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QUOTIENT GROUPS OF LINEAR TOPOLOGICAL SPACES

BY

JANUSZ GRABOWSKI AND WOJCIECH WOJTYŃSKI (WARSZAWA)

0. Introduction. A theorem in finite dimensional Lie group theory states that the class of abelian Lie groups coincides with the class of quotient groups of finite dimensional real vector spaces. (Hereafter by a *quotient group* we shall understand a quotient group with respect to a closed subgroup.)

In this paper we consider an infinite dimensional analogue of this situation and we study properties of quotient groups of linear topological spaces from the point of view of Lie theory.

All the concepts of infinite dimensional Lie groups (e.g., Banach, Fréchet, IHL; cf. [3], [4]) are too restrictive for our purpose. In particular, when we drop the assumption that the dimension of G is finite, none of the above-mentioned classes of Lie groups is closed under the operation of passing to a quotient group.

In the sequel we introduce the class of "weak Lie groups" (abbreviated to WLG) which is defined by imposing conditions on the family of one-parameter subgroups of a group. In this paper we consider only abelian groups. The discussion of WLG in general will be given elsewhere.

1. Preliminaries. Let G be a topological group (all the groups dealt with are supposed to be Hausdorff). By $C(\boldsymbol{R}, G)$ we denote the topological group of all continuous G-valued functions on the real line $\boldsymbol{R}$ with the pointwise multiplication and the compact open topology $\mathscr{M}$. We shall define the exponential map

$$\operatorname{Exp}: C(\boldsymbol{R}, G) \to G$$

putting $\operatorname{Exp}(\phi) = \phi(1)$. There is also defined a natural multiplication by reals:

$$\boldsymbol{R} \times C(\boldsymbol{R}, G) \ni (s, \phi) \to s\phi \in C(\boldsymbol{R}, G),$$

where $(s\phi)(t) = \phi(st)$. Observe that both the exponential map and the multiplication by reals are continuous with respect to the topology $\mathscr{M}$.

Let $\Lambda(G)$ denote the family of all one-parameter subgroups of G, i.e., the family of all continuous homomorphisms of the additive group of reals into G.

$\Lambda(G)$ is a closed subset of $C(\boldsymbol{R}, G)$, and for an abelian G it is also a subgroup of $C(\boldsymbol{R}, G)$. Moreover, $\Lambda(G)$ is invariant under multiplication by reals. Restricting to $\Lambda(G)$ the compact-open topology $\mathscr{M}$, the group multiplication and the multiplication by reals, we get on $\Lambda(G)$ (for an abelian G) the structure of a real topological vector space. We shall refer to $\Lambda(G)$ together with this structure as to the *Lie algebra of* G. It is easy to observe that the topology $\mathscr{M}$ is the weakest of the real topological vector space topologies on $\Lambda(G)$ such that the exponential map Exp: $\Lambda(G) \to G$ is continuous.

Clearly, the *kernel* of the exponential map is a closed subgroup of $\Lambda(G)$, which will be denoted by $K(G)$.

We shall omit simple proofs of the following propositions.

1.1. PROPOSITION. *Let G_i, $i = 1, 2$, be abelian topological groups and let p: $G_1 \to G_2$ be a continuous homomorphism. Then the induced mapping*

$$\hat{p}\colon \Lambda(G_1) \ni \phi \to p \circ \phi \in \Lambda(G_2)$$

is continuous and linear. Moreover,

$$p \circ \mathrm{Exp} = \mathrm{Exp} \circ \hat{p} \quad \text{and} \quad \hat{p}(K(G_1)) \subset K(G_2).$$

Observe that if p is injective, then so is $\hat{p}$, but the analogous statement for surjectivity does not hold in general (cf. Examples 2.12 and 2.14 below). Also $\widehat{p_1 \circ p_2} = \hat{p}_1 \circ \hat{p}_2$, i.e., the correspondences $G \to \Lambda(G)$ and $p \to \hat{p}$ constitute a covariant functor from the category of abelian topological groups into the category of topological vector spaces.

1.2. PROPOSITION. *Let $\{U_\alpha\}_{\alpha\in\Omega}$ be a basis of (open) neighbourhoods of the neutral element e in an abelian topological group G. For each $\alpha \in \Omega$ let*

$$\hat{U}_\alpha = \{\phi \in \Lambda(G)\colon \phi(t) \in U_\alpha \text{ for } -1 \leqslant t \leqslant 1\}.$$

Then $\{\hat{U}_\alpha\}_{\alpha\in\Omega}$ constitutes a basis of (open) neighbourhoods of 0 in $\Lambda(G)$.

