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ALGEBRAIC PROPERTIES OF THE E-TRANSFORMATION

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Let (S_n) be a sequence of real or complex numbers. The E-transformation consists in transforming (S_n) into a set of sequences (or equivalently, into a two dimensional array) whose entries are given by

$$E_k^{(n)} = \frac{\begin{vmatrix} S_n & & S_{n+k} \\ g_1(n) & & g_1(n+k) \\ \dots & \dots & \dots \\ g_k(n) & \dots & g_k(n+k) \end{vmatrix}}{\begin{vmatrix} 1 & & 1 \\ g_1(n) & & g_1(n+k) \\ \dots & \dots & \dots \\ g_k(n) & & g_k(n+k) \end{vmatrix}},$$

where the g_i's are given auxiliary sequences, which can also depend on (S_n). This formalism is quite general since it contains most of the sequence transformations actually used to accelerate the convergence of the given sequence (S_n). It also includes the problem of interpolation by a linear family forming a Chebyshev system [7], the general interpolation problem [3] and least squares interpolation and extrapolation [4].

The $E_k^{(n)}$'s can be recursively computed by the so-called E-algorithm, thus avoiding the computation of the determinants in the above formula. The E-algorithm is a particular case of an algorithm due to Mühlbach [7] which generalizes the Neville–Aitken scheme. The E-algorithm is as follows [2], [6] (the operator Δ operates on the upper index n):

$$E_0^{(n)} = S_n, \quad g_{0,i}^{(n)} = g_i(n), \quad n = 0, 1, \dots,$$
$$E_k^{(n)} = E_{k-1}^{(n)} - g_{k-1,k}^{(n)} \Delta E_{k-1}^{(n)} / \Delta g_{k-1,k}^{(n)}, \quad k = 1, 2, \dots,$$
$$g_{k,i}^{(n)} = g_{k-1,i}^{(n)} - g_{k-1,k}^{(n)} \Delta g_{k-1,i}^{(n)} / \Delta g_{k-1,k}^{(n)} \quad i \geqslant k.$$

The aim of this paper is to give some algebraic properties of the E-transformation. To this purpose we need a more convenient notation explicitly indicating that $E_k^{(n)}$ depends on the sequence (S_n) and on $(g_1(n)), \ldots, (g_k(n))$. Thus we write

$$E_k(S_n;\, g_1(n), \ldots, g_k(n)) = E_k^{(n)}.$$

The denominators appearing in $E_k^{(n)}$ will always be assumed to be different from zero.

The first three properties are obvious consequences of properties of determinants.

PROPERTY 1. $\forall a_i \neq 0$ *for* $i = 1, \ldots, k$

$$E_k(S_n;\, a_1 g_1(n), \ldots, a_k g_k(n)) = E_k(S_n;\, g_1(n), \ldots, g_k(n)).$$

PROPERTY 2. $\forall a_1 \neq 0, a_2, \ldots, a_k$

$$E_k(S_n;\, a_1 g_1(n) + \ldots + a_k g_k(n), g_2(n), \ldots, g_k(n)) = E_k(S_n;\, g_1(n), \ldots, g_k(n)).$$

Obviously a similar result holds if g_i is replaced by $a_1 g_1 + \ldots + a_k g_k$ with $a_i \neq 0$ or if some g_i's are replaced by linear combinations of the others. $E_k(S_n;\, g_1(n), \ldots, g_k(n))$ is a symmetric function of $g_1, \ldots, g_k$.

