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On the existence and regularity of solutions of linear equations in spaces $H_k(G)$

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Abstract. Let G be an open set in $\mathbf{R}^n$. Furthermore, let k: $\mathbf{R}^n \to \mathbf{R}$ be a positive continuous weight function such that $c(1+|\xi|^2)^{m/2} \leqslant k(\xi) \leqslant C(1+|\xi|^2)^{M/2}$ with some $c > 0$, C, m and $M \in \mathbf{R}$. The space $H_k(G)$ denotes (essentially) the completion of $C_0^\infty(G)$ with respect to the norm

$$\|\varphi\|_k = \big((2\pi)^{-n} \int_{\mathbf{R}^n} |(F\varphi)(\xi)\, k(\xi)|^2\big)^{1/2},$$

where F is the Fourier transform. By $\mathbf{H}_k(G)$ one denotes the subspace of $D'(G)$ such that u lies in $\mathbf{H}_k(G)$ if there exists $f_u \in H_k(\mathbf{R}^n)$ with $u = f_u|_G$. The spaces $\mathbf{H}_k(G)$ and $H_k(G)$ are equipped with a Hilbert space structure.

The paper considers the maximal extension $L^{\#}_{k_1,k_2}$: $\mathbf{H}_{k_1}(G) \to \mathbf{H}_{k_2}(G)$ of a linear operator L: $C_0^\infty(G) \to C_0^\infty(G)$. Criteria for the existence of a continuous one-sided inverse and of a compact one-sided inverse of $L^{\#}_{k_1,k_2}$ are obtained. Furthermore, conditions for the existence of a Fredholm realization $L^{\wedge}_{k_1,k_2}$ (with ind $L^{\wedge}_{k_1,k_2} = 0$) are established. The presented theory is applied to smooth partial differential operators $L(x, D)$ to obtain conditions for the existence of one-sided inverses, for the existence of Fredholm realizations and for the local regularity of solutions of the maximal realizations of $L(x, D)$.

1. Introduction. Let G be an open set in $\mathbf{R}^n$ and k: $\mathbf{R}^n \to \mathbf{R}$ be a positive continuous function so that

$$c(1+|\xi|^2)^{m/2} \leqslant k(\xi) \leqslant C(1+|\xi|^2)^{M/2} \quad \text{for } \xi \in \mathbf{R}^n \tag{1.1}$$

with some $c > 0$, $C > 0$, m and $M \in \mathbf{R}$. Denote by $H_k(G)$ the completion of $C_0^\infty(G)$ (imbedded into the space S' of all tempered distributions in the natural way) with respect to the norm

$$\|\varphi\|_k = \big((2\pi)^{-n} \int_{\mathbf{R}^n} |(F\varphi)(\xi)\, k(\xi)|^2\big)^{1/2}.$$

Here F: $S \to S$ denotes the Fourier transform. Furthermore, denote by $\mathbf{H}_k(G)$ the subspace of the distribution space $D'(G)$ such that for each $u \in \mathbf{H}_k(G)$ one finds an element $f_u \in H_k(\mathbf{R}^n)$ with the property $u = f_u|_G$. The spaces can be equipped with the Hilbert space structure (cf. 2.1 and 2.2). $\mathbf{K}$ denotes the set of functions satisfying (1.1).

Suppose that L is a linear operator $C_0^\infty(G)\to C_0^\infty(G)$ and suppose that the minimal closed extension $L^{\sim}_{1/k_2^\vee,1/k_1^\vee}: H_{1/k_2^\vee}(G)\to H_{1/k_1^\vee}(G)$ exists (where $k^\vee\in \boldsymbol{K}$; $k^\vee(\xi)=k(-\xi)$). Furthermore, assume that

$$\|L\varphi\|_{1/k_1^\vee} \geqslant c\|\varphi\|_{1/k_2^\vee} \quad \text{for all } \varphi\in C_0^\infty(G). \tag{1.2}$$

Then one finds a continuous linear operator Q: $\boldsymbol{H}_{k_2}(G)\to \boldsymbol{H}_{k_1}(G)$ such that (cf. Theorem 3.2)

$$L^{\#}_{k_1,k_2}\circ Q = I_{k_2}, \tag{1.3}$$

where $L^{\#}_{k_1,k_2}$: $\boldsymbol{H}_{k_1}(G)\to \boldsymbol{H}_{k_2}(G)$ is the maximal extension of L and where I_{k_2} denotes the identity operator $\boldsymbol{H}_{k_2}(G)\to \boldsymbol{H}_{k_2}(G)$.

Assuming that $k^{\sim}\geqslant 1$, that the imbedding $H_{(k^{\sim}/k_2)^\vee}(G)\to H_{1/k_2^\vee}(G)$ is compact, that $L^{\sim}_{(k^{\sim}/k_2)^\vee,1/k_1^\vee}$ exists and that

$$\|L\varphi\|_{1/k_1^\vee} \geqslant c\|\varphi\|_{(k^{\sim}/k_2)^\vee} \quad \text{for } \varphi\in C_0^\infty(G), \tag{1.4}$$

we show that one finds a compact operator K: $\boldsymbol{H}_{k_2}(G)\to \boldsymbol{H}_{k_1}(G)$ such that

$$L^{\#}_{k_1,k_2}\circ K = I_{k_2}. \tag{1.5}$$

This result implies a sufficient condition for the existence of the Fredholm realization $L^{\wedge}_{k_1,k_2}$ of $L^{\#}_{k_1,k_2}$ with

$$\operatorname{ind} L^{\wedge}_{k_1,k_2} = 0 \tag{1.6}$$

(cf. Theorems 3.5 and 4.1). Furthermore, one gets a sufficient condition for the inclusion

$$D(L^{\#}_{k_1,k_2}) \subset H^{\mathrm{loc}}_{k_1 k_1^{\sim}}(G), \tag{1.7}$$

when the solutions of the homogeneous equation $L^{\#}_{k_1,k_2}u+Cu=0$ lie in $H^{\mathrm{loc}}_{k_1k_1^{\sim}}(G)$ (cf. Corollary 3.7 and Theorem 4.3).

Let G be a bounded open set and let $k^{\sim}\in \boldsymbol{K}'$ (for the definition of $\boldsymbol{K}'$ cf. 2.1) so that $k^{\sim}(\xi)\to\infty$ with $|\xi|\to\infty$. Furthermore, let $L(x, D)$ be a smooth linear partial differential operator such that

$$\operatorname{Re}(L'(x, D)\varphi, \varphi)_0 \geqslant C_1\|\varphi\|^2_{k^{\sim\vee}} - C_2\|\varphi\|_0^2 \tag{1.8}$$

and

$$\|L(x, D)\varphi\|_{1/k^{\sim\vee}} \leqslant C\|\varphi\|_{k^{\sim\vee}} \quad \text{for all } \varphi\in C_0^\infty(G). \tag{1.9}$$

Then the presented theory below implies that one can find a Fredholm realization $L'^{\wedge}$ of $L'^{\#} := L'^{\#}_{1,1}$ such that

$$\operatorname{ind} L'^{\wedge} = 0 \quad \text{and} \quad D(L'^{\wedge})\subset \boldsymbol{H}_{k^{\sim}}(G). \tag{1.10}$$

In the case when $N(L'^{\#}+C_2 I)\subset H^{\mathrm{loc}}_{k^{\sim}}(G)$ one has (cf. Corollary 4.4)

$$D(L'^{\#})\subset H^{\mathrm{loc}}_{k^{\sim}}(G). \tag{1.11}$$

For sufficient algebraic criteria of (1.8) cf. [9], p. 55; [8]; [1], p. 19; [5]; [7] and [11].