1.3. PROPOSITION. *If $\{d_\alpha\}_{\alpha\in\Omega}$ is a family of (complete) right-invariant pseudo-metrics defining the topology on a topological group G, and if Γ is a closed subgroup of G, then $\{\varrho_\alpha\}_{\alpha\in\Omega}$, where*

$$\varrho_\alpha([x], [y]) = \inf_{z\in\Gamma} d(xy^{-1}, z),$$

is a family of (complete) right-invariant pseudo-metrics on G/Γ defining the quotient topology. On the other hand,

$$\delta_\alpha(\phi, \psi) = \sup_{t\in[-1,1]} d_\alpha(\phi(t), \psi(t)), \qquad \alpha \in \Omega,$$

is a family of (complete) invariant pseudo-metrics on $\Lambda(G)$ defining the compact-open topology.

1.4. PROPOSITION. *Let Γ be a topological subgroup of a topological group G. The canonical embedding $i\colon \Gamma \to G$ induces a continuous injective linear mapping $\hat{i}\colon \Lambda(\Gamma) \to \Lambda(G)$ which is also an embedding. Moreover, the image of $\hat{i}$ is closed if Γ is closed.*

2. Abelian weak Lie groups.

2.1. DEFINITION. Let G be an abelian topological group with the Lie algebra $\Lambda(G)$.

(a) G is said to be *exponential* iff $\operatorname{Exp}(\Lambda(G)) = G$.

(b) G is said to be an *abelian weak Lie group* (*AWL-group*) iff $\operatorname{Exp}\colon \Lambda(G) \to G$ is an open map.

(c) An AWL-group G is said to be *Hilbert* (respectively, *Banach*, *Fréchet*, *nuclear*, etc.) if $\Lambda(G)$ is a Hilbert (respectively, Banach, Fréchet, nuclear, etc.) topological vector space.

2.2. Remarks. (a) Exponential groups have to be arcwise connected, and the connected component of unity in an AWL-group is an open and exponential subgroup.

(b) An abelian exponential topological group G is an AWL-group iff the continuous group isomorphism

$$[\operatorname{Exp}]\colon \Lambda(G)/K(G) \to G$$

induced by $\operatorname{Exp}\colon \Lambda(G) \to G$ is an isomorphism of topological groups. This follows easily from the commutativity of the diagram

$$\begin{tikzcd} \Lambda(G) \arrow[d, "\pi"'] \arrow[dr, "\operatorname{Exp}"] & \\ \Lambda(G)/K(G) \arrow[r, "{[\operatorname{Exp}]}"'] & G \end{tikzcd}$$

Indeed, since the canonical projection π is open, Exp is open iff [Exp] is open.

(c) It is not true that each abelian exponential group is an AWL-group. As an example, consider the additive group of reals $\boldsymbol{R}$ furnished with the topology induced by a dense homomorphic injection of $\boldsymbol{R}$ into a torus T^2. Clearly, this topology fails to have a basis of connected neighbourhoods of 0.

2.3. THEOREM. (a) *Every real topological vector space X is an AWL-group with respect to its additive structure. Moreover,* $\operatorname{Exp}\colon \Lambda(X) \to X$ *is an isomorphism of topological vector spaces.*

(b) *If G is an abelian topological group, then*

$$\widehat{\operatorname{Exp}}\colon \Lambda(\Lambda(G)) \to \Lambda(G)$$

is an isomorphism of topological vector spaces. Moreover,

$$\Lambda(K(G)) = \{0\}.$$

(c) *If Γ is a closed subgroup of a topological vector space X, then*

$$\operatorname{Exp}\colon \Lambda(\Gamma) \to \Gamma$$

is an embedding (*i.e.*, Exp *induces the original topology on* $\Lambda(\Gamma)$) *and* $\operatorname{Exp}(\Lambda(\Gamma))$ *is a closed subgroup of* Γ (*and thus of* X). *Moreover,*

$$\Lambda\big(\Gamma/\operatorname{Exp}(\Lambda(\Gamma))\big) = \{0\}.$$

Proof. (a) Exp: $\Lambda(X) \to X$ is a continuous homomorphism of topological groups. Take $\phi \in \Lambda(X)$ and put

$$x_n = \phi\left(\frac{1}{n}\right) = \operatorname{Exp}\left(\frac{1}{n}\phi\right) \quad \text{for } n = 1, 2, \dots$$

Since $nx_n = \phi(1) = x_1$, we have

$$\phi\left(\frac{m}{n}\right) = mx_n = \frac{m}{n}x_1 \quad \text{for } m, n = \pm 1, \pm 2, \dots,$$

i.e., $qx_1 = \phi(q)$ for all rationals q. Therefore $\phi(t) = tx_1 = t\operatorname{Exp}(\phi)$ for all $t \in \boldsymbol{R}$ by the continuity of ϕ, which shows that ϕ is linear and injective. It is easy to see that the mapping r: $X \to \Lambda(X)$ defined by $r(x)(t) = tx$ is the inverse of Exp. By the continuity of multiplication by reals it follows that X has a basis $\{U_\alpha\}_{\alpha\in\Omega}$ of star-shaped symmetric neighbourhoods of 0. By Proposition 1.2, $\{\hat{U}_\alpha\}_{\alpha\in\Omega}$ is a basis of neighbourhoods of 0 in $\Lambda(X)$ and we have

$$r^{-1}(\hat{U}_\alpha) = \{x \in X\colon\ tx \in U_\alpha \text{ for } -1 \leqslant t \leqslant 1\} = U_\alpha,$$

i.e., r is continuous.

