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BERNOULLI CONVOLUTIONS — AN APPLICATION OF SET THEORY IN ANALYSIS

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1. Introduction. A problem in real analysis and Fourier transforms is investigated, leading in an entirely natural way to descriptive set theory. In particular, Choquet's theorem on capacitability of analytic sets in invoked. Here are the necessary definitions.

A finite Borel measure μ in $\mathbf{R}$ belongs to the *Rajchman class* $\underline{R}$ if the Fourier–Stieltjes transform

$$\hat{\mu}(u) \equiv \int e(-ut)\,\mu(dt) \qquad (e(s) \equiv e^{2\pi i s})$$

vanishes at ∞. When $\mu \in \underline{R}$ then $f \cdot \mu \in \underline{R}$ for test functions f, and hence also for any f in $L^1(|\mu|)$. A closed F is an M_0-*set* if F carries a measure $\mu \neq 0$ in $\underline{R}$. In view of the remark about $\underline{R}$, we can assume that μ is a probability measure. The class $\underline{D}^1$ is the class of functions ϕ in $C^1(\mathbf{R})$ such that $\phi' > 0$ everywhere.

(a) A set F is of type $M_0(\underline{D}^1)$ if $\phi(F)$ is of type M_0 for *every* ϕ in $\underline{D}^1$.

(b) A set F is of type $M_{00}(\underline{D}^1)$ if F carries a measure $\mu \neq 0$ such that $\phi^*\mu$ is in $\underline{R}$ for *every* $\phi \in \underline{D}^1$.

Clearly $M_{00}(\underline{D}^1) \subseteq M_0(\underline{D}^1)$. Moreover, the classical examples of M_0-sets of measure 0 turn out to be $M_{00}(\underline{D}^1)$ ([7]).

One more definition: a set S in a metric space (X, d) has *property* (L) if there is a sequence $\varepsilon_n \to 0+$, and for each n a partition $S = \bigcup_i S_n^i$ (finite or countable) such that $\operatorname{diam}(S_n^i) \leq \varepsilon_n$ for each n and i, but $d(S_n^i, S_n^j) \geq n\varepsilon_n$ for every n and $i \neq j$. For every compact linear set F of type (L) there exist (in great abundance) functions ϕ in $(\underline{D}^1)$ such that $\phi(F)$ is very far from a set of type M_0 ([6], [4], [3, pp. 89–96]). We are now in a position to combine all of these concepts.

THEOREM. *There exists a compact linear set F of type $M_0(\underline{D}^1)$ such that every probability measure in F has positive mass in some closed set of type (L).*

COROLLARY. $M_0(\underline{D}^1) \neq M_{00}(\underline{D}^1)$.

Indeed, given any signed measure $\lambda \neq 0$ in F, $|\lambda|$ has positive mass in some compact set $F_1 \subseteq F$ of type (L). Let $\phi \in \underline{D}^1$ be such that $\phi(F_1)$ is not M_0. Since $|\phi^*\lambda|$ has positive mass in $\phi(F_1)$, $\phi^*\lambda$ cannot be in the class $\underline{R}$. (A Rajchman measure has positive mass only in M_0-sets; this property *characterizes* the class $\underline{R}$, by a theorem of R. Lyons [9].)

To conclude the introduction we state a fact which is used in the construction of F: there is a number θ in $(0,1/2)$ such that $\sum_{n=1}^{\infty} \sin^2 \pi\lambda\theta^{-n} = +\infty$ for every $\lambda > 0$. This can be found in [5, Ch. VI], [13, pp. 147–156] or [11, pp. 35–36]. By relatively straightforward means one can show that $\limsup |\sin \pi\lambda\theta^{-n}| \geq \sin \pi/7$ when $\theta = 2/5$ (as in [1], [8], [2]). All linear sets are subsets of the Cantor set $\{\sum_{m=1}^{\infty} \varepsilon_m \theta^m : \varepsilon_m = 0 \text{ or } 1\}$.