Suppose that $L(D)$ is a partial differential operator with constant coefficients, and that G is bounded. Then we show that there exists a continuous operator Q: $H_k(G) \to H_{kL^\sim}(G)$ such that (here we denote $L^\sim(\xi) = (\sum_{|\alpha| \leqslant r} |L^{(\alpha)}(\xi)|^2)^{1/2}$ and $k \in K'$)

$$L'^{\#}_{kL^\sim,k} \circ Q = I_k \tag{1.12}$$

(cf. [4], p. 31). In the case when $L^\sim(\xi) \to \infty$ with $|\xi| \to \infty$ one obtains that there exists a compact operator K: $H_k(G) \to H_k(G)$ such that

$$L'^{\#}_{k,k} \circ K = I_k \tag{1.13}$$

(cf. Theorem 4.7). In this case also the inclusion

$$D(L'^{\#}_{k,k}) \subset H^{\mathrm{loc}}_{kL'/L'^\sim}(G) \tag{1.14}$$

is verified, where $L'^\sim(\xi) = (\sum_{|\alpha| > 0} |L^{(\alpha)}(\xi)|^2)^{1/2}$.

2. Notation and preliminaries

2.1. Let G be an open set in $\boldsymbol{R}^n$. For the definition and basic properties of the spaces $C_0^\infty(G)$, $C^\infty(G)$, $D'(G)$, $E'(G)$, S and S' we refer to monograph [3], pp. 1–53. The Fourier transform F: $S \to S$ is defined by

$$(F\varphi)(\xi) = \int_{\boldsymbol{R}^n} \varphi(x) e^{-i(x,\xi)} dx. \tag{2.1}$$

The Fourier transform $S' \to S'$ (which we also denote by F) is defined by

$$(Fu)(\varphi) = u(F\varphi) \qquad \text{for } \varphi \in S. \tag{2.2}$$

Denote by $\boldsymbol{K}$ the class of positive continuous weight functions k: $\boldsymbol{R}^n \to \boldsymbol{R}$ such that for each $k \in \boldsymbol{K}$ one finds constants $c > 0$, $C > 0$, m, $M \in \boldsymbol{R}$ with which the estimate

$$ck_m(\xi) \leqslant k(\xi) \leqslant Ck_M(\xi) \qquad \text{for all } \xi \in \boldsymbol{R}^n \tag{2.3}$$

holds, where we write $k_s(\xi) = (1 + |\xi|^2)^{s/2}$; $s \in \boldsymbol{R}$. By $\boldsymbol{K}'$ we denote the subclass of $\boldsymbol{K}$ such that besides of (2.3) each $k \in \boldsymbol{K}'$ satisfies the estimate

$$k(\xi + \eta) \leqslant C' k_{M'}(\xi) k(\eta) \qquad \text{for all } \xi,\ \eta \in \boldsymbol{R}^n. \tag{2.4}$$

Let k be in $\boldsymbol{K}$. Define a scalar product in $C_0^\infty(G)$ by

$$(\varphi, \psi)_k = (2\pi)^{-n} \int_{\boldsymbol{R}^n} (F\varphi)(\xi) \overline{(F\psi)(\xi)} k^2(\xi) d\xi. \tag{2.5}$$

Furthermore, let $H_k^\sim(G)$ be the completion of $C_0^\infty(G)$ with respect to the norm $\|\varphi\|_k := ((\varphi, \varphi)_k)^{1/2}$. Let u be in $H_k^\sim(G)$ and let $\{\varphi_n\}$ be a representative of u. In virtue of the Banach–Steinhaus Theorem, the Parseval identity and the Hölder inequality one sees that the quantity λu defined by

$$(2.6)\qquad (\lambda u)(\varphi) = \lim_{n\to\infty} \int_G \varphi_n(x)\,\varphi(x) =: \lim_{n\to\infty} \varphi_n(\varphi) \quad \text{for } \varphi \in S$$

lies in S'. One sees that λ is an injective linear mapping $H_k^{\sim}(G) \to S'$. We denote

$$H_k(G) = \lambda(H_k^{\sim}(G)).$$

The linear space $H_k(G)$ is equipped with the norm $\|v\|_k := \|\lambda^{-1}(v)\|_k := \lim_{n\to\infty} \|\varphi_n\|_k$, where $\{\varphi_n\} \in \lambda^{-1}(v)$. Then $H_k(G)$ is a Hilbert space with the scalar product

$$(2.7)\qquad (u, v)_k := \lim_{n\to\infty} (\varphi_n, \psi_n)_k, \quad \text{where } \{\varphi_n\} \in u \text{ and } \{\psi_n\} \in v.$$

In the case when $G = \boldsymbol{R}^n$ we write $H_k(\boldsymbol{R}^n) = H_k$. One sees that a tempered distribution $u \in S'$ lies in H_k if and only if $Fu \in L_1^{\mathrm{loc}}(G)$ and $(Fu)k \in L_2(\boldsymbol{R}^n)$. In addition, one has

$$(2.8)\qquad (u, v)_k = (2\pi)^{-n} \int_{\boldsymbol{R}^n} (Fu)(\xi)\overline{(Fv)(\xi)}k^2(\xi)d\xi.$$

Due to the Hölder inequality we get

$$(2.9)\qquad |u(\varphi)| \leqslant \|u\|_k \|\varphi\|_{1/k^\vee} \quad \text{for } u \in H_k(G) \text{ and } \varphi \in S,$$

where $k^\vee \in \boldsymbol{K}$; $k^\vee(\xi) = k(-\xi)$.

The Riesz Theorem implies the following characterization of the dual space H_k^* of H_k

THEOREM 2.1. *Let T be in H_k^*. Then there exists a unique element $u \in H_{1/k^\vee}$ such that*

$$T\varphi = u(\varphi) \quad \textit{for all } \varphi \in S.$$

Conversely, suppose that u belongs to $H_{1/k^\vee}$. Then the linear form $T\colon S \to \boldsymbol{C}$ defined by

$$T\varphi = u(\varphi)$$

has a unique continuous extension from $H_k \to \boldsymbol{C}$. In addition one has

$$(2.10)\qquad \|T\| = \|u\|_{1/k^\vee}. \quad \square$$

Remark 2.2. In virtue of Theorem 2.1 we can define a linear isometric isomorphism $j_{1/k^\vee} : H_{1/k^\vee} \to H_k^*$ by

$$j_{1/k^\vee}(u) = T.$$

2.2. Let A be a set in $\boldsymbol{R}^n$. We denote by $H_k^\circ(A)$ the subspace of H_k such that for each $u \in H_k^\circ(A)$ one has

$$\operatorname{supp} u \subset A,$$

where $\operatorname{supp} u$ is the support of $u \in S'$. In the case when A is closed in $\boldsymbol{R}^n$ the space $H_k^\circ(A)$ is closed in H_k.

Let $\boldsymbol{H}_k^\sim(G)$ be the factor space $H_k/H_k^\circ(\boldsymbol{R}^n \backslash G)$. Since $H_k^\circ(\boldsymbol{R}^n \backslash G)$ is closed, $\boldsymbol{H}_k^\sim(G)$ becomes a Banach space with the norm

$$|||U|||_k^\sim = \inf_{u \in U} \|u\|_k.$$

The linear mapping J: $\boldsymbol{H}_k^\sim(G) \to D'(G)$ defined by

$$J(U) = u|_G, \quad \text{where } u \in U,$$

is an injection. Here $u|_G$ denotes the restriction of u on G. We define

$$\boldsymbol{H}_k(G) = J(\boldsymbol{H}_k^\sim(G)) \quad \text{and} \quad |||V|||_k = |||J^{-1}(V)|||_k^\sim$$

Then $\boldsymbol{H}_k(G)$ is a Banach space. $\boldsymbol{H}_k(G)$ can also be equipped with the Hilbert space structure by defining the scalar product by

$$(U, V)_k := (u^\perp, v^\perp)_k,$$

where $U = [u] = [u^\perp \oplus z]$ and $V = [v^\perp \oplus y]$. Here the orthogonal sum $\oplus$ refers to the decomposition $H_k = H_k^\circ(\boldsymbol{R}^n \backslash G)^\perp \oplus H_k^\circ(\boldsymbol{R}^n \backslash G)$.

Furthermore, one sees that a distribution $V \in D'(G)$ lies in $\boldsymbol{H}_k(G)$ if and only if one finds an element f_V from H_k so that

$$V(\varphi) = f_V(\varphi) \quad \text{for all } \varphi \in C_0^\infty(G). \tag{2.11}$$

Let $C_{(0)}^\infty(G)$ be the subspace of $C^\infty(G)$ such that for each $\psi \in C_{(0)}^\infty(G)$ one finds an element f_ψ from $C_0^\infty := C_0^\infty(\boldsymbol{R}^n)$ with which

$$\psi = f_\psi|_G. \tag{2.12}$$

Then $C_{(0)}^\infty(G)$ is dense in $\boldsymbol{H}_k(G)$, since C_0^∞ is dense in H_k.