(b) We have the following commutative diagram:

$$\begin{tikzcd}
\Lambda(\Lambda(G)) \arrow[r, "\widehat{\operatorname{Exp}}"] \arrow[d, shift right, "\operatorname{Exp}"'] & \Lambda(G) \arrow[d, "\operatorname{Exp}"] \\
\Lambda(G) \arrow[r, "\operatorname{Exp}"] \arrow[u, shift right, "r"'] & G
\end{tikzcd}$$

Since $\operatorname{Exp}\big(t(\widehat{\operatorname{Exp}}\circ r)(\phi)\big) = (\operatorname{Exp}\circ\widehat{\operatorname{Exp}}\circ r)(t\phi) = \operatorname{Exp}(t\phi)$ for all $t \in \boldsymbol{R}$, we get

$$\widehat{\operatorname{Exp}}\circ r = \mathrm{id}_{\Lambda(G)}.$$

Hence $\widehat{\operatorname{Exp}}$ is an isomorphism of topological vector spaces since such is r by (a). Let i: $K(G) \to \Lambda(G)$ be the canonical embedding. From the commutative diagram

$$\begin{tikzcd}
\Lambda(K(G)) \arrow[r, "\hat{i}"] \arrow[d, "\operatorname{Exp}"'] & \Lambda(\Lambda(G)) \arrow[r, "\widehat{\operatorname{Exp}}"] \arrow[d, shift right, "\operatorname{Exp}"'] & \Lambda(G) \arrow[d, "\operatorname{Exp}"] \\
K(G) \arrow[r, "i"] & \Lambda(G) \arrow[r, "\operatorname{Exp}"] \arrow[u, shift right, "r"'] & G
\end{tikzcd}$$

we have

$$e = \operatorname{Exp}\circ i\circ\operatorname{Exp}(t\phi) = \operatorname{Exp}\circ\widehat{\operatorname{Exp}}\circ\hat{i}(t\phi) = \operatorname{Exp}\big(t\widehat{\operatorname{Exp}}\circ\hat{i}(\phi)\big)$$

for all $\phi \in \Lambda(K(G))$ and $t \in \boldsymbol{R}$, i.e.,

$$\widehat{\operatorname{Exp}} \circ \hat{i}(\phi) = 0 \quad \text{for all } \phi \in \Lambda(K(G)).$$

Since $\hat{i}$ and $\widehat{\operatorname{Exp}}$ are injective, $\phi = 0$.

(c) Let $i\colon \Gamma \to X$ be the canonical embedding. From the commutative diagram

$$\begin{tikzcd} \Lambda(\Gamma) \arrow[r, "\hat{i}"] \arrow[d, "\operatorname{Exp}"'] & \Lambda(X) \arrow[d, "\operatorname{Exp}"] \\ \Gamma \arrow[r, "i"] & X \end{tikzcd}$$

where $\operatorname{Exp}\colon \Lambda(X) \to X$ is an isomorphism by (a) and $\hat{i}$ is a linear embedding by Proposition 1.4, one can see that $\operatorname{Exp}\colon \Lambda(\Gamma) \to \Gamma$ is an embedding. Since $\operatorname{Exp} \circ \hat{i}(\Lambda(\Gamma))$ is closed in X, $\operatorname{Exp}(\Lambda(\Gamma))$ is closed in Γ. Finally, let $\Gamma_1 = \operatorname{Exp}(\Lambda(\Gamma))$. We have a homomorphic embedding

$$i_1\colon \Gamma/\Gamma_1 \to X/\Gamma_1$$

inducing the embedding

$$\hat{i}_1\colon \Lambda(\Gamma/\Gamma_1) \to \Lambda(X/\Gamma_1) \simeq X/\Gamma_1.$$

If $\phi \in \Lambda(\Gamma/\Gamma_1)$, then there is $x \in X$ such that

$$\hat{i}_1(\phi)(t) = t[x] = [tx] \in X/\Gamma_1.$$

Clearly, $tx \in \Gamma$ for all $t \in \boldsymbol{R}$, so $x \in \Gamma_1$ and $[x] = 0$. By the injectivity of $\hat{i}_i$ we have $\phi = 0$.

By Theorem 2.3 every topological vector space is an AWL-group. The following two propositions provide new examples of AWL-groups.

2.4. PROPOSITION. *The direct product $\prod_{\alpha\in\Omega} G_\alpha$ (respectively, the direct sum $\bigoplus_{\alpha\in\Omega} G_\alpha$) of a family $\{G_\alpha\}_{\alpha\in\Omega}$ of abelian exponential groups is an abelian exponential group with the Lie algebra $\prod_{\alpha\in\Omega} \Lambda(G_\alpha)$ (respectively, $\bigoplus_{\alpha\in\Omega} \Lambda(G_\alpha)$). The same is true for a family of AWL-groups.*

The proof is standard.