2. Let S be the set of strictly increasing sequences $\sigma = (s_k)_{k=1}^{\infty}$ of positive integers. To each element σ of S we attach a set $E(\sigma)$ as follows. Let $k_N = 0$ if $s_1 > N$ and otherwise let k_N be the largest integer k such that $s_k \leq N$ $(N = 1, 2, \ldots)$. Thus $k_N \leq N$ since $s_N \geq N$ and $k_N \to +\infty$ as $N \to +\infty$. Now $E(\sigma)$ is the set of integers in $\bigcup_{N=1}^{\infty} [4^N, 4^N + k_N]$. If $\sigma = (s_k)$ and $\sigma^* = (s_k^*)$ are related by the inequality $\sigma \leq \sigma^*$, i.e. $s_k \leq s_k^*$ for each k, then plainly $E(\sigma) \supseteq E(\sigma^*)$. The part of $E(\sigma)$ in $[1, 4^N + N]$ is determined entirely by $s_1, \ldots, s_N$ (since $s_{N+1} > N$) so the mapping from σ to $E(\sigma)$ is continuous between the usual product topologies. Before proceeding we record an inequality about the size of the intersections $E(\sigma) \cap [p, 2p]$. For each integer p, at most one interval $[4^N, 4^N + N]$ can meet $[p, 2p]$; and that is possible only if $4^N \leq 2p$, whence $N < N \log 4 \leq \log 2p < p$. Thus $E(\sigma) \cap [p, 2p]$ has size $1 + k_N \leq 1 + k_p$; and $k_p \leq r$ when $s_r \geq p \geq 1$.

Let $A(\sigma)$ be the set of all sums $\sum_{m=1}^{\infty} \varepsilon_m \theta^m$, in which $\varepsilon_m = 0$ or 1, and $\varepsilon_m = 0$ for $m \in E(\sigma)$. The mapping from σ to $A(\sigma)$ is continuous, when the sets $A(\sigma)$ are provided with the Hausdorff metric on compact, linear sets. The operator α is defined over subsets B of S by the formula

$$\alpha(B) = \bigcup \{A(\sigma) : \sigma \in B\}.$$

Then $\alpha(S)$ is an analytic set (cf. [12]). By Choquet's capacitability theorem we see that for every positive Borel measure μ in $\mathbf{R}$

$$\mu^*(\alpha(S)) = \sup\{\mu^*(\alpha(C)) : C \text{ compact}\}.$$

Thus $\alpha(S)$ is universally measurable, since the sets $\alpha(C)$ in the supremum are compact. The map from σ to $A(\sigma)$ is increasing:

$$\sigma \leq \sigma^* \Rightarrow E(\sigma^*) \subseteq E(\sigma) \Rightarrow A(\sigma) \subseteq A(\sigma^*).$$

LEMMA 1. *A measure $\mu \geq 0$, with positive mass in $\alpha(S)$, has positive mass in some compact set of type (L).*

Let us recall that $\alpha(S)$ is μ-measurable. By the capacitability theorem, we have only to verify that each set $\alpha(C)$, with C a compact set in S, is of type (L). The elements of C have a single majorant σ_0 in S, so that $\alpha(C) \subseteq A(\sigma_0)$. Now the set $E(\sigma_0)$ contains segments of consecutive integers of unbounded lengths. Since $0 < \theta < 1/2$, a sketch confirms that $A(\sigma_0)$ is of type (L), and so also $\alpha(C)$.

3. To explain the next step in our construction we remark that the set $\overline{\alpha(S)}$ is too large; in fact it is of type $M_{00}(\underline{D}^1)$, as may be proved by the considerations in the last paragraph. Hence we shall replace each set $A(\sigma)$ by a subset $A'(\sigma)$, obtaining thereby a new operator α', etc. Before defining $A'(\sigma)$, we consider a number t in the set $\overline{\alpha'(S)}\setminus\alpha(S)$. Thus $t = \lim t_j$, where $t_j \in A'(\sigma_j)$ and (σ_j) is a sequence in S. If the sequence (σ_j) admitted an accumulation point σ in S, then it would follow that $t \in \alpha(S)$, contrary to our assumption. Thus there must be some fixed integer $k \geq 1$, and a subsequence along which $s_k^j \to +\infty$. This gives some control over the "residual set" $\overline{\alpha'(S)} \setminus \alpha(S)$. In the process of replacing each set $A(\sigma)$ by a smaller set, we cannot make it so small that the $M_0(\underline{D}^1)$ property of $\overline{\alpha(S)}$ is lost. This necessity imposes an extra complication in the definition of $A'(\sigma)$.

Let $w(k, \ell, n)$ be a one-one function into the positive integers, defined for $k \geq 1$, $\ell \geq 1$, $n \geq 1$. Then $J(k, \ell, n)$ is defined to be the interval $[2 \cdot 4^N, 2 \cdot 4^N + N]$, with $N = w(k, \ell, n)$. These intervals are disjoint from each other, and from all the intervals $[4^N, 4^N + N]$ used in defining the sets $A(\sigma)$. The set $M_k^\ell(\sigma)$ is defined to be the set of all sums $\sum_{m=1}^{\infty} \varepsilon_m \theta^m$ in which $\varepsilon_m = 0$ or 1, and $\varepsilon_m = 0$ for all integers m in all the intervals $J(k, \ell, n)$, $1 \leq n \leq s_k$. Let $M_k(\sigma) = \bigcup_{\ell=1}^{k} M_k^\ell(\sigma)$, $M(\sigma) = \bigcap_{k=1}^{\infty} M_k(\sigma)$, and $A'(\sigma) = A(\sigma) \cap M(\sigma)$.