Let V be in $\boldsymbol{H}_k(G)$ and let $u \in J^{-1}(V)$. Then for all $w \in H_k^\circ(\boldsymbol{R}^n \backslash G)$ and $\varphi \in C_0^\infty(G)$ one has by (2.9)

$$|V(\varphi)| = |u(\varphi)| = |(u+w)(\varphi)| \leqslant \|u+w\|_k \, \|\varphi\|_{1/k^\vee}$$

and then

$$|V(\varphi)| \leqslant |||V|||_k \, \|\varphi\|_{1/k^\vee} \quad \text{for all } V \in \boldsymbol{H}_k(G),\ \varphi \in C_0^\infty(G). \tag{2.13}$$

The Hahn–Banach Theorem and (2.13) implies the following characterization for the dual $H_{1/k^\vee}^*(G)$ of $H_{1/k^\vee}(G)$

THEOREM 2.3. *Let T be in $H_{1/k^\vee}^*(G)$. Then there exists a unique element $U \in \boldsymbol{H}_k(G)$ such that*

$$T\varphi = U(\varphi) \quad \textit{for all} \quad \varphi \in C_0^\infty(G).$$

Conversely, suppose that U belongs to $\boldsymbol{H}_k(G)$. Then the linear form T: $C_0^\infty(G) \to \boldsymbol{C}$ defined by

$$T\varphi = U(\varphi)$$

has a unique continuous extension from $H_{1/k^\vee}(G) \to C$. *In addition, one has*

$$\|T\| = |||U|||_k. \qquad \square \tag{2.14}$$

Remark 2.4. Due to Theorem 2.3 there exists an isometrical isomorphism J_k: $H_k(G) \to H^*_{1/k^\vee}(G)$ such that

$$(J_k U)(\varphi) = U(\varphi) \quad \text{for all } \varphi \in C_0^\infty(G). \tag{2.15}$$

Furthermore, we obtain for the dual $H_k^*(G)$ of $H_k(G)$

THEOREM 2.5. *Let T be in $H_k^*(G)$. Then there exists a unique element $v \in H_{1/k^\vee}(G)$ such that*

$$T(\varphi|_G) = v(\varphi) \quad \textit{for all } \varphi = C_0^\infty.$$

Conversely, suppose that v belongs to $H_{1/k^\vee}(G)$. Then the linear form T: $C_{(0)}^\infty(G) \to C$ defined by

$$T\psi = v(f_\psi)$$

has a unique continuous extension from $H_k(G) \to C$. *In addition, one has*

$$\|T\| = \|v\|_{1/k^\vee}. \tag{2.16}$$

Proof. (A) Let T be in $H_k^*(G)$. Since J_k: $H_k(G) \to H^*_{1/k^\vee}(G)$ is an isometrical isomorphism, one sees that the dual operator J_k^*: $H^{**}_{1/k^\vee}(G) \to H_k^*(G)$ is an isometrical isomorphism. Let $\varkappa_{1/k^\vee}: H_{1/k^\vee}(G) \to H^{**}_{1/k^\vee}(G)$ be the canonical isomorphism. Then the element $v := \varkappa^{-1}_{1/k^\vee}(J_k^*)^{-1}(T)$ lies in $H_{1/k^\vee}(G)$. Choose a sequence $\{\varphi_n\}$ so that $\|\varphi_n - v\|_{1/k^\vee} \to 0$ with $n \to \infty$. Then one sees

$$\begin{aligned} v(\varphi) &= \lim_{n\to\infty} \varphi_n(\varphi) = \lim_{n\to\infty} (\varphi|_G)(\varphi_n) \\ &= \lim_{n\to\infty} (J_k(\varphi|_G))(\varphi_n) = \lim_{n\to\infty} (\varkappa_{1/k^\vee}(\varphi_n))(J_k(\varphi|_G)) \\ &= \lim_{n\to\infty} ((J_k^* \circ \varkappa_{1/k^\vee})(\varphi_n))(\varphi|_G) = (J_k^*(\varkappa_{1/k^\vee} v))(\varphi|_G) \\ &= T(\varphi|_G) \end{aligned}$$

and

$$\|v\|_{1/k^\vee} = \|T\|. \tag{2.18}$$

(B) Suppose that v belongs to $H_{1/k^\vee}(G)$. Choose a sequence $\{\varphi_n\}$ so that $\|\varphi_n - v\|_{1/k^\vee} \to 0$, with $n \to \infty$. Then one has for all $w \in H_k^\circ(R^n \backslash G)$

$$|v(f_\psi)| = \lim_{n\to\infty} \varphi_n(f_\psi) = \lim_{n\to\infty} |(f_\psi + w)(\varphi_n)| \leqslant \|f_\psi + w\|_k \|v\|_{1/k^\vee} \tag{2.19}$$

and then

$$|T\psi| \leqslant |||\psi|||_k \|v\|_{1/k^\vee}. \tag{2.20}$$

This completes the proof. $\square$

Remark 2.6. Due to Theorem 2.5 there exists an isometrical isomorphism $j_{1/k^{\vee}}: H_{1/k^{\vee}}(G) \to H_k^*(G)$ such that

$$(j_{1/k^{\vee}}\, v)(\varphi|_G) = v(\varphi) \quad \text{for all } \varphi \in C_0^\infty. \tag{2.21}$$

We obtain the following

COROLLARY 2.7. *Let k_1 and k_2 be in $\boldsymbol{K}$ so that $k_1 \geqslant k_2$. Then the imbedding λ_1: $\boldsymbol{H}_{k_1}(G) \to \boldsymbol{H}_{k_2}(G)$ is compact if and only if the imbedding λ_2: $H_{1/k_2^{\vee}}(G) \to H_{1/k_1^{\vee}}(G)$ is compact.*

Proof. Suppose that λ_1 is compact. Then the dual operator λ_1^*: $\boldsymbol{H}_{k_2}^*(G) \to \boldsymbol{H}_{k_1}^*(G)$ is also compact. Furthermore, one has for all $\varphi \in C_0^\infty$

$$\begin{aligned}(\lambda_2 v)(\varphi) &= v(\varphi) = (j_{1/k_2^{\vee}}\, v)(\varphi|_G) = (j_{1/k_2^{\vee}}\, v)\big(\lambda_1(\varphi|_G)\big) \\ &= \big(\lambda_1^*(j_{1/k_2^{\vee}}\, v)\big)(\varphi|_G) = \big((j_{1/k_1^{\vee}}^{-1} \circ \lambda_1^* \circ j_{1/k_2^{\vee}})v\big)(\varphi)\end{aligned} \tag{2.22}$$

and then $\lambda_2 = j_{1/k_1^{\vee}}^{-1} \circ \lambda_1^* \circ j_{1/k_2^{\vee}}$ is compact. Similarly one sees that $\lambda_1 = J_{k_2}^{-1} \circ \lambda_2^* \circ J_{k_1}$ is compact when λ_2 is compact. □

Remark 2.8. (A) One also sees that $\boldsymbol{H}_{k_1}(G) \subset \boldsymbol{H}_{k_2}(G)$ if and only if $H_{1/k_2^{\vee}}(G) \subset H_{1/k_1^{\vee}}(G)$.

(B) Suppose that k_1 and k_2 lie in the Hörmander class K of weight functions (cf. [4], p. 4). Furthermore, assume that G is bounded. Since $H_k(G) \subset H_k^\circ(\bar{G})$ one sees that the imbedding λ_2: $H_{1/k_2^{\vee}}(G) \to H_{1/k_1^{\vee}}(G)$ is compact if and only if

$$k_1(\xi)/k_2(\xi) \to 0 \quad \text{with } |\xi| \to \infty \tag{2.23}$$

(cf. [4], pp. 8–9).

