^{-n(x,m)})'(y)}{F_{\nu(x,m)}^{-n(x,m)}(y)}\right| \leqslant 3K\sum_{x\in(F^*)^{-m}(z)}|(F_{\nu(x,m)}^{-n(x,m)})'(y_m)|=3KP^m(1)(y_m)\leqslant 3KM(z) \tag{4.16}$$

for every $m\geqslant 1$ and $y\in B(z,r)$.

Hence this series defines on $B(z,r)$ a holomorphic function for which we keep the name $P^m(1)$. It follows again from (4.16) that

$$\Big|m^{-1}\sum_{j=0}^{m-1}P^j(1)(y)\Big|\leqslant 3KM(z)\quad\text{for every } m\geqslant 1\text{ and } y\in B(z,r).$$

Thus, by Vitali's theorem, the family $\{m^{-1}\sum_{j=0}^{m-1}P^j(1)\}_{m=1}^{\infty}$ of holomorphic functions on $B(z,r)$ is normal in the sense of Montel and therefore one can find a subsequence $\{m_k\}_{k=1}^{\infty}$ increasing to infinity such that $m_k^{-1}\sum_{j=0}^{m_k-1}P^j(1)$ converges on $B(z,r/2)$ uniformly to an analytic functon, say $H\colon B(z,r/2)\to \mathbf{C}$. Hence, in view of (4.8), $\varphi = H$ almost everywhere on $S^1\cap B(z,r/2)$. The proof is finished. ∎

Remark 4.7. The study of smoothness of densities of invariant measures is an important part of the theory of differentiable dynamical systems on manifolds. The results closest to Lemma 4.6 have been obtained in [H], [Krz] and [PUZ, II]. All of them concern, however, the case of finitely many pieces of monotonicity, although in [H] the possibility of generalization to the "infinite case" has been mentioned. The main reason for presenting in this paper a detailed proof of Lemma 4.6 was to show that the basic ideas worked out in the "finite case" extend to the "infinite case" indeed.

§ 5. The main result. In this section, extending slightly the idea of [B] and [S1] (comp. also [P3]), we prove our main result. We now assume that J is a Jordan curve, conditions (a)–(e) are satisfied and condition (f) is fulfilled for both connected components A_1 and A_2 of $\bar{\mathbf{C}}\backslash J$. Let, like in the previous sections, $R_1\colon \{z\colon |z|\leqslant 1\}\to \bar{A}_1$, $R_2\colon \{z\colon |z|\geqslant 1\}\to \bar{A}_2$ be the extensions of conformal homeomorphisms defined respectively on the open sets $\{z\colon |z|<1\}$ and $\{z\colon |z|>1\}$. Let F_i, $i=1,2$, denote the corresponding liftings of the map T. Since $R_i\circ(F_i|S^1) = R\circ(R_i|S^1)$, $i=1,2$, we get $h\circ F_1 = F_2\circ h$ where $h = R_2^{-1}\circ(R_1|S^1)$. Since moreover $h(a_j^{(1)}) = a_j^{(2)}$, $j=0,\ldots,d-1$ (the points a_j are defined just before formulas (4.2) and (4.3)) we obtain

$$h\circ F_1^* = F_2^*\circ h. \tag{5.1}$$

LEMMA 5.1. *If the homeomorphism $h\colon S^1\to S^1$ is absolutely continuous then it is real-analytic on $S^1\backslash\{\Phi\}$.*

Proof. Let $\mu_1 = \varphi_1 l$, $\mu_2 = \varphi_2 l$ be the measures given by Lemma 4.5 applied to F_1^* and F_2^* respectively. In view of Lemma 4.6 we can assume that the functions φ_1 and φ_2 are real-analytic on $S^1\backslash\{\Phi\}$. As h is absolutely continuous, it follows from (5.1) that $h_*(\varphi_1 l)$ is a probability F_2^*-invariant measure absolutely continuous with respect to Lebesgue measure l. Therefore, in view of Lemma 4.5,

$$h_*(\varphi_1 l) = \varphi_2 l. \tag{5.2}$$

Now define the functions $M, N\colon S^1\to S^1$ setting

$$\begin{aligned} M(z) &= \exp\big(2\pi i\varphi_1 l([\Phi,z])\big) = \exp\Big(2\pi i\int_{\Phi}^{z}\varphi_1\,dl\Big),\\ N(z) &= \exp\big(2\pi i\varphi_2 l([\Phi,z])\big) = \exp\Big(2\pi i\int_{\Phi}^{z}\varphi_2\,dl\Big). \end{aligned} \tag{5.3}$$