2.5. PROPOSITION. *Given an AWL-group G and its closed subgroup Γ, the quotient group G/Γ is an AWL-group. Moreover, the canonical projection $p\colon G \to G/\Gamma$ induces a continuous linear mapping*

$$\hat{p}\colon \Lambda(G) \to \Lambda(G/\Gamma)$$

with dense image and the kernel equal to $\Lambda(\Gamma)$.

Proof. From the commutative diagram

$$\begin{tikzcd} \Lambda(G) \arrow[r, "\hat{p}"] \arrow[d, "\mathrm{Exp}"'] & \Lambda(G/\Gamma) \arrow[d, "\mathrm{Exp}"] \\ G \arrow[r, "p"] & G/\Gamma \end{tikzcd}$$

where p, and hence $p\circ\mathrm{Exp} = \mathrm{Exp}\circ\hat{p}$ are open, we conclude that

$$\mathrm{Exp}\colon \Lambda(G/\Gamma) \to G/\Gamma$$

is open and that

$$\begin{aligned}\ker(\hat{p}) &= \{\phi\in\Lambda(G)\colon \mathrm{Exp}(t\hat{p}(\phi)) = e \text{ for all } t\in\mathbf{R}\}\\ &= \{\phi\in\Lambda(G)\colon \mathrm{Exp}\circ\hat{p}(t\phi) = p\circ\mathrm{Exp}(t\phi) = e \text{ for all } t\in\mathbf{R}\}\\ &= \{\phi\in\Lambda(G)\colon \mathrm{Exp}(t\phi)\in\Gamma \text{ for all } t\in\mathbf{R}\} = \Lambda(\Gamma).\end{aligned}$$

To show the density take $\psi\in\Lambda(G/\Gamma)$ and a neighbourhood U of e in G/Γ. Our aim is to find an element of $\hat{p}(\Lambda(G))\in(\psi+\hat{U})$. (We use the additive notation, since G is abelian.)

Take a neighbourhood W of the identity in G/Γ such that $W-W\subset U$. Since $p\circ\mathrm{Exp}$ is open, the set $V = p\circ\mathrm{Exp}(\hat{p}^{-1}(\hat{W}))$ is open, and there is a positive integer n such that $\psi(t)\in V$ for $t\in[-1/n, 1/n]$. Let $\phi_{1/n}$ be an element in $(\hat{p}^{-1}(\hat{W}))$ such that

$$p\circ\mathrm{Exp}(\phi_{1/n}) = \psi(1/n).$$

Put $\phi = n\phi_{1/n}$. For each $t\in[-1, 1]$ there are an integer m and $r\in[-1, 1]$ such that $t = m/n+r/n$, so $t\phi = m\phi_{1/n}+r\phi_{1/n}$. Hence

$$\begin{aligned}\hat{p}(\phi)(t) &= \mathrm{Exp}\circ\hat{p}(t\phi) = p\circ\mathrm{Exp}(t\phi)\\ &= p\big(m\mathrm{Exp}(\phi_{1/n})+\mathrm{Exp}(r\phi_{1/n})\big) = \psi(m/n)+p\circ\mathrm{Exp}(r\phi_{1/n})\\ &= \psi(t)+p\circ\mathrm{Exp}(r\phi_{1/n})-\psi(r/n)\end{aligned}$$

and we get

$$\hat{p}(\phi)(t)-\psi(t) = p\circ\mathrm{Exp}(r\phi_{1/n})-\psi(r/n)\in W-W\subset U \quad \text{for all } t\in[-1, 1],$$

i.e., $\hat{p}(\phi)\in(\psi+\hat{U})$.

2.6. THEOREM. *Let G be an abelian exponential topological group with the Lie algebra $\Lambda(G)$. Let G_α denote the underlying group of G and let Ω be the family of all topological group topologies on G_α providing $\Lambda(G)$ as the corresponding Lie algebra. Then there is the strongest topology $\mathscr{M}$ in Ω, which is the unique AWL-group topology on G_α.*

Proof. Let $[\mathrm{Exp}]\colon \Lambda(G)/K(G)\to G$ be the continuous homomorphism induced by $\mathrm{Exp}\colon \Lambda(G)\to G$. Since $[\mathrm{Exp}]$ is a group isomorphism, we can identify G_α with $\Lambda(G)/K(G)$. Put $\mathscr{M}$ to be the quotient topology. Since $[\mathrm{Exp}]$ is continuous, it is obvious that $\mathscr{M}$ is stronger than any group topology on G_α providing $\Lambda(G)$ as the corresponding Lie algebra. It suffices to show that

$$\Lambda(\Lambda(G)/K(G)) \simeq \Lambda(G).$$

We have the following commutative diagram of continuous homomorphisms:

$$\begin{tikzcd}
 & & \Lambda(\Lambda(G)) \arrow[d, "\mathrm{Exp}"] \arrow[dddll, "\hat{p}"'] \arrow[dddrr, "\mathrm{Exp}"] & & \\
 & & \Lambda(G) \arrow[dl, "p"'] \arrow[dr, "\mathrm{Exp}"] & & \\
 & \Lambda(G)/K(G) \arrow[rr, "{[\mathrm{Exp}]}"] & & G & \\
\Lambda(\Lambda(G)/K(G)) \arrow[ur, "\mathrm{Exp}"] \arrow[rrrr, "{[\hat{\mathrm{Exp}}]}"] & & & & \Lambda(G) \arrow[ul, "\mathrm{Exp}"']
\end{tikzcd}$$

where $\mathrm{Exp}: \Lambda(\Lambda(G)) \to \Lambda(G)$ is an isomorphism by Theorem 2.3 (b), and $[\widehat{\mathrm{Exp}}]$ is injective since $[\mathrm{Exp}]$ is injective. Hence $\hat{p} \circ (\mathrm{Exp})^{-1}$ is a continuous linear map which is the inverse of $[\widehat{\mathrm{Exp}}]$, and this shows that

$$\Lambda(G) \simeq \Lambda(\Lambda(G)/K(G)).$$

2.7. THEOREM. *Let G_i, $i = 1, 2$, be connected AWL-groups with Lie algebras $\Lambda(G_i)$ and let $\varphi: \Lambda(G_1) \to \Lambda(G_2)$ be a continuous linear mapping. Then there exists a continuous homomorphism $\phi: G_1 \to G_2$ such that $\hat{\phi} = \varphi$ if and only if*

$$\varphi(K(G_1)) \subset K(G_2).$$

Proof. Since $\hat{\phi}(K(G_1)) \subset K(G_2)$ by 1.1, suppose that $\varphi(K(G_1)) \subset K(G_2)$. Then φ induces a continuous homomorphism

$$[\varphi]: \Lambda(G_1)/K(G_1) \to \Lambda(G_2)$$

by the commutative diagram

$$\begin{tikzcd}
\Lambda(G_1) \arrow[d, "p"] \arrow[dr, "\varphi"] \arrow[dd, bend right=60, "\mathrm{Exp}"'] & & \\
\Lambda(G_1)/K(G_1) \arrow[r, "{[\varphi]}"] \arrow[d, "{[\mathrm{Exp}]}"] & \Lambda(G_2) \arrow[r, "\mathrm{Exp}"] & G_2 \\
G_1 \arrow[urr, "\phi"] & &
\end{tikzcd}$$

It is easy to see that $\phi = \mathrm{Exp} \circ [\varphi] \circ [\mathrm{Exp}]^{-1}$ is the desired homomorphism.

The following theorem gives a useful description of AWL-groups:

2.8. THEOREM. *Let G be a connected abelian topological group. Then the following are equivalent:*

(a) *G is an AWL-group.*

(b) *G is a quotient group of a real topological vector space.*

Proof. (a) $\Rightarrow$ (b) follows by Remark 2.2 (b).

(b) $\Rightarrow$ (a) results from Theorem 2.3 (a) and Proposition 2.5.

Thus the class of connected AWL-groups is the same as the class of quotient groups of topological vector spaces. Observe that if Γ is a closed subgroup of a real topological vector space X, then by 2.3 we can consider $\Lambda(\Gamma)$

as a closed linear subspace of X included in Γ. We have then

$$X/\Gamma = (X/\Lambda(\Gamma))/(\Gamma/\Lambda(\Gamma)), \quad \text{where } \Lambda(\Gamma/\Lambda(\Gamma)) = \{0\},$$

i.e., we can reduce the vector space X and the subgroup Γ to the case where Γ includes no linear subspace. This motivates the following definition:

2.9. DEFINITION. Let G be a connected AWL-group. A pair (X, Γ), where X is a real topological vector space and Γ is a closed subgroup of X such that $\Lambda(\Gamma) = \{0\}$, will be called a *realization* of G if X/Γ and G are isomorphic topological groups. We shall also write X/Γ instead of (X, Γ) in this context. Two realizations X_1/Γ_1 and X_2/Γ_2 are called *isomorphic* if there is an isomorphism $\phi\colon X_1 \to X_2$ of topological vector spaces such that $\phi(\Gamma_1) = \Gamma_2$.

By $R(G)$ we shall denote the class of all realizations of G up to isomorphism.

Every connected AWL-group has at least one realization, namely $\Lambda(G)/K(G)$, called the *canonical realization*.