2.3. Let $\{K_l\}$ be a sequence of compact subsets of G such that int $K_l \subset K_{l+1}$ and that $G = \bigcup_{l=1}^{\infty} K_l$. Choose $\psi_l \in C_0^\infty(G)$ so that $\psi_l(x) \equiv 1$, $x \in K_l$. Define a semi-norm $q_{l,k}$: $C_0^\infty(G) \to \boldsymbol{R}$ by

$$q_{l,k}(\varphi) = \sup_{1 \leqslant j \leqslant l} \|\psi_j \varphi\|_k,$$

where $k \in \boldsymbol{K}'$. Let d_k: $C_0^\infty(G) \times C_0^\infty(G) \to \boldsymbol{R}$ be the metric

$$d_k(\psi, \varphi) = \sum_{l=1}^{\infty} (1/2^l)\big(q_{l,k}(\psi - \varphi)/(1 + q_{l,k}(\psi - \varphi))\big).$$

Denote by $\tilde{H}_k^{\mathrm{loc}}(G)$ the completion of $C_0^\infty(G)$ with respect to the metric d_k. Then the quantity λu defined by

$$(\lambda u)(\varphi) = \lim \varphi_n(\varphi)$$

(where $\{\varphi_n\} \in u$) belongs to $D'(G)$ (due to the Banach–Steinhaus Theorem). We write $H_k^{\text{loc}}(G) = \lambda(\tilde{H}_k^{\text{loc}}(G))$. Then $H_k^{\text{loc}}(G)$ is a complete space with the metric

$$d_k(u, v) = \lim_{n\to\infty} d_k(\varphi_n, \psi_n), \quad \text{when } \{\varphi_n\} \in \lambda^{-1}(u) \text{ and } \{\psi_n\} \in \lambda^{-1}(v).$$

Clearly one has $C^\infty(G) \subset H_k^{\text{loc}}(G)$. Furthermore, one obtains

THEOREM 2.9. *A distribution $u \in D'(G)$ lies in $H_k^{\text{loc}}(G)$ if and only if $\psi_l u \in H_k$ for each $l \in N$. The metric d_k is given by*

$$d_k(u, v) = \sum_{l=1}^{\infty} \left(\frac{1}{2^l}\right) \frac{\sup\limits_{0\leqslant j\leqslant l} \|\psi_j(u-v)\|_k}{1+\sup\limits_{0\leqslant j\leqslant l} \|\psi_j(u-v)\|_k}. \quad \square \tag{2.24}$$

Remark 2.10. (A) The topology in $H_k^{\text{loc}}(G)$ is equivalent with the topology defined by the semi-norms $u \to \|\psi_l u\|_k$; $l \in N$.

(B) Since by (2.4) one has

$$\|\psi v\|_k \leqslant C\Big((2\pi)^{-1} \int_{R^n} |(F\psi)(\xi) k_{M'}(\xi)| \, d\xi\Big) \|v\|_k =: C \|\psi\|_{1,k_{M'}} \|v\|_k \tag{2.25}$$

for $\psi \in C_0^\infty(G)$ and $v \in H_k$

one sees that the metric d_k does not depend on the sequence $\{K_l\}$.

(C) In view of (2.25) one sees that a distribution $u \in D'(G)$ lies in $H_k^{\text{loc}}(G)$ if and only if $\psi u \in H_k$ for each $\psi \in C_0^\infty(G)$.

(D) $H_k^{\text{loc}}(G)$ is a Fréchet space with the quasi-norm $q_k(u) = d_k(u, 0)$.

2.4. Let k_1 and k_2 be in K. In the sequel L denotes a linear operator from $C_0^\infty(G) \to C_0^\infty(G)$. We say that L is *(k_1, k_2)-closable*, when the dense operator L_{k_1,k_2}: $H_{k_1}(G) \to H_{k_2}(G)$ defined by

$$D(L_{k_1,k_2}) = C_0^\infty(G), \quad L_{k_1,k_2}\varphi = L\varphi \tag{2.26}$$

is closable from $H_{k_1}(G)$ to $H_{k_2}(G)$, in other words, L satisfies the following condition: Let $\{\varphi_n\} \subset C_0^\infty(G)$ be a sequence so that with $f \in H_{k_2}(G)$, $\|\varphi_n\|_{k_1} + \|L\varphi_n - f\|_{k_2} \to 0$ when $n \to \infty$. Then $f = 0$. The smallest closed extension of L_{k_1,k_2} (cf. [12], p. 78) is denoted by $\tilde{L}_{k_1,k_2}$.

Remark 2.11. (A) Suppose that $L(x, D) = \sum_{|\sigma|\leqslant r} a_\sigma(x) D^\sigma$ is a linear partial differential operator with $C^\infty(R^n)$ coefficients. Then $L(x, D)$ maps $C_0^\infty(G)$ into $C_0^\infty(G)$, the formal transpose $L'(x, D)$: $C_0^\infty(G) \to C_0^\infty(G)$ exists and $L(x, D)$ (and $L'(x, D)$) is closable from $H_{k_1}(G)$ to $H_{k_2}(G)$ (is closable from $H_{k_1}(G)$ to $H_{k_2}(G)$, resp.).

(B) Suppose that $L(x, D)$ is a linear partial differential operator with $C^\infty(G)$-coefficients and that G' is an open set of G such that $\bar{G}'$ is compact. Then L: $H_{k_1}(G') \to H_{k_2}(G')$ is closable (and L': $H_{k_1}(G') \to H_{k_2}(G')$ is closable).

(C) Suppose that $k_1 = k_2 = 1$ and that $L(x, D)$ is a properly supported pseudo-differential operator of the Beals and Fefferman type (cf. [2], p. 176). Then $L^{\sim}_{k_1,k_2}$ and $L'^{\sim}_{k_1,k_2}$ exists.

(D) Let $L(x, D)$ be a properly supported pseudo-differential operator of the Beals and Fefferman type such that with some $m \in N$ and $C > 0$ one has

$$\|L(x, D)\varphi\|_{k-m} \leqslant C\|\varphi\|_{k_1}.$$

Then $L^{\sim}_{k_1,k_2}$ exists.

We define still a linear operator $L^{\#}_{k_1,k_2}$: $H_{k_1}(G) \to H_{k_2}(G)$ with the requirement

$$\begin{aligned} D(L^{\#}_{k_1,k_2}) = \{u \in H_{k_1}(G) | \text{ there exists } f \in H_{k_2}(G) \\ \text{such that } u(L\varphi) = f(\varphi) \text{ for all } \varphi \in C_0^\infty(G)\}, \\ L^{\#}_{k_1,k_2} u = f. \end{aligned} \tag{2.27}$$

In virtue of (2.13) one sees that $L^{\#}_{k_1,k_2}$ is a closed operator.

Suppose that L is $(1/k_2^\vee, 1/k_1^\vee)$-closable. Let $L^*_{1/k_2^\vee,1/k_1^\vee}$: $H^*_{1/k_1^\vee}(G) \to H^*_{1/k_2^\vee}(G)$ be the dual operator of L: $H_{1/k_2^\vee}(G) \to H_{1/k_1^\vee}(G)$. Then one has

THEOREM 2.12. *The operators $L^{\#}_{k_1,k_2}$ and $L^{\#}_{1/k_2^\vee,1/k_1^\vee}$ obey the relation*

$$L^{\#}_{k_1,k_2} = J_{k_2}^{-1} \circ (L^*_{1/k_2^\vee,1/k_1^\vee}) \circ J_{k_1}. \tag{2.28}$$

Proof. Let u be in $D(L^{\#}_{k_1,k_2})$ and let $L^{\#}_{k_1,k_2} u = f$. Then one has by (2.15)

$$(J_{k_1} u)(L\varphi) = u(L\varphi) = f(\varphi) = (J_{k_2} f)(\varphi) \tag{2.29}$$

for all $\varphi \in D(L_{1/k_2^\vee,1/k_1^\vee}) = C_0^\infty(G)$. Hence $J_{k_1} u \in D(L^*_{1/k_2^\vee,1/k_1^\vee})$ and $L^*(J_{k_1} u) = J_{k_2} f$, that is, $u \in D\big(J_{k_2}^{-1} \circ (L^*_{1/k_2^\vee,1/k_1^\vee}) \circ J_{k_1}\big)$ and $\big(J_{k_2}^{-1} \circ (L^*_{1/k_2^\vee,1/k_1^\vee}) \circ J_{k_1}\big) u = f$.