In view of (5.2) we get

$$M(h(z)) = N(z). \tag{5.4}$$

Since the measures $\varphi_1 l$ and $\varphi_2 l$ are equivalent to l, the maps M and N are homeomorphisms. Therefore (5.4) can be rewritten in the form

$$h = M^{-1}\circ N. \tag{5.5}$$

Thus, as φ_1 and φ_2 are real-analytic on $S^1\backslash\{\Phi\}$, the real-analyticity of h on $S^1\backslash\{\Phi\}$ follows from (5.3). The proof is finished. ∎

THEOREM 5.2. *If $T\colon U\to\bar{\mathbf{C}}$ satisfies conditions* (a)–(e), *J is a Jordan curve and condition* (f) *is fulfilled for both connected components of $\bar{\mathbf{C}}\backslash J$ then the set J is either a real-analytic curve or* $\mathrm{HD}(J) > 1$.

Proof. Suppose that $\mathrm{HD}(J) = 1$. Then, in view of Theorem 3.1, the 1-dimensional Hausdorff measure of J is finite. Thus J is rectifiable and by the F. Riesz and M. Riesz theorem (see [RR], comp. also [CL]), $R_i\colon S^1\to J$, $R_i^{-1}\colon J\to S^1$, $i=1,2$, are absolutely continuous. Therefore the map $h = R_2^{-1}\circ R_1\colon S^1\to S^1$ is also absolutely continuous. Hence, in view of Lemma 5.1, h is real-analytic on $S^1\backslash\{\Phi\}$. Consequently, fixing an arbitrary $x\in S^1\backslash\{\Phi\}$, one can find $\varepsilon>0$ (depending on x) so small that $h|S^1\cap B(x,\varepsilon)$ extends to a holomorphic map $\tilde{h}\colon B(x,\varepsilon)\to\bar{\mathbf{C}}$.

Since $h\colon S^1\to S^1$ is an orientation preserving homeomorphism, $\tilde{h}(B(x,\varepsilon)\cap\{z\colon |z|>1\})\subset\{z\colon |z|>1\}$. We may assume ε to be so small that $\{z\colon |z|<1\}\cap B(x,\varepsilon)\subset R_1^{-1}(U)$ and $\{z\colon |z|>1\}\cap\tilde{h}(B(x,\varepsilon))\subset R_2^{-1}(U)$. Consequently we get two continuous maps $R^1|\{z\colon |z|\leqslant 1\}\cap B(x,\varepsilon)$ and $R_2\circ\tilde{h}|\{z\colon |z|\geqslant 1\}\cap B(x,\varepsilon)$ which coincide on the common real-analytic boundary $S^1\cap B(x,\varepsilon)$ and which are holomorphic on the sets $\{z\colon |z|<1\}\cap B(x,\varepsilon)$ and $\{z\colon |z|>1\}\cap B(x,\varepsilon)$ respectively. Hence they glue together to a holomorphic map $H\colon B(x,\varepsilon)\to\bar{\mathbf{C}}$. As $R_1\colon \{z\colon |z|\leqslant 1\}\to A_1$ is injective, there are $y\in S^1\cap B(x,\varepsilon)$ and $\eta>0$ so small that $B(y,\eta)\subset B(x,\varepsilon)$ and $H|B(y,\eta)$ is a holomorphic homeomorphism.