2.10. THEOREM. *Let G be a connected AWL-group. Let X/Γ be a realization of G and let $p\colon X \to X/\Gamma$ be the canonical projection. Then p induces a continuous linear injection $\hat{p}\colon X \to \Lambda(G)$, where $\hat{p}(x)(t) = p(tx)$, with dense image. Moreover, $\hat{p}(\Gamma) \subset K(G)$ and $\hat{p}(\Gamma)$ is dense in $K(G)$.*

In other words, we have the following exact commutative diagram of continuous homomorphisms:

$$\begin{tikzcd}
0 \arrow[r] & K(G) \arrow[r] & \Lambda(G) \arrow[dr, "\mathrm{Exp}"] & \\
 & & & G \\
0 \arrow[r] & \Gamma \arrow[r] \arrow[uu, "\hat{p}|\Gamma"] & X \arrow[ur, "p"'] \arrow[uu, "\hat{p}"'] &
\end{tikzcd}$$

where $\hat{p}$ and $\hat{p}|\Gamma$ have dense images.

Proof. Since X is an AWL-group, $\hat{p}\colon X \to \Lambda(G)$ is a dense injection by 2.5. Clearly, $\hat{p}(\Gamma) \subset K(G)$. To prove the density of $\hat{p}(\Gamma)$ in $K(G)$ suppose that there is $\gamma \in K(G)$ and a neighbourhood U of e in G such that

$$(\gamma + \hat{U}) \cap \hat{p}(\Gamma) = \emptyset.$$

Take a neighbourhood W of e in G such that $W - W \subset U$. We have $\hat{W} - \hat{W} \subset \hat{U}$. Because $V := \hat{p}^{-1}(\hat{W})$ is a neighbourhood of 0 in X and p is open, $p(V)$ is a neighbourhood of e in G. Since $\mathrm{Exp}(\gamma + \hat{W})$ is also a neighbourhood of e in G and since $\hat{p}(X)$ is dense in $\gamma + \hat{W}$, there exists $y \in \hat{p}^{-1}(\gamma + \hat{W})$ such that

$$\mathrm{Exp} \circ \hat{p}(y) = p(y) \in p(V).$$

Take $x \in V$ such that $p(x) = p(y)$. We have then $p(y - x) = e$, i.e., $y - x \in \Gamma$, and

$$\hat{p}(y - x) = \hat{p}(y) - \hat{p}(x) \in \gamma + \hat{W} - \hat{W} \subset \gamma + \hat{U},$$

a contradiction.

Now we discuss some examples of AWL-groups and their algebras.

2.11. EXAMPLE. Let (Ω, Σ, μ) be a measure space and let $h\colon \Omega \to \boldsymbol{R}$ be a measurable function. Put

$$L^p = L^p(\Omega, \Sigma, \mu) \quad \text{for } 0 \leqslant p \leqslant \infty$$

and let

$$K_h^p = \{f \in L^p\colon f(\omega) \in \boldsymbol{Z}\cdot h(\omega) \text{ for } \mu\text{-almost all } \omega\},$$

where $\boldsymbol{Z}$ is the set of all integers. (We consider only real-valued functions.) It is easy to see that K_h^p is a closed subgroup of L^p and that $\Lambda(K_h^p) = \{0\}$. Consider the AWL-group $G_h^p = L^p/K_h^p$. We shall show that for $0 < p < \infty$ the algebra $\Lambda(K_h^p)$ is the completion of L^p furnished with the Fréchet norm $\|\cdot\|_{p,h}$ induced by the metrizing modular $\varrho_{p,h}\colon L^p \to \boldsymbol{R}_+$ defined by

$$\varrho_{p,h}(f) = \int_{\Omega} \min(|f|^p, |\bar{h}|^p)\,d\mu,$$

where

$$\bar{h}(\omega) = \begin{cases} h(\omega) & \text{if } h(\omega) \neq 0, \\ +\infty & \text{if } h(\omega) = 0. \end{cases}$$

It is easy to observe that $\|\cdot\|_{p,h}$ can be explicitly defined by

$$\|f\|_{p,h} = \inf\{\varepsilon > 0\colon \varrho_{p,h}(f/\varepsilon) < \varepsilon\}.$$

Since L^p has a complete Fréchet norm $\|\cdot\|_p$, G_h^p and hence $\Lambda(G_h^p)$ have complete invariant metrics by Proposition 1.3. We have a dense linear injection $\hat{p}\colon L^p \to \Lambda(G_h^p)$ induced by the canonical projection

$$p\colon L^p \to L^p/K_h^p \simeq G_h^p,$$

so we can identify the linear metric space $\Lambda(G_h^p)$ with the completion of L^p induced by the injection $\hat{p}$. Writing $|||f|||_{p,h}$ instead of $d_{p,h}(f, 0)$ for $f \in L^p$, by 1.3 we have

$$|||f|||_{p,h} = \sup_{t \in [-1,1]} \inf_{k \in K_h^p} \|tf - k\|_p,$$

where

$$\|g\|_p = \Big(\int_{\Omega} |g|^p d\mu\Big)^{\min(1/p,1)}$$

is the F-norm on L^p, $0 < p < \infty$.