The converse is similarly shown. □

Remark 2.13. Since $H_k(G)$, $k \in \boldsymbol{K}$, is a Hilbert space, one has the identity

$$L^{\sim}_{1/k_2^\vee,1/k_1^\vee} = \varkappa_2^{-1} \circ (L^{**}_{1/k_2^\vee,1/k_1^\vee}) \circ \varkappa_1, \tag{2.30}$$

where $\varkappa_1$: $H_{1/k_2^\vee}(G) \to H^{**}_{1/k_2^\vee}(G)$ and $\varkappa_2$: $H_{1/k_1^\vee}(G) \to H^{**}_{1/k_1^\vee}(G)$ are the canonical isomorphisms and where $L^{**}_{1/k_2^\vee,1/k_1^\vee}$: $H^{**}_{1/k_1^\vee}(G) \to H^{**}_{1/k_2^\vee}(G)$ is the dual operator of $L^*_{1/k_2^\vee,1/k_1^\vee}$ (cf. [6], p. 168).

3. One-sided inverses and realizations

3.1. In this chapter we everywhere suppose that the operator L maps $C_0^\infty(G)$ into $C_0^\infty(G)$. At first we establish

THEOREM 3.1. *Suppose that L is $(1/k_2^\vee, 1/k_1^\vee)$-closable and that there exists a constant $c > 0$ such that*

$$\|L\varphi\|_{1/k_1^\vee} \geqslant c\|\varphi\|_{1/k_2^\vee} \quad \textit{for all } \varphi \in C_0^\infty(G). \tag{3.1}$$

Then the relation

$$R(L^{\#}_{k_1,k_2}) = \boldsymbol{H}_{k_2}(G) \tag{3.2}$$

holds.

Proof. In virtue of (3.1) the range $R(L^{\sim}_{1/k_2^{\vee},1/k_1^{\vee}})$ is closed and

$$N(L^{\sim}_{1/k_2^{\vee},1/k_1^{\vee}}) = \{0\}. \tag{3.3}$$

Hence one has

$$R(L^{\sim *}_{1/k_2^{\vee},1/k_1^{\vee}}) = H^{*}_{1/k_2^{\vee}}(G) \tag{3.4}$$

(cf. [6], p. 234). One sees easily that $L^{\sim *}_{1/k_2^{\vee},1/k_1^{\vee}} = L^{*}_{1/k_2^{\vee},1/k_1^{\vee}}$ and then by (3.4)

$$R(L^{*}_{1/k_2^{\vee},1/k_1^{\vee}}) = H^{*}_{1/k_2^{\vee}}(G). \tag{3.5}$$

Thus assertion (3.2) follows from relation (2.28). □

Furthermore, we obtain

THEOREM 3.2. *Suppose that L is $(1/k_2^{\vee}, 1/k_1^{\vee})$-closable and that inequality (3.1) holds. Then there exists a continuous operator Q: $\boldsymbol{H}_{k_2}(G) \to \boldsymbol{H}_{k_1}(G)$ such that*

$$L^{\#}_{k_1,k_2} \circ Q = I_{k_2} \tag{3.6}$$

where I_{k_2}: $\boldsymbol{H}_{k_2}(G) \to \boldsymbol{H}_{k_2}(G)$ is the identity operator.

Proof. Since $L^{\#}_{k_1,k_2}$ is a closed operator, the kernel $N(L^{\#}_{k_1,k_2})$ is closed in $\boldsymbol{H}_{k_1}(G)$. Let N be the orthogonal complement of $N(L^{\#}_{k_1,k_2})$. Then the linear operator $\mathscr{L}_{k_1,k_2}$: $N \cap D(L^{\#}_{k_1,k_2}) \to \boldsymbol{H}_{k_2}(G)$ such that

$$\mathscr{L}_{k_1,k_2} = L^{\#}_{k_1,k_2}|_{N \cap D(L^{\#}_{k_1,k_2})}$$

is closed. Furthermore the kernel $N(\mathscr{L}_{k_1,k_2}) = \{0\}$ and by (3.2) one has $R(\mathscr{L}_{k_1,k_2}) = D(\mathscr{L}^{-1}_{k_1,k_2}) = \boldsymbol{H}_{k_2}(G)$. Let Q be the operator $\mathscr{L}^{-1}_{k_1,k_2}$. Then Q is closed, since $\mathscr{L}_{k_1,k_2}$ is closed. In addition, $D(Q) = \boldsymbol{H}_{k_2}(G)$. Hence Q is bounded (in view of the Closed Graph Theorem). One observes easily that Q obeys relation (3.6), which finishes the proof. □

COROLLARY 3.3. *Suppose that $k^{\sim} \geqslant 1$ and that the imbedding $H_{(k^{\sim}/k_2)^{\vee}}(G) \to H_{1/k_2^{\vee}}(G)$ is compact. Furthermore, assume that L is $((k^{\sim}/k_2)^{\vee}, 1/k_1^{\vee})$-closable and that there exists a constant $c > 0$ such that*

$$\|L\varphi\|_{1/k_1^{\vee}} \geqslant c\|\varphi\|_{(k^{\sim}/k_2)^{\vee}} \quad \textit{for all } \varphi \in C_0^{\infty}(G). \tag{3.7}$$

Then there exists a compact operator K: $\boldsymbol{H}_{k_2}(G) \to \boldsymbol{H}_{k_1}(G)$ so that

$$L^{\#}_{k_1,k_2} \circ K = I_{k_2}. \tag{3.8}$$

Proof. In virtue of Theorem 3.2 there exists a continuous operator $Q\colon \boldsymbol{H}_{k_2/k^\sim}(G) \to \boldsymbol{H}_{k_1}(G)$ such that

$$L^{\#}_{k_1,k_2/k^\sim} \circ Q = I_{k_2/k^\sim}. \tag{3.9}$$

Since the imbedding $H_{(k^\sim/k_2)^\vee}(G) \to H_{1/k_2^\vee}(G)$ is compact, Corollary 2.7 implies that the imbedding $\lambda_{k_2,k_2/k^\sim}\colon \boldsymbol{H}_{k_2}(G) \to \boldsymbol{H}_{k_2/k^\sim}(G)$ is compact. Choosing $K := Q \circ \lambda_{k_2,k_2/k^\sim}$ one sees that K is compact (since Q is continuous and $\lambda_{k_2,k_2/k^\sim}$ is compact) and that (3.8) is valid. This completes the proof. $\square$

3.2. We say that a closed linear operator $L^{\wedge}_{k_1,k_2}\colon \boldsymbol{H}_{k_1}(G) \to \boldsymbol{H}_{k_2}(G)$, which is a restriction of $L^{\#}_{k_1,k_2}$, is a realization of $L^{\#}_{k_1,k_2}$. The proof of Theorem 3.2 gives us immediately

THEOREM 3.4. *Suppose that L is $(1/k_2^\vee, 1/k_1^\vee)$-closable and that inequality* (3.1) *holds. Then there exists a realization $L^{\wedge}_{k_1,k_2}$ of $L^{\#}_{k_1,k_2}$ so that*

$$R(L^{\wedge}_{k_1,k_2}) = \boldsymbol{H}_{k_2}(G) \tag{3.10}$$

and

$$N(L^{\wedge}_{k_1,k_2}) = \{0\}. \tag{3.11}$$

In particular, $L^{\wedge}_{k_1,k_2}$ is a Fredholm operator with

$$\operatorname{ind} L^{\wedge}_{k_1,k_2} = 0. \tag{3.12}$$

Proof. Choose $L^{\wedge}_{k_1,k_2} = \mathscr{L}_{k_1,k_2}$, where $\mathscr{L}_{k_1,k_2}$ is the operator, which appeared in the proof of Theorem 3.2. $\square$