Now let z be an arbitrary point in J. By Corollary 2.7 there is $w\in H(S^1\cap B(y,\eta))$ such that $T^n(w) = z$ for some integer $n\geqslant 0$. Choose $\sigma>0$ so small that $B(w,\sigma)\subset H(B(y,\eta))$ and $T^n|B(w,\sigma)$ is injective. Hence $T^n\circ H|H^{-1}(B(w,\sigma))$ is a holomorphic homeomorphism onto an open neighbourhood of z, such that $T^n\circ H\big(S^1\cap H^{-1}(B(w,\sigma))\big)\subset J$. So every point of J admits a real-analytic local parametrization. Thus J is a real-analytic curve and the proof is finished. ∎

§ 6. Examples. In this section we describe a class of examples for which the conditions from the introduction are satisfied. These examples have in fact motivated our whole work.

Let $f\colon \bar{\mathbf{C}}\to\bar{\mathbf{C}}$ be an analytic endomorphism of the Riemann sphere $\bar{\mathbf{C}}$ ($f|\mathbf{C}$ is a rational function). Suppose that f has a rationally indifferent periodic point ω. This means that if $p = p(\omega,f)$ is the minimal period of ω then $(f^p)'(\omega) = e^{2\pi i\alpha}$ with some rational number α. Let $K = K(\omega,f)$ be the closure of the basin of immediate attraction to ω and let C denote the set of all critical points for f, i.e. those points c for which $f'(c) = 0$. We shall prove the following.

LEMMA 6.1. *Let $T = f^p$ and let A be a connected simply connected component of* $\operatorname{Int} K$ *such that $J = \partial A$ is a Jordan curve. If $J \cap \overline{\bigcup_{n=0}^{\infty} f^n(C)} = \{\omega\}$ then, with a suitable choice of U, conditions* (a)–(e) *are satisfied and condition* (f) *is fulfilled for A and $\bar{\mathbf{C}} \setminus \bar{A}$.*

Proof. Since $T(J) = J$ and $J \cap \overline{\bigcup_{n=0}^{\infty} f^n(C)} = \{\omega\}$, we see that conditions (a) and (b) are satisfied. In particular, there exists $\eta > 0$ such that

$$T|B(z,\eta) \text{ is injective for every } z \in J. \tag{6.1}$$

Therefore, as $T\colon \bar{\mathbf{C}} \to \bar{\mathbf{C}}$ is open and J is a Jordan curve, we deduce that

$$\forall_{\varepsilon>0} \forall_{z\in J} \exists_{\sigma>0} \quad T\big(B(z,\varepsilon) \cap \bar{A}\big) \supset B\big(T(z),\sigma\big) \cap \bar{A}.$$

As J is compact and T is open, we easily see that σ can be chosen independently of z. In particular, there exists $\sigma > 0$ such that for every $z \in J$

$$T\big(B(z,\eta) \cap \bar{A}\big) \supset B\big(T(z),\sigma\big) \cap \bar{A} \tag{6.2}$$

and let $\alpha \leqslant \eta$ be so small that for every $z \in J$

$$T\big(B(z,\alpha)\big) \subset B\big(T(z),\sigma\big). \tag{6.3}$$

In view of Leau's theorem ([L], the local topological picture around a rationally indifferent fixed point) we can find $r > 0$ such that

$$\text{If } T^n(z) \in B(\omega, 2r) \text{ for every } n \geqslant 0 \text{ then } \lim_{n\to\infty} T^n(z) = \omega. \tag{6.4}$$

If moreover $z \in J = \partial A$ then $z = \omega$.

Now we shall copy an argument used in the proof of Theorem 2.6. Since $J \cap \bigcup_{n=0}^{\infty} T^n(C) = \{\omega\}$, there exists $\varrho > 0$ such that

$$B\big(J \setminus B(\omega,r), 2\varrho\big) \cap \Big(\bigcup_{n=0}^{\infty} T^n(C)\Big) = \emptyset. \tag{6.5}$$