For $f \in L^p$ put

$$f_h(\omega) = \begin{cases} f(\omega) & \text{if } h(\omega) = 0, \\ I(|f(\omega)/h(\omega)|)h(\omega) & \text{if } h(\omega) \neq 0, \end{cases}$$

where

$$I(a) = \begin{cases} [a] & \text{if } a \leqslant [a] + \tfrac{1}{2}, \\ [a]+1 & \text{if } a > [a] + \tfrac{1}{2}, \end{cases}$$

and $[a]$ is the integer part of a. Observe that $|f_h| \leqslant 2|f|$, so $f_h \in K_h^p$, and that

$$\inf_{k \in K_h^p} \|f - k\|_p = \|f - f_h\|_p.$$

Thus we can write

$$|||f|||_{p,h} = \sup_{t\in[-1,1]} \|tf-(tf)_h\|_p.$$

Put

$$A = \{\omega\in\Omega\colon\ |f(\omega)| \leqslant \tfrac{1}{2}|I(\omega)|\},$$

$$B = \Omega\backslash A \quad \text{and} \quad F(\omega, t) = |tf(\omega)-(tf)_h(\omega)|.$$

F is a measurable function on $\Omega\times \boldsymbol{R}$. If $\omega\in B$, then

$$F(\omega, t) = \begin{cases} t|f(\omega)| & \text{for } |t| \leqslant \frac{1}{2}|h(\omega)/f(\omega)|,\\ |h(\omega)|-t|f(\omega)| & \text{for } \frac{1}{2}|h(\omega)/f(\omega)| < t \leqslant |h(\omega)/f(\omega)|,\end{cases}$$

and $F(\omega, t+|h/f|(\omega)) = F(\omega, t)$. Hence for $\omega\in B$ we have

$$\int_0^1 f(\omega, t)^p dt \geqslant \left[2\left|\frac{f(\omega)}{h(\omega)}\right|\right] \int_0^{|h(\omega)/f(\omega)|/2} |tf(\omega)|^p dt = \left[2\left|\frac{f(\omega)}{h(\omega)}\right|\right]\frac{|h(\omega)|^p|h(\omega)|}{2^{p+1}(p+1)|f(\omega)|}.$$

Since

$$\left[2\left|\frac{f(\omega)}{h(\omega)}\right|\right]\cdot\left|\frac{h(\omega)}{f(\omega)}\right| \geqslant 1,$$

we get

$$\int_0^1 F(\omega, t)^p dt \geqslant C_p|h(\omega)|^p \quad \text{for } \omega\in B \text{ and } C_p = \frac{1}{2^{p+1}(p+1)}.$$

On the other hand, for $\omega\in A$ we have $F(\omega, t) = t|f(\omega)|$, so

$$\int_0^1 F(\omega, t)^p dt = \frac{|f(\omega)|^p}{p+1},$$

and finally

$$\begin{aligned}\int_\Omega\Big(\int_0^1 F(\omega, t)^p dt\Big)d\mu(\omega) &\geqslant \int_A \frac{|f(\omega)|^p}{p+1}d\mu(\omega)+\int_B C_p|h(\omega)|^p d\mu(\omega)\\ &\geqslant C_p\Big(\int_A |2f|^p d\mu+\int_B |h|^p d\mu\Big).\end{aligned}$$

Changing the order of integration, we conclude that there is $t_0\in[0, 1)$ such that

$$\begin{aligned}\int_\Omega F(\omega, t_0)^p d\mu(\omega) &\geqslant C_p\Big(\int_{\substack{\{\omega\in\Omega:|2f(\omega)|\\ \leqslant|\bar{h}(\omega)|\}}} |2f(\omega)|^p d\mu(\omega)+\int_{\substack{\{\omega\in\Omega:|2f(\omega)|\\ >|\bar{h}(\omega)|\}}}\Big)\\ &= C_p\int_\Omega \min(|2f|^p, |\bar{h}|^p)d\mu\\ &= C_p\int_\Omega \min(|2f|^p, |\bar{h}|^p)d\mu \geqslant C_p\int_\Omega \min(|f|^p, |h|^p)d\mu\\ &\geqslant C_p\varrho_{p,h}(f).\end{aligned}$$

But

$$|||f|||_{p,h}^{\max(p,1)} \geqslant \|t_0 f-(t_0 f)_h\|_p^{\max(p,1)} = \int_\Omega F(\omega, t_0)^p d\mu(\omega),$$

so

$$|||f|||_{p,h}^{\max(p,1)} \geqslant C_p \varrho_{p,h}(f).$$

On the other hand, since

$$|tf(\omega)-(tf)_h(\omega)| = |tf(\omega)| \leqslant |2f(\omega)| \quad \text{for } \omega\in A \text{ and } -1 \leqslant t \leqslant 1$$

and

$$|tf(\omega)-(tf)_h(\omega)| \leqslant |h(\omega)| \quad \text{for } \omega\in B \text{ and } t\in \boldsymbol{R},$$