We show the following perturbation result

THEOREM 3.5. *Suppose that $k^\sim \geqslant 1$, $k_1 \geqslant k_2$ and that the imbedding $H_{(k^\sim/k_2)^\vee}(G) \to H_{1/k_2^\vee}(G)$ is compact. Furthermore, assume that L is $((k^\sim/k_2)^\vee, 1/k_1^\vee)$-closable and that there exist constants $C \in \boldsymbol{C}$ and $c > 0$ such that*

$$\|(L+C)\varphi\|_{1/k_1^\vee} \geqslant c\,\|\varphi\|_{(k^\sim/k_2)^\vee} \quad \textit{for all } \varphi \in C_0^\infty(G). \tag{3.13}$$

Then there exists a realization $L^{\wedge}_{k_1,k_2}$ of $L^{\#}_{k_1,k_2}$ which is a Fredholm operator with

$$\operatorname{ind} L^{\wedge}_{k_1,k_2} = 0. \tag{3.14}$$

Proof. In virtue of Theorem 3.4 there exists a realization $\bar{L}_{k_1,k_2/k^\sim}$ of $L^{\#}_{k_1,k_2/k^\sim} + C\lambda_{k_1,k_2/k^\sim}$ such that $\bar{L}_{k_1,k_2/k^\sim}$ is a Fredholm operator with

$$R(\bar{L}_{k_1,k_2/k^\sim}) = \boldsymbol{H}_{k_2/k^\sim}(G) \quad \text{and} \quad N(\bar{L}_{k_1,k_2/k^\sim}) = \{0\}, \tag{3.15}$$

where $\lambda_{k_1,k_2/k^\sim}$ denotes the imbedding $\boldsymbol{H}_{k_1}(G) \to \boldsymbol{H}_{k_2/k^\sim}(G)$.

Since $\bar{L}_{k_1,k_2/k^\sim}$ is a closed operator and since (3.15) holds, we obtain with some $\gamma > 0$

$$\gamma |||u|||_{k_1} \leqslant |||\bar{L}_{k_1,k_2/k^\sim} u|||_{k_2/k^\sim} \quad \text{for all } u \in D(\bar{L}_{k_1,k_2/k^\sim}). \tag{3.16}$$

Define an operator $\bar{L}_{k_1,k_2}$: $\boldsymbol{H}_{k_1}(G) \to \boldsymbol{H}_{k_2}(G)$ by

$$\bar{L}_{k_1,k_2} := \bar{L}_{k_1,k_2/k^\sim}|_{D(\bar{L}_{k_1,k_2/k^\sim}) \cap \bar{L}^{-1}_{k_1 k_2/k^\sim}(\boldsymbol{H}_{k_2}(G))}.$$

Then one has

$$(3.17) \qquad R(\bar{L}_{k_1,k_2}) = \boldsymbol{H}_{k_2}(G) \quad \text{and} \quad N(\bar{L}_{k_1,k_2}) = \{0\}$$

and in view of (3.16) we obtain

$$(3.18) \qquad \gamma |||u|||_{k_1} \leqslant |||\bar{L}_{k_1,k_2} u|||_{k_2/k^\sim} \quad \text{for all } u \in D(\bar{L}_{k_1,k_2}).$$

The operator $\bar{L}_{k_1,k_2}$ is also (trivially) closed. Hence especially $\bar{L}_{k_1,k_2}$ is a Fredholm operator with

$$\operatorname{ind} \bar{L}_{k_1,k_2} = 0.$$

The operator $A := -C\lambda_{k_1,k_2}$ is $\bar{L}_{k_1,k_2}$-compact (cf. [6], p. 194): Let $\{u_n\} \subset D(\bar{L}_{k_1,k_2})$ be a sequence such that $|||\bar{L}_{k_1,k_2} u_n|||_{k_2} \leqslant M$. Since the imbedding $\boldsymbol{H}_{k_2}(G) \to \boldsymbol{H}_{k_2/k^\sim}(G)$ is compact we find a subsequence $\{u_{n_j}\}$ so that $\{\bar{L}_{k_1,k_2} u_{n_j}\}$ is a Cauchy sequence in $\boldsymbol{H}_{k_2/k^\sim}(G)$. Thus by (3.18) $\{u_{n_j}\}$ is a Cauchy sequence in $\boldsymbol{H}_{k_1}(G)$ and then one finds $u \in \boldsymbol{H}_{k_1}(G)$ such that $|||u_{n_j} - u|||_{k_1} \to 0$ with $j \to \infty$. Since λ_{k_1,k_2} is bounded we see that $|||Au_{n_j} - Au|||_{k_2} \to 0$ with $j \to \infty$. Hence A is $\bar{L}_{k_1,k_2}$-compact. Since $\bar{L}_{k_1,k_2}$ is a Fredholm operator we get that the operator $\hat{L}_{k_1,k_2} := \bar{L}_{k_1,k_2} + A$ is a Fredholm operator with

$$(3.19) \qquad \operatorname{ind}(\hat{L}_{k_1,k_2}) = 0$$

(cf. [6], p. 238). Furthermore, one has

$$(L^{\wedge}_{k_1,k_2} u)(\varphi) = ((\bar{L}_{k_1,k_2} + A)u)(\varphi) = u((L+C)\varphi) - Cu(\varphi) = u(L\varphi)$$

for all $\varphi \in C_0^\infty(G)$ and then $L^{\wedge}_{k_1,k_2}$ is a realization of $L^{\#}_{k_1,k_2}$. This completes the proof. □

Corollary 3.6. *Suppose that $k_1^\sim \geqslant 1$, $k_2^\sim \geqslant 1$, $k_1 \geqslant k_2$ and that the imbedding $H_{(k_2^\sim/k_2)^\vee}(G) \to H_{1/k_2^\vee}(G)$ is compact. Furthermore, assume that L is $((k_2^\sim/k_2)^\vee, 1/(k_1 k_1^\sim)^\vee)$-closable and that there exist constants $C \in \boldsymbol{C}$ and $c > 0$ such that*

$$(3.20) \qquad \|(L+C)\varphi\|_{1/(k_1,k_1^\sim)^\vee} \geqslant C \|\varphi\|_{(k_2^\sim/k_2)^\vee} \quad \textit{for all } \varphi \in C_0^\infty(G).$$

Then there exists a realization $L^{\wedge}_{k_1,k_2}$ of $L^{\#}_{k_1,k_2}$ which is a Fredholm operator with

$$(3.21) \qquad \operatorname{ind}(L^{\wedge}_{k_1,k_2}) = 0$$

and

$$(3.22) \qquad D(L^{\wedge}_{k_1,k_2}) \subset \boldsymbol{H}_{k_1 k_1^\sim}(G).$$

Proof. Theorem 3.5 implies that there exists a Fredholm realization $L^{\wedge}_{k_1k_1^{\sim},k_2}$ of $L^{\#}_{k_1k_1^{\sim},k_2}$ such that

$$\operatorname{ind} L^{\wedge}_{k_1k_1^{\sim},k_2} = 0. \tag{3.23}$$

We define $L^{\wedge}_{k_1,k_2}$: $\boldsymbol{H}_{k_1}(G) \to \boldsymbol{H}_{k_2}(G)$ by

$$L^{\wedge}_{k_1,k_2} = L^{\#}_{k_1,k_2}|_{D(L^{\wedge}_{k_1k_1^{\sim},k_2})}.$$

Then $L^{\wedge}_{k_1,k_2}$ is closed. In virtue of the proof of Theorem 3.5, $L^{\wedge}_{k_1k_1^{\sim},k_2} = \bar{L}_{k_1k_1^{\sim},k_2} - C\lambda_{k_1k_1^{\sim},k_2}$, where $\bar{L}_{k_1k_1^{\sim},k_2}$ obeys

$$\gamma |||u|||_{k_1k_1^{\sim}} \leqslant |||\bar{L}_{k_1k_1^{\sim},k_2} u|||_{k_2/k_2^{\sim}} \quad \text{for all } u \in D(\bar{L}_{k_1k_1^{\sim},k_2}). \tag{3.24}$$

Hence we obtain (since $k_2^{\sim} \geqslant 1$ and $k_1 \geqslant k_2$)

$$\gamma |||u|||_{k_1k_1^{\sim}} \leqslant |||L^{\wedge}_{k_1,k_2} u|||_{k_2} + C|||u|||_{k_1}, \tag{3.25}$$

which implies that $L^{\wedge}_{k_1,k_2}$ is closed. Trivially one has $\operatorname{ind} L^{\wedge}_{k_1,k_2} = 0$, which completes the proof. □