PROPERTY 3. $\forall a_1, \ldots, a_k$

$$E_k(S_n + a_1 g_1(n) + \ldots + a_k g_k(n);\, g_1(n), \ldots, g_k(n)) = E_k(S_n;\, g_1(n), \ldots, g_k(n)).$$

This property must be properly understood if the g_i depend on the initial sequence (S_n). In that case, this result is true when (S_n) is replaced by $(S_n + a_1 g_1(n) + \ldots + a_k g_k(n))$ but the $g_1(n), \ldots, g_k(n)$ remain unchanged. For example, if $g_i(n) = \Delta S_{n+i-1}$ then the E-transformation reduces to Shanks transformation (the ε-algorithm). If we replace (S_n) by $(S_n + g_1(n) + \ldots$ $\ldots + g_i(n) = S_{n+i})$ for $i \leqslant k$ then, by Property 3, we get [1], p. 172,

$$E_k(S_n;\, \Delta S_n, \ldots, \Delta S_{n+k-1}) = E_k(S_{n+i};\, \Delta S_n, \ldots, \Delta S_{n+k-1}).$$

We shall now point out some properties in which the quasi-linearity of the g_i's play a fundamental role. We now assume that one of the three following conditions is satisfied:

(i) g_i's are independent of (S_n).

(ii) $\forall i, \exists a_i \neq 0$ such that $g_i(n;\, a S_n + b) = a_i g_i(n;\, S_n)$ where $a \neq 0$ and b are constants.

(iii) the E-transformation is applied with the same g_i's.

PROPERTY 4. *Under one of the conditions* (i), (ii) *or* (iii), $\forall a \neq 0$ *and* $\forall b$

$$E_k(aS_n + b;\, g_1(n), \ldots, g_k(n)) = aE_k(S_n;\, g_1(n), \ldots, g_k(n)) + b.$$

PROPERTY 5. *Under one of the conditions* (i) *or* (iii), *if* $\forall n$, $d_n \neq 0$ *then*

$$E_k(S_n;\, d_n g_1(n), \ldots, d_n g_k(n)) = \frac{E_k(S_n d_n^{-1};\, g_1(n), \ldots, g_k(n))}{E_k(d_n^{-1};\, g_1(n), \ldots, g_k(n))}.$$

Proof. In the formula expressing of $E_k(S_n;\, d_n g_1(n), \ldots, d_n g_k(n))$ as a quotient of two determinants, let us divide the first columns of the numerator and the denominator by d_n, the second ones by d_{n+1} and so on. The claim is then obtained by multiplying the numerator and the denominator by the determinant appearing in the denominator of $E_k(S_n;\, g_1(n), \ldots, g_k(n))$. ∎

This property is Theorem 2.2 of Håvie [6]. It shows how to compute the $E_k(S_n;\, d_n g_1(n), \ldots, d_n g_k(n))$ from the $E_k(S_n d_n^{-1};\, g_1(n), \ldots, g_k(n))$, the $E_k(d_n^{-1};\, g_1(n), \ldots, g_k(n))$, and the $g_{k-1,k}^{(n)}$ obtained from $g_{0,i}^{(n)} = g_i(n)$ without computing new auxiliary $g_{k,i}^{(n)}$.

This property can also be written as

$$E_k(d_n S_n;\, g_1(n), \ldots, g_k(n)) = \frac{E_k(S_n;\, d_n^{-1} g_1(n), \ldots, d_n^{-1} g_k(n))}{E_k(d_n^{-1};\, d_n^{-1} g_1(n), \ldots, d_n^{-1} g_k(n))}.$$

The following properties can be obtained either directly or as consequences of the preceding one.

PROPERTY 6. *Under one of the conditions* (i) *or* (iii), *if* $\forall n$, $S_n \neq 0$ *then*

$$E_{m+p}(S_n; f_1(n), \ldots, f_m(n),\, S_n h_1(n), \ldots, S_n h_p(n))$$
$$= \frac{1}{E_{m+p}(S_n^{-1};\, S_n^{-1} f_1(n), \ldots, S_n^{-1} f_m(n),\, h_1(n), \ldots, h_p(n))}.$$

Proof. Obvious from the second form of Property 5 and in view of $E_k(1;\, g_1(n), \ldots, g_k(n)) = 1$. ∎

For $m = 0$, we have

PROPERTY 7. *Under one of the conditions* (i) *or* (iii), *if* $\forall n$, $S_n \neq 0$ *then*

$$E_k(S_n;\, S_n g_1(n), \ldots, S_n g_k(n)) = \frac{1}{E_k(S_n^{-1};\, g_1(n), \ldots, g_k(n))}.$$