LEMMA 2. *The set $\overline{\alpha'(S)}\setminus\alpha(S)$ is a countable union of sets of type (L).*

Proof. Each point in the difference is obtained as $\lim t_j$, where $t_j \in A'(\sigma_j)$ and the sequence (σ_j) has the property that the kth coordinate $s_k^j \to +\infty$, for a certain fixed $k \geq 1$. We shall prove that those numbers t, attained with a *fixed* k, are contained in at most k sets of type (L). Each t_j belongs to a set $M_k^{\ell_j}(\sigma_j)$, with $1 \leq \ell_j \leq k$. Plainly we can suppose that $\ell_j = \ell_0$, the same for all j. Thus $t_j = \sum_{m=1}^{\infty} \varepsilon_m^j \theta^m$, with $\varepsilon_m^j = 0$ for all m in the intervals $J(k, \ell_0, n)$, $1 \leq n \leq s_k^j$. Inasmuch as $s_k^j \to +\infty$, we see that $t = \sum_{m=1}^{\infty} \varepsilon_m \theta^m$, where $\varepsilon_m = 0$ for all m in the intervals $J(k, \ell_0, n)$, $1 \leq n < +\infty$. The interval $J(k, \ell_0, n)$ has length $1 + w(k, \ell_0, n) \to +\infty$. Thus, for a fixed k and ℓ_0, we obtain an (L)-set, and the union of these sets contains $\overline{\alpha'(S)} \setminus \alpha(S)$. (This device for handling the closure of an analytic set is adapted from Mazurkiewicz and Sierpiński [10].)

We can now define $F = \overline{\alpha'(S)}$ and obtain from Lemmas 1 and 2

LEMMA 3. *Each probability measure in F has positive mass in some set of type (L).*

4. The remaining task is to prove that for each $\phi \in \underline{D}^1$ there is some σ such that $\phi(A'(\sigma))$ is an M_0-set. We do this for the larger set $\phi(A(\sigma))$ in 4a, and then for $\phi(A'(\sigma))$ in 4b. The analysis is similar to that in [7]. Each set $A(\sigma)$ carries a product measure $\prod' \frac{1}{2}(\delta(0) + \delta(\theta^m))$, where the prime in a product (or sum) means that $m \geq 1$, $m \notin E(\sigma)$. Sets like $A(\sigma)$, and the associated (convolution) product measure, are called *standard*. (The convergence of the product is of course assured by $\sum_{m=1}^{\infty} \theta^m < 1$.)

4a. Since all sets in question are contained in $[0,1]$, we can assume that $0 < a_1 \leq \phi' \leq a_2 < \infty$, and ϕ' is uniformly continuous, with modulus of continuity ω. Now $\omega(0+) = 0$ and $\omega(2t) \leq 2\omega(t)$ for $t > 0$. Therefore there is an integer $r = r(u)$ such that $u\theta^r \to +\infty$ but $u\theta^r\omega(\theta^r) \to 0$ as $u \to +\infty$. In case $\theta^r \geq u^{-2/3}$, it will be convenient to increase r so that $\theta^{r-1} \geq u^{-2/3} > \theta^r$; it will remain true that $u\theta^r \to +\infty$, $u\theta^r\omega(\theta^r) \to 0$. Also, $p = p(u)$ is defined by the inequalities $1 \leq u\theta^p < \theta^{-1}$. Thus $p(u) - r(u) \to +\infty$ as $u \to +\infty$ but $p(u) < 2r(u)$ for large u. There is a function $h(u) \to \infty$ as $u \to +\infty$ such that

$$\sum_{j=0}^{p-r} \sin^2 \pi\lambda\theta^{-j} \geq h(u), \qquad a_1 \leq \lambda \leq a_2\theta^{-1};$$

this is a consequence of Dini's theorem on monotone sequences of continuous functions.

The sequence σ is chosen so that, for large u, and $q = [h(u)/2]$, we have $s_q > p(u)$. This is possible because $h(u) \to +\infty$ as $u \to +\infty$. The integer k_p, defined through σ, therefore satisfies $k_p < [h(u)/2]$. Thus the interval $[r(u), p(u)]$ contains at most $h(u)/2$ elements of σ.