COROLLARY 3.7. *Suppose that $k_1 \in \boldsymbol{K}'$, $k_1^{\sim} \in \boldsymbol{K}'$, $k_2 \in \boldsymbol{K}$; $k_1^{\sim} \geqslant 1$, $k_1 \geqslant k_2$, that L is $(1/k_2^{\vee}, 1/(k_1 k_1^{\sim})^{\vee})$-closable and that there exist constants $C \in \boldsymbol{C}$ and $c > 0$ such that*

$$\|(L+C)\varphi\|_{1/(k_1k_1^{\sim})^{\vee}} \geqslant c\,\|\varphi\|_{1/k_2^{\vee}} \quad \textit{for all } \varphi \in C_0^{\infty}(G). \tag{3.26}$$

Furthermore, assume that the inclusion

$$N(L^{\#}_{k_1,k_2} + C\lambda_{k_1,k_2}) \subset H^{\text{loc}}_{k_1k_1^{\sim}}(G) \tag{3.27}$$

holds. Then every solution of the equation

$$L^{\#}_{k_1,k_2} u = f; \quad f \in \boldsymbol{H}_{k_2}(G)$$

lies in $H^{\text{loc}}_{k_1k_1^{\sim}}(G)$.

Proof. In virtue of Theorem 3.4 there exists a realization $L^{\wedge}_{k_1k_1^{\sim},k_2}$ of $L^{\#}_{k_1k_1^{\sim},k_2} + C\lambda_{k_1k_1^{\sim},k_2}$ such that

$$R(L^{\wedge}_{k_1k_1^{\sim},k_2}) = \boldsymbol{H}_{k_2}(G). \tag{3.28}$$

Let u be in $D(L^{\#}_{k_1,k_2})$ and let $L^{\#}_{k_1,k_2} u = f$. Then $f + Cu$ lies in $\boldsymbol{H}_{k_2}(G)$ (since $k_1 \geqslant k_2$) and so we find an element $w \in D(L^{\wedge}_{k_1k_1^{\sim},k_2}) \subset \boldsymbol{H}_{k_1k_1^{\sim}}(G)$ such that

$$L^{\wedge}_{k_1k_1^{\sim},k_2} w = f + Cu.$$

For all $\varphi \in C_0^\infty(G)$ we obtain

$$w((L+C)\varphi) = (L^{\wedge}_{k_1 k_1^{\sim}, k_2} w)(\varphi) = (f+Cu)(\varphi) = u((L+C)\varphi)$$

and then $u-w \in N(L^{\#}_{k_1,k_2} + C\lambda_{k_1,k_2}) \subset H^{\mathrm{loc}}_{k_1 k_1^{\sim}}(G)$. Since $H_{k_1 k_1^{\sim}}(G) \subset H^{\mathrm{loc}}_{k_1 k_1^{\sim}}(G)$ we see that $u \in H^{\mathrm{loc}}_{k_1 k_1^{\sim}}(G)$ which finishes the proof. □

4. Applications to partial differential operators

4.1. In this section we suppose that $k_1 = k_2 \equiv 1$, that is, $H_{k_1}(G) = H_{k_2}(G) = \boldsymbol{H}_{k_1}(G) = \boldsymbol{H}_{k_2}(G) = L_2(G)$.

We have

THEOREM 4.1. *Suppose that $k^{\sim} \geqslant 1$ and that the imbedding $H_{k^{\sim}}(G) \to L_2(G)$ is compact. Furthermore, assume that L: $C_0^\infty(G) \to C_0^\infty(G)$ is $(k^{\sim\vee}, 1/k^{\sim\vee})$-closable and that there exist constants $C_1 > 0$ and $C_2 \geqslant 0$ such that*

$$\text{(4.1)} \qquad \mathrm{Re}(L\varphi, \varphi)_0 \geqslant C_1 \|\varphi\|^2_{k^{\sim\vee}} - C_2 \|\varphi\|_0^2 \quad \textit{for all } \varphi \in C_0^\infty(G).$$

Then there exists a realization $L^{\wedge}$ of $L^{\#} := L^{\#}_{1,1}$ which is a Fredholm operator with

$$\text{(4.2)} \qquad \mathrm{ind}\, L^{\wedge} = 0$$

and

$$\text{(4.3)} \qquad D(L^{\wedge}) \subset \boldsymbol{H}_{k^{\sim}}(G).$$

Proof. In virtue of (4.1) we have

$$\|(L+C_2)\varphi\|_{1/k^{\sim\vee}} \|\varphi\|_{k^{\sim\vee}} \geqslant |((L+C_2)\varphi, \varphi)_0| \geqslant \mathrm{Re}((L+C_2)\varphi, \varphi)_0 \geqslant C_1 \|\varphi\|^2_{k^{\sim\vee}}$$

and then

$$\text{(4.4)} \qquad \|(L+C_2)\varphi\|_{1/k^{\sim\vee}} \geqslant C_1 \|\varphi\|_{k^{\sim\vee}} \quad \text{for all } \varphi \in C_0^\infty(G).$$

Hence the assertion follows from Corollary 3.6. □

Remark 4.2. (A) In the case where $C_2 = 0$ one has

$$R(L^{\wedge}) = L_2(G) \quad \text{and} \quad N(L^{\wedge}) = \{0\}$$

(cf. the proof of Theorem 3.5).

(B) We recall that in the case where G is bounded and $k^{\sim}(\xi) \to \infty$ with $|\xi| \to \infty$, the imbedding $H_{k^{\sim}}(G) \to L_2(G)$ is compact.

THEOREM 4.3. *Suppose that $k^{\sim} \in \boldsymbol{K}'$; $k^{\sim} \geqslant 1$, that L: $C_0^\infty(G) \to C_0^\infty(G)$ is $(1, 1/k^{\sim\vee})$-closable and that inequality* (4.1) *holds. Furthermore, assume that*

$$\text{(4.5)} \qquad N(L^{\#} + C_2 I) \subset H^{\mathrm{loc}}_{k^{\sim}}(G).$$

Then the inclusion

$$D(L^{\#}) \subset H^{\mathrm{loc}}_{k^{\sim}}(G) \tag{4.6}$$

is valid.

Proof. The assertion follows immediately from (4.4) and from Corollary 3.7. □

COROLLARY 4.4. *Suppose that G is bounded and that $k^{\sim} \in \boldsymbol{K}'$; $k^{\sim}(\xi) \to \infty$ with $|\xi| \to \infty$. Let $L(x, D) = \sum_{|\sigma| \leqslant r} a_\sigma(x) D^\sigma$ be a linear partial differential operator with $C^\infty(G)$-coefficients such that*

$$\operatorname{Re}(L(x, D)\varphi, \varphi)_0 \geqslant C_1 \|\varphi\|^2_{k^{\sim\vee}} - C_2 \|\varphi\|^2_0 \quad \text{for all } \varphi \in C_0^\infty(G) \tag{4.7}$$

and that

$$\|L(x, D)\varphi\|_{1/k^{\sim\vee}} \leqslant C \|\varphi\|_{k^{\sim\vee}} \quad \text{for all } \varphi \in C_0^\infty(G). \tag{4.8}$$

Then there exists a realization $L^{\wedge}$ of $L^{\#}$ which is a Fredholm operator with

$$\operatorname{ind} L^{\wedge} = 0 \tag{4.9}$$

and

$$D(L^{\wedge}) \subset H_{k^{\sim}}(G).$$

In addition, in the case where

$$N(L^{\#} + C_2 I) \subset H^{\mathrm{loc}}_{k^{\sim}}(G)$$

one has

$$D(L^{\#}) \subset H^{\mathrm{loc}}_{k^{\sim}}(G).$$

Proof. In virtue of (4.8) one obtains that L is $(k^{\sim\vee}, 1/k^{\sim\vee})$-closable. Hence the assertion follows from Theorems 4.1 and 4.3. □

Remark 4.5. (A) Relation (4.9) follows without the assumption (4.8) since $L(x, D)$ is always $(k_1^{\sim}, 1)$-closable (cf. Theorem 3.5).