Therefore for every $z \in J \setminus B(\omega,r)$ and $n \geqslant 0$, all the holomorphic inverse branches $\{T_v^{-n}\colon B(z,2\varrho) \to \bar{\mathbf{C}}\}_{v\in\mathscr{B}_n(z)}$ of T^n are well-defined on $B(z,2\varrho)$. Now note that if the Julia set of a rational map contains a rationally indifferent periodic point then it is not equal to the whole sphere $\bar{\mathbf{C}}$. Therefore, as $z \in J$ and J is contained in the Julia set of T, it follows from Theorem 6.2 of [Br] that the family $\{T_v^{-n}\colon n \geqslant 0,\ v \in \mathscr{B}_n(z)\}$ is normal and all its accumulation points are constant functions. Consequently for every $z \in J \setminus B(\omega,r)$

$$\lim_{n\to\infty} \max\{\operatorname{diam} T_v^{-n}\big(B(z,\varrho)\big)\colon v \in \mathscr{B}_n(z)\} = 0. \tag{6.6}$$

Since $J \setminus B(\omega,r)$ is compact, there exists a finite set $E \subset J \setminus B(\omega,r)$ such that

$$\bigcup_{z\in E} B(z,\varrho/2) \supset J \setminus B(\omega,r). \tag{6.7}$$

Since $J \setminus B(\omega,r)$ is compact, there exists a finite set $E \subset J \setminus B(\omega,r)$ such that

$$\bigcup_{z\in E} B(z,\varrho/2) \supset J \setminus B(\omega,r). \tag{6.7}$$

Since E is finite, it follows from (6.6) that there exists $m \geqslant 1$ such that for every $z \in E$, $n \geqslant m$ and $v \in \mathscr{B}_n(z)$

$$\operatorname{diam} T_v^{-n}\big(B(z,\varrho)\big) < \tfrac{1}{2}\alpha. \tag{6.8}$$

Choose now $0 < \beta \leqslant \frac{1}{2}\varrho$ so small that for every $x \in J \setminus B(\omega,r)$, every $0 \leqslant n < m$ and every holomorphic inverse branch $T_v^{-n}\colon B(x,2\varrho) \to \bar{\mathbf{C}}$ defined on $B(x,2\varrho)$ we have

$$\operatorname{diam} T_v^{-n}\big(B(x,\beta)\big) < \tfrac{1}{2}\alpha. \tag{6.9}$$

From (6.7) it follows that for every $x \in J \setminus B(\omega,r)$ one can find $z \in E$ such that $B(x,\frac{1}{2}\varrho) \subset B(z,\varrho)$. Therefore, as $\beta \leqslant \frac{1}{2}\varrho$, we conclude from (6.8) and (6.9) that

$$\operatorname{diam} T_v^{-n}\big(B(y,\beta)\big) < \tfrac{1}{2}\alpha \tag{6.10}$$

for every $n \geqslant 0$, $y \in J \setminus B(\omega,r)$ and $v \in \mathscr{B}_n(y)$. Take now $k \geqslant 2$ so large that

$$2\alpha/k < \beta \quad \text{and} \quad \alpha/k < r \tag{6.11}$$

and put $\delta = \alpha/k$.

Our aim is to prove that conditions (a)–(f) are satisfied with $U = B(J,\delta)$. Condition (f) for the component A follows immediately from the formula $T(A) = A$. In order to prove it for $\bar{\mathbf{C}} \setminus \bar{A}$ suppose that for some $x \in B(J,\alpha) \cap (\bar{\mathbf{C}} \setminus \bar{A})$ we have $T(x) \in \bar{A}$. Then there is $z \in J$ such that $x \in B(z,\alpha)$ and, because of (6.3), $T(x) \in B\big(T(z),\sigma\big) \cap \bar{A}$. Thus, applying (6.2), one finds $y \in B(z,\eta) \cap \bar{A}$ such that $T(y) = T(x)$. Since $y \neq x$ and $y, x \in B(z,\eta)$, this contradicts (6.1) and shows that

$$T\big(B(J,\alpha) \cap (\bar{\mathbf{C}} \setminus \bar{A})\big) \subset \bar{\mathbf{C}} \setminus \bar{A}.$$