PROPERTY 8. *Under one of the conditions* (i) *or* (iii), *if* $\forall n$, $d_n \neq 0$, *then*

$$E_k(S_n;\, d_n g_1(n), g_2(n), \ldots, g_k(n)) = \frac{E_k(S_n d_n^{-1};\, g_1(n), d_n^{-1} g_2(n), \ldots, d_n^{-1} g_k(n))}{E_k(d_n^{-1};\, g_1(n), d_n^{-1} g_2(n), \ldots, d_n^{-1} g_k(n))}.$$

A similar property holds if g_i is replaced by $d_n g_i(n)$ or if several g_i's are changed accordingly.

PROPERTY 9. *Under one of the conditions* (i) *or* (iii), *if* $\forall n$, $d_n \neq 0$ *then*

$$E_k(d_n S_n;\, d_n g_1(n), \ldots, d_n g_k(n)) = \frac{E_k(S_n;\, g_1(n), \ldots, g_k(n))}{E_k(d_n^{-1};\, g_1(n), \ldots, g_k(n))}.$$

In the particular case of $g_i(n) = \Delta S_{n+i-1}$ (Shanks transformation or the ε-algorithm) let us replace (S_n) by $(\lambda^n S_n)$. By the preceding property,

$$E_k(\lambda^n S_n;\ \lambda^n(\lambda S_{n+1}-S_n), \ldots, \lambda^{n+k-1}(\lambda S_{n+k}-S_{n+k-1}))$$
$$= \frac{E_k(S_n;\ \lambda S_{n+1}-S_n, \ldots, \lambda^{k-1}(\lambda S_{n+k}-S_{n+k-1}))}{E_k(\lambda^{-n},\ \lambda S_{n+1}-S_n, \ldots, \lambda^{k-1}(\lambda S_{n+k}-S_{n+k-1}))}$$
$$= \frac{E_k(S_n;\ \lambda S_{n+1}-S_n, \ldots, \lambda S_{n+k}-S_{n+k-1})}{E_k(\lambda^{-n},\ \lambda S_{n+1}-S_n, \ldots, \lambda S_{n+k}-S_{n+k-1})}$$

from Property 1. Thus

$$e_k(\lambda^n S_n) = \frac{\lambda^n \lambda^k H_{k+1}(S_n)}{\lambda^{-k} H_k(v_n(\lambda))}$$

where $H_k(u_n)$ is the usual Hankel determinant and

$$v_n(\lambda) = \lambda^2 S_{n+2} - 2\lambda S_{n+1} + S_n.$$

If we set

$$w_n(\lambda) = S_{n+2} - 2\lambda S_{n+1} + \lambda^2 S_n$$

then

$$v_n(\lambda) = \lambda^2 w_n(\lambda^{-1})$$

and we obtain an already known result [1], p. 175,

$$e_k(\lambda^n S_n) = \frac{\lambda^n H_{k+1}(S_n)}{H_k(w_n(\lambda^{-1}))}.$$

These numbers can recursively obtained by applying the ε-algorithm to the sequence $(\lambda^n S_n)$ or an algorithm due to Guzinski [5] to (S_n).

Let us now state some properties of the E-algorithm. They can be deduced from the rules of the algorithm and the determinantal definition of $E_k^{(n)}$.

From Sylvester's determinantal identity we immediately obtained

PROPERTY 10.

$$\frac{\Delta E_{k-1}^{(n)}}{\Delta g_{k-1,k}^{(n)}} = \frac{\begin{vmatrix} S_n & & S_{n+k} \\ g_1(n) & & g_1(n+k) \\ \cdots & & \cdots\cdots \\ g_{k-1}^{(n)} & & g_{k-1}(n+k) \\ 1 & \cdots & 1 \end{vmatrix}}{\begin{vmatrix} g_k(n) & & g_k(n+k) \\ g_1(n) & & g_1(n+k) \\ & & \cdots\cdots \\ g_{k-1}(n) & & g_{k-1}(n+k) \\ 1 & & 1 \end{vmatrix}}$$