(B) Suppose that the formal transpose L': $C_0^\infty(G) \to C_0^\infty(G)$ of L exists. Then $L'^{\#}$ is the extension (a so-called maximal realization of L) of L. We have for all φ, $\psi \in C_0^\infty(G)$

$$(L\psi)(\varphi) = \psi(L'\varphi).$$

In the case where L' exists, the operator $L_{1,1}$: $L_2(G) \to L_2(G)$ is closable (since $C_0^\infty(G)$ is dense in $L_2(G)$). The minimal closed extension $L^{\sim}$ (a so-called minimal realization of L) satisfies $L^{\sim} \subset L^{\wedge} \subset L'^{\#}$ for any realization $L^{\wedge}$ of $L'^{\#}$.

4.2. Let $L(D) = \sum_{|\sigma| \leqslant r} a_\sigma D^\sigma$ be a (non-trivial) linear partial differential operator with constant coefficients $a_\sigma \in \boldsymbol{C}$. Then L: $H_{k_1}(G) \to H_{k_2}(G)$ is closable when k_1 and $k_2 \in \boldsymbol{K}$. Define a function $L^{\sim} \in \boldsymbol{K}$ by

$$(4.10) \qquad L^{\sim}(\xi) = \Big(\sum_{|\alpha| \leq r} |L^{(\alpha)}(\xi)|^2\Big)^{1/2},$$

where $L^{(\alpha)}(\xi) = \dfrac{\partial^\alpha L}{\partial \xi^\alpha}(\xi)$. We have

LEMMA 4.6. *Let k be in $\boldsymbol{K}'$ and let G be a bounded set in $\boldsymbol{R}^n$. Then there exists a constant $c > 0$ such that*

$$(4.11) \qquad \|L'(D)\varphi\|_{1/kL^{\sim})^{\vee}} \geqslant c\|\varphi\|_{1/k^{\vee}} \quad \textit{for all } \varphi \in C_0^\infty(G). \quad \square$$

For the proof cf. [10] (Lemma 3.3 and Theorem 3.4; we remark that in [10] k belongs to the Hörmander class $K \subset \boldsymbol{K}'$ of weight functions, but one easily sees that the proofs runs similarly when k lies in $\boldsymbol{K}'$).

Theorem 3.2, Corollary 3.3 and Lemma 4.6 yields us immediately

THEOREM 4.7. *Let k be in $\boldsymbol{K}'$ and let G be a bounded set in $\boldsymbol{R}^n$. Then one has*

(i) *there exists a continuous linear operator Q: $\boldsymbol{H}_k(G) \to \boldsymbol{H}_{kL^{\sim}}(G)$ such that*

$$(4.12) \qquad L'^{\#}_{kL^{\sim},k} \circ Q = I_k.$$

(ii) *Suppose that*

$$(4.13) \qquad L^{\sim}(\xi) \to \infty \quad \textit{with} \quad |\xi| \to \infty.$$

Then there exists a compact operator K: $\boldsymbol{H}_k(G) \to \boldsymbol{H}_k(G)$ such that

$$(4.14) \qquad L'^{\#}_{k,k} \circ K = I_k.$$

Proof. Conclusion (i) follows immediately from Theorem 3.2. Choosing $k = k/L^{\sim}$ in (4.11) one finds that

$$(4.15) \qquad \|L(D)\varphi\|_{1/k^{\vee}} \geqslant c\|\varphi\|_{(L^{\sim}/k)^{\vee}} \quad \text{for all } \varphi \in C_0^\infty(G).$$

Since $L^{\sim}(\xi) \to \infty$ with $|\xi| \to \infty$ and since G is bounded, the imbedding $H_{(L^{\sim}/k)^{\vee}}(G) \to H_{1/k^{\vee}}(G)$ is compact. Hence assertion (ii) follows from Corollary 3.3. $\square$

Suppose that $k \in \boldsymbol{K}'$ and that G is bounded. In virtue of Theorem 3.4 and inequality (4.11) there exists a realization $\tilde{L}_{kL^{\sim},k}$ of $L'^{\#}_{kL^{\sim},k}$ so that

$$(4.16) \qquad R(\hat{L}_{kL^{\sim},k}) = \boldsymbol{H}_k(G), \quad N(\hat{L}_{kL^{\sim},k}) = \{0\}.$$

In virtue of the Closed Graph Theorem the inequality

$$|||u|||_{kL^{\sim}} \leqslant C|||\hat{L}_{kL^{\sim},k}u|||_k \quad \text{for } u \in D(\hat{L}_{kL^{\sim},k})$$

holds.

Supposing that $k^{\sim} \in \boldsymbol{K}'$; $k^{\sim} \leqslant CL^{\sim}$ and that

$$(4.17) \qquad N(L'^{\#}_{k,k}) \subset H^{\mathrm{loc}}_{kk^{\sim}}(G)$$

one sees by (4.16) that

$$D(L'^{\#}_{k,k}) \subset H^{\mathrm{loc}}_{kk^{\sim}}(G). \tag{4.18}$$

We show the following

COROLLARY 4.8. *Suppose that $k \in \boldsymbol{K}'$ and that G is a bounded open set in $\boldsymbol{R}^n$. Then the inclusion*

$$D(L'^{\#}_{k,k}) \subset H^{\mathrm{loc}}_{kL^{\sim}/L'^{\sim}}(G) \tag{4.19}$$

holds, where $L^{\sim} \in \boldsymbol{K}'$ is defined by

$$L^{\sim}(\xi) = \Big(\sum_{|\alpha|>0} |L^{(\alpha)}(\xi)|^2\Big)^{1/2}. \tag{4.20}$$

Proof. As we mentioned above it suffices to establish the inclusion

$$N(L'^{\#}_{k,k}) \subset H^{\mathrm{loc}}_{kL^{\sim}/L'^{\sim}}(G). \tag{4.21}$$

Choose $u \in N(L'^{\#}_{k,k})$ and let $\Phi \in C_0^{\infty}(G)$. Furthermore, let G' be an open subset of G such that $\operatorname{supp} \Phi \subset G'$ and that $\bar{G}' \subset_K G$. Choose $\Phi' \subset C_0^{\infty}(G)$ so that $\Phi'(x) = 1$ for all $x \in \bar{G}'$. Then one has by the Leibniz formula (for distributions) (cf. also [4], p. 65)

$$\begin{aligned} (\Phi u)\big(L'(D)\varphi\big) &= (\Phi\Phi' u)\big(L'(D)\varphi\big) = \big(L(D)(\Phi\Phi' u)\big)(\varphi) \\ &= \big(\Phi L(D)(\Phi' u)\big)(\varphi) + \sum_{|\alpha|>0} \frac{1}{\alpha!}\big(D^{\alpha}\Phi \cdot L^{\alpha}(D)(\Phi' u)\big)(\varphi) \\ &= 0 + g(\varphi) \quad \text{for all } \varphi \in C^{\infty}(\boldsymbol{R}^n), \end{aligned} \tag{4.22}$$

where

$$g := \sum_{|\alpha|>0} \frac{1}{\alpha!}\big(D^{\alpha}\Phi \cdot L^{(\alpha)}(D)(\Phi' u)\big) \in H^{c}_{k/L'^{\sim}}(\boldsymbol{R}^n)$$

(note that $k \in \boldsymbol{K}'$). Applying (4.22) with $\varphi = e^{-i(\xi,\cdot)}$ one finds that

$$L(\xi)\, F(\Phi u)(\xi) = (Fg)(\xi) \quad \text{for all } \xi \in \boldsymbol{R}^n. \tag{4.23}$$

Noting that $L^{\sim}(\xi) \leqslant |L(\xi)| + L^{\sim}(\xi)$ one sees by (4.23) that $(kL^{\sim}/L'^{\sim})\, F(\Phi u) \in L_2$ and then $u \in H^{\mathrm{loc}}_{kL^{\sim}/L'^{\sim}}(G)$. □

Remark 4.9. We remark that by repeating the reasoning at the end of the proof of Corollary 4.8, we get

$$N(L'^{\#}_{k,k}) \subset H^{\mathrm{loc}}_{k(L^{\sim}/L'^{\sim})^N}(G) \quad \text{for each } N \in \boldsymbol{N}. \tag{4.24}$$

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