Consequently, as $U = B(J,\delta) \subset B(J,\alpha)$, condition (f) is fulfilled for both connected components A and $\bar{\mathbf{C}} \setminus \bar{A}$. Moreover,

$$B(J,\alpha) \cap T^{-1}(J) = J. \tag{6.12}$$

In view of (6.4) condition (e) is satisfied with $W = B(\omega,r) \cap U$. So we only need to prove (c) and (d). For this suppose that $T^n(z) \in U$ for every $n \geqslant 0$ but $T^n(z)$ does not converge to ω. Then, in view of (6.4), there exists a sequence $\{n_j\}_{j=1}^{\infty}$ of integers increasing to infinity such that

$$T^{n_j}(z) \notin B(\omega, 2r) \quad \text{for every } j \geqslant 1. \tag{6.13}$$

Let $y \in \bar{U} \setminus B(\omega,2r)$ be an accumulation point of $\{T^{n_j}(z)\}_{j=1}^{\infty}$. Passing, if necessary, to a subsequence we can assume that

$$T^{n_j}(z) \in B(y,\beta/2) \quad \text{for every } j \geqslant 1. \tag{6.14}$$

As, by (6.11), $\delta < \min(\frac{1}{2}\beta, r)$, we therefore can find $\zeta \in J \backslash B(\omega, r)$ such that

$$y, T^{n_j}(z) \in B(\zeta, \beta). \tag{6.15}$$

Fix now $j \geqslant 1$. In view of (6.5) there exists a holomorphic inverse branch $T_v^{-n_j}$: $B(\zeta, 2\varrho) \to \bar{\mathbf{C}}$ of T^{n_j} defined on $B(\zeta, 2\varrho)$ and determined by the condition

$$T_v^{-n_j}(T^{n_j}(z)) = z. \tag{6.16}$$

We claim that

$$T_v^{-n_j}(\zeta) \in J. \tag{6.17}$$

Suppose that (6.17) is not true. Then, as $\zeta \in J$, there exists $0 \leqslant l \leqslant n_j - 1$ such that $T^{l+1} \circ T_v^{-n_j}(\zeta) \in J$ and $T^l \circ T_v^{-n_j}(\zeta) \notin J$. Hence, because of (6.12),

$$T^l \circ T_v^{-n_j}(\zeta) \notin B(J, \alpha). \tag{6.18}$$

On the other hand, since $T^l \circ T_v^{-n_j}$: $B(\zeta, 2\varrho) \to \bar{\mathbf{C}}$ is a holomorphic inverse branch of $T^{n_j - l}$, it follows from (6.10) that $\operatorname{diam} T^l \circ T_v^{-n_j}(B(\zeta, \beta)) < \frac{1}{2}\alpha$. Moreover, since any nonnegative iteration of z belongs to U, it follows from (6.16) and (6.15) that $T^l(z) \in B(J, \delta) \cap T^l \circ T_v^{-n_j}(B(\zeta, \beta))$. Consequently $T^l \circ T_v^{-n_j}(\zeta) \in B(J, \delta + \frac{1}{2}\alpha)$ and, as $k \geqslant 2$, we conclude that $T^l \circ T_v^{-n_j}(\zeta) \in B(J, \alpha)$. This contradicts (6.18) and proves (6.17). As $\zeta \in J \backslash B(\omega, r)$, letting $j \to \infty$, it follows from (6.15)–(6.17) and (6.6) that $z \in J$. Therefore condition (c) is satisfied. Condition (d) follows now immediately from (c) and Leau's local picture around a rationally indifferent fixed point. The proof is finished. ∎

Remark 6.2. Note that we have needed the assumption for J to be a Jordan curve only to have formula (6.2).

The following result follows immediately from Lemma 6.1 and Theorem 5.2.

THEOREM 6.3. *If A is simply connected component of* $\operatorname{Int} K$ *and if* $\partial A \cap \overline{\bigcup_{n=0}^{\infty} f^n(C)} = \{\omega\}$ *then ∂A is either a real-analytic curve or* $\operatorname{HD}(\partial A) > 1$.

Adopting now the method of the proof of Lemma 9.1 of [Br] we can deduce from Theorem 6.3 the following stronger statement.