With this choice of σ, the standard measure μ in $A(\sigma)$ is defined, and we proceed to estimate the Fourier transform of $\phi^*\mu$. For each $u > 1$ we factor μ into a product $\lambda_1 * \lambda_2$ of standard measures: λ_1 is the product over the integers $m \notin E(\sigma)$, $m \geq r(u)$, and λ_2 over the integers $m \notin E(\sigma)$, $m < r(u)$. We use the formulas

$$\Big| \int e(-u\phi(t))\, \mu(dt)\Big| = \Big| \int\int e(-u\phi(x+y))\, \lambda_1(dx)\, \lambda_2(dy)\Big|$$
$$\leq \sup_y \Big| \int e(-u\phi(x+y)\, \lambda_1(dx)\Big|.$$

The support of λ_1 has diameter $< 2\theta^r$, so that $\phi(x+y)$ can be estimated by Taylor's formula and the modulus of continuity of ϕ', yielding an upper bound

$$\sup_y |\hat{\lambda}_1(u\phi'(y))| + 4\pi u\theta^r \omega(\theta^r).$$

By the choice of $r = r(u)$, we have $u\theta^r\omega(\theta^r) \to 0$ as $u \to +\infty$. Hence it is sufficient to estimate $|\hat{\lambda}_1(u\phi'(y))|$.

We use the formulas

$$|\hat{\lambda}_1(t)|^2 = \prod_{m=r}^{\infty}{}' \cos^2 \pi t\theta^m \le \exp\Big(-\sum_{m=r}^{\infty}{}' \sin^2 \pi t\theta^m\Big)$$
$$\le \exp\Big(-\sum_{m=r}^{p}{}' \sin^2 \pi t\theta^m\Big).$$

Since the intersection $E(\sigma) \cap [r(u), p(u)]$ has size $< h(u)/2$, we can write

$$|\hat{\lambda}_1(t)|^2 < \exp\Big(-\sum_{m=r}^{p} \sin^2 \pi t\theta^m\Big) \exp(h(u)/2).$$

In this formula we substitute $t = u\phi'(y)\theta^p$; we see that $|\hat{\lambda}_1(t)|^2 < \exp(-h(u)) \times \exp(h(u)/2)$, or $|\hat{\lambda}_1(t)| < \exp(-h(u)/4) = o(1)$ as $u \to +\infty$. Thus we have proved that $\phi^*\mu$ is in the Rajchman class and $\phi(A(\sigma))$ is an M_0-set. Before passing to 4b, we observe that this estimation depends only on the intersection $[r(u), p(u)] \cap E(\sigma)$, and $r(u) < p(u) < 2r(u)$.

4b. To complete the proof that $\phi(A'(\sigma))$ is an M_0-set, we first examine the set $M(\sigma)$. It is an uncountable union of intersections $M_1^{\ell_1}(\sigma) \cap M_2^{\ell_2}(\sigma) \cap \ldots \cap M_k^{\ell_k}(\sigma) \cap \ldots$ where $\ell_1 = 1$, $1 \le \ell_2 \le 2, \ldots, 1 \le \ell_k \le k, \ldots$ Intersecting each of these sets with $A(\sigma)$ yields a standard set and a standard product measure. The measure μ is then obtained by averaging the standard measures over all choices of $\ell_1, \ell_2, \ldots$

Each set in the average is obtained by suppressing certain intervals $[2 \cdot 4^N, 2 \cdot 4^N + N]$ from the standard measure over $A(\sigma)$. For each $u > 1$, the estimation in 4a is unchanged unless one of the intervals $[2 \cdot 4^N, 2 \cdot 4^N + N]$ omitted intersects $[r(u), 2r(u)]$, and this defines at most one integer $N = N(u)$. This N occurs only if $N = w(k, \ell_k, n)$, wherein $1 \le \ell_k \le k$, $1 \le n \le s_k$. We recall that $w(\ell, k, n)$ is one-one. As $u \to +\infty$, then $r(u) \to +\infty$ and $N(u) \to +\infty$. Since the numbers $s_1, s_2, \ldots, s_k, \ldots$ do not depend on $\ell_1, \ell_2, \ldots, \ell_k, \ldots$, we must have $k \to +\infty$ with u. Thus the estimate in 4a is effective except in a proportion $1/k(u)$, and $k(u) \to +\infty$ with u. Therefore $\phi^*\mu$ belongs to the Rajchman class $\underline{R}$, and we have proved that F is of type $M_0(\underline{D}^1)$. At the same time, the second property of F, stated in the theorem, is just the one in Lemma 3.

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