In the case of the ε-algorithm this property gives

$$\frac{1}{\Delta\varepsilon_{2k-1}^{(n)}} = -\frac{g_{k-1,k}^{(n+1)}\Delta\varepsilon_{2k-2}^{(n)}}{\Delta g_{k-1,k}^{(n)}}.$$

PROPERTY 11.

$$g_{k,i}^{(n)} = E_k(g_i(n); g_1(n), \ldots, g_k(n)).$$

Thus the preceding properties can be applied to the $g_{k,i}^{(n)}$'s.

An important property is the following one:

PROPERTY 12.

$$E_{k+m}^{(n)} = \begin{vmatrix} E_m^{(n)} & \cdots & E_m^{n+k} \\ g_{m,m+1}^{(n)} & \cdots & g_{m,m+1}^{(n+k)} \\ \cdots & \cdots & \cdots \\ g_{m,m+k}^{(n)} & \cdots & g_{m,m+k}^{(n+k)} \end{vmatrix} \Bigg/ \begin{vmatrix} 1 & \cdots & 1 \\ g_{m,m+1}^{(n)} & \cdots & g_{m,m+1}^{(n+k)} \\ \cdots & \cdots & \cdots \\ g_{m,m+k}^{(n)} & \cdots & g_{m,m+k}^{(n+k)} \end{vmatrix}$$

and a similar formula resulting by interchanging m and k.

Proof. Let $\tilde{E}_k^{(n)}$ be the quantities obtained by applying the E-algorithm with the initial values

$$\tilde{E}_0^{(n)} = E_m^{(n)}, \qquad \tilde{g}_{0,i}^{(n)} = g_{m,m+i}^{(n)}, \qquad m \text{ fixed}.$$

Thus $\tilde{E}_k^{(n)}$ is given by the quotient of determinants on the right hand side of the asserted equality. But, in the E-algorithm, each step is obtained from the preceding ones and thus $\tilde{E}_k^{(n)}$ is the $(k+m)$th step with the usual initializations. Thus $\tilde{E}_k^{(n)} = \tilde{E}_{k+m}^{(n)}$. ■

In view of Property 11, a similar result holds for $g_{k+m,i}^{(n)}$.

For $m = 0$, Property 12 reduces to the usual determinantal formula for $E_k^{(n)}$. If $k = 1$, we recover the rules of the E-algorithm. When m is arbitrary, we get a rule for computing directly the elements of column $m+k$ from those of column m without computing intermediate columns. Such a rule can be used to avoid a division by zero in the algorithm thus providing a singular rule for the E-algorithm. Skipping over some columns that have almost equal neighbouring elements can also help to avoid numerical instability.

For example if

$$g_{k-1,k}^{(n)} = g_{k-1,k}^{(n+1)} \neq g_{k-1,k}^{(n+2)},$$

we have

$$E_{k+1}^{(n)} = E_k^{(n+1)} - \frac{g_{k,k+1}^{(n+1)}\Delta E_{k-1}^{(n)}}{\Delta g_{k-1,k+1}^{(n)}}.$$

From Property 12 we obtain

PROPERTY 13. *If* $\forall n$,

$$E_m^{(n)} = S + a_{m+1}\, g_{m,m+1}^{(n)} + a_{m+2}\, g_{m,m+2}^{(n)} + \dots$$

then $\forall n$,

$$E_{k+m}^{(n)} = S + a_{k+m+1}\, g_{k+m,k+m+1}^{(n)} + a_{k+m+2}\, g_{k+m,k+m+2}^{(n)} + \dots$$

PROPERTY 14. *A necessary and sufficient condition in order that* $\forall n$, $E_{k+m}^{(n)} = S$, *is that* $\forall n$,

$$E_m^{(n)} = S + a_{m+1}\, g_{m,m+1}^{(n)} + \dots + a_{m+k}\, g_{m,m+k}^{(n)}.$$

These two properties generalize well-known results when $m = 0$.

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