COROLLARY 6.4. *If the assumptions of Theorem* 6.3 *are satisfied then either* $\operatorname{HD}(\partial A) > 1$ *or, up to a biholomorphic change of coordinates*, f: $\bar{\mathbf{C}} \to \bar{\mathbf{C}}$ *is a finite Blaschke product.*

Now, using the work of Douady and Hubbard ([DH]), we shall describe an efficient condition for the assumptions of Theorem 6.3 to be satisfied. So, let f: $\bar{\mathbf{C}} \to \bar{\mathbf{C}}$ be a polynomial of degree $d \geqslant 2$ and recall that the *filled-in Julia set* $K(f)$ of f is defined to be $\{z \in \mathbf{C}$: $\{f^n(z)\}_{n=0}^{\infty}$ is bounded$\}$. Note that $K(f)$ is compact and its boundary $\partial K(f)$ is the usual Julia set of f.

Let S_1 be the set of all periodic sources of f, S_2 the set of all periodic sinks and S_3 the set of all rationally indifferent periodic points. We say that the

polynomial f satisfies *condition* $(*)$ if and only if for every critical point c of f at least one of the following two conditions is satisfied:

$$\text{the } \omega\text{-limit set of } c \text{ is contained in } S_2 \cup S_3, \tag{6.19}$$

$$T^n(c) \in S_1 \quad \text{for some } n \geqslant 1. \tag{6.20}$$

If $(*)$ is satisfied then, as states the result of lecture 10 of [DH], the filled-in Julia set $K(f)$ is connected and locally connected. Consequently, as $\bar{\mathbf{C}} \backslash K(f)$ is connected, applying Proposition 3, p. 13 of [DH], we conclude that each connected component of $\operatorname{Int} K(f)$ is simply connected and its boundary is a Jordan curve. Since every connected component of $\operatorname{Int} K(\omega, f)$, where ω is a rationally indifferent periodic point of f, is also a connected component of $\operatorname{Int} K(f)$, we therefore have proved the following result providing some sufficient conditions for Theorem 6.3 and Corollary 6.4 to be applicable.

LEMMA 6.5. *If f: $\bar{\mathbf{C}} \to \bar{\mathbf{C}}$ is a polynomial which satisfies condition* $(*)$, *if ω is a rationally periodic point of f and if A is a connected component of* $\operatorname{Int} K(\omega, f)$ *then the assumptions of Theorem* 6.3 *are satisfied.*

Remark 6.6. In particular, Theorem 6.3 and Corollary 6.4 apply to each polynomial of degree 2 which has a rationally indifferent fixed point.

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Harmonic, Gibbs and Hausdorff measures on repellers for holomorphic maps, II

by

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Abstract. We prove the hard part of the Refined Volume Lemma, postponed from Part I, leading to the following dichotomy:

For a simply connected domain $\Omega \subset \hat{\mathbb{C}}$ with the boundary $\partial\Omega$ preserved by a holomorphic map defined on its neighbourhood, repelling on the side of Ω, either $\partial\Omega$ is a real-analytic circle or interval or else a harmonic measure ω on $\partial\Omega$ viewed from Ω is singular with respect to the Hausdorff measure Λ_{Φ_c} with Makarov's function $\Phi_c(t) = t\exp\left(c\sqrt{\log(1/t)\log\log\log(1/t)}\right)$ for $c > c(\omega) = \sqrt{2\sigma^2/\chi} \neq 0$ ($\sigma^2 = \sigma^2(\omega)$ a certain asymptotic variance and χ a Lyapunov characteristic exponent) and ω is absolutely continuous for $c < c(\omega)$.

We also prove the above for $\partial\Omega$ a mixing piecewise repeller including the case of the limit set for a quasi-Fuchsian group, the boundary of the "snowflake" and more generally Carleson's fractal Jordan curves.

Finally, we study complex 1-parameter families of mixing repellers. In particular, if $\partial\Omega$ is the boundary of the basin of attraction to ∞ for the iteration of $z \mapsto z^2 + a$ we prove that $\sigma^2(\omega)$ is a subharmonic and real-analytic function of a, compute its quadratic part at $a = 0$ and estimate all other coefficients of the power series expansion with respect to a.

Contents

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