we get

$$\int_\Omega |tf(\omega)-(tf)_h(\omega)|^p d\mu(\omega) \leqslant \int\limits_{\substack{\{\omega\in\Omega: |2f(\omega)| \\ \leqslant |\bar{h}(\omega)|\}}} |2f(\omega)|^p d\mu(\omega) + \int\limits_{\substack{\{\omega\in\Omega: |2f(\omega)| \\ \geqslant |\bar{h}(\omega)|\}}}$$

$$= \int_\Omega \min(|2f|^p, |\bar{h}|^p) d\mu \leqslant 2^p \varrho_{p,h}(f),$$

i.e.,

$$|||f|||_{p,h} = \sup_{t\in[-1,1]} \int_\Omega |tf(\omega)-(tf)_h(\omega)|^p d\mu(\omega) \leqslant 2^p \varrho_{p,h}(f).$$

This shows that $|||\cdot|||_{p,h}$ and $\|\cdot\|_{p,h}$ induce the same topology. We shall reformulate the result of the above example in the three special situations.

2.12. Example. Suppose that $\mu(\Omega) < \infty$ and put $h(\bar{\omega}) = 1$. We shall show that $\varrho_{p,h}(f) \to 0$ iff $f_n \to 0$ in measure. Let

$$\varrho_{p,h}(f_n) = \int_\Omega \min(|f_n|^p, 1) d\mu \to 0$$

and take $0 < \varepsilon < 1$. Since

$$\int\limits_{\{|f_n|\leqslant\varepsilon\}} |f_n|^p d\mu + \int\limits_{\{\varepsilon<|f_n|<1\}} |f_n|^p d\mu + \mu\{|f_n| \geqslant 1\} \to 0$$

and

$$\varepsilon^p \mu(\{\varepsilon < |f_n| < 1\}) \leqslant \int\limits_{\{\varepsilon<|f_n|<1\}} |f_n|^p d\mu,$$

we get

$$\mu(\{|f_n| > \varepsilon\}) \to 0.$$

On the other hand, let $f_n \to 0$ in measure. Take $0 < \varepsilon < 1$ and $\delta > 0$. We have

$$\mu(\{|f_n| > \varepsilon\}) < \delta \quad \text{for } n \geqslant 0.$$

Hence

$$\varrho_{p,h}(f_n) = \int_\Omega \min(|f_n|^p, 1) d\mu \leqslant \int\limits_{\{|f_n|\leqslant\varepsilon\}} |f_n|^p + \mu(\{|f_n| > \varepsilon\})$$

$$\leqslant \varepsilon^p \mu(\Omega) + \delta \quad \text{for } n \geqslant 0.$$

Since ε and δ may be chosen arbitrarily small, $\varrho_{p,h}(f_n) \to 0$.

This shows that, in this case, $\Lambda(G_h^p)$ equals L^0, the space of all measurable functions on Ω with the topology of convergence in measure. Note that the same arguments apply if we take h such that $0 < m < |h(\omega)| < M < \infty$ for some $m, M \in \boldsymbol{R}$ and μ-almost all $\omega \in \Omega$ instead of $h = 1$. Since the kernel $K(G_h^p)$ of $\operatorname{Exp}: \lambda(G_h^p) \to G_h^p$ is the closure in L^0 of $K_h^p \cap L^0$, we have

$$K(G_p^h) = \{f \in L^0: f(\omega) \in \boldsymbol{Z} \cdot h(\omega)\}.$$

In particular, G_h^p are isomorphic for all $0 \leqslant p < \infty$.

2.13. Example. Let $L^p = l^p$ (i.e., Ω is countably infinite and μ is purely atomic with atoms of measure 1) and let $h = 1$. Then

$$\varrho_{p,h}(f) = \sum_{m=1}^{\infty} \min(|f(m)|^p, 1) \leqslant \|f\|_p^{\max(p,1)}.$$

On the other hand, if $\|f_n\|_p \to 0$, then for sufficiently large n we have $|f_n(k)| < 1$ for $k = 1, 2, \ldots$, so

$$\varrho_{p,h}(f_n) = \|f_n\|_p^{\max(p,1)} \to 0.$$

This shows that $\Lambda(G_h^p) = l^p$ in this case.

2.14. Example. Let $L^p = l^p$ and let $h \in l^p$, $h(k) \neq 0$, $k = 1, 2, \ldots$ If

$$\varrho_{p,h}(f_n) = \sum_{k=1}^{\infty} \min(|f_n(k)|^p, |h(k)|^p) \to 0,$$

then clearly $f_n(k) \underset{n}{\to} 0$ for each $k = 1, 2, \ldots$ On the other hand, if

$$f_n(k) \underset{n}{\to} 0 \quad \text{for } k = 1, 2, \ldots,$$

then for each $\varepsilon > 0$ there is k_0 such that

$$\sum_{k=k_0}^{\infty} |h(k)|^p < \varepsilon/2$$

and there is n_0 such that

$$\sum_{k=1}^{n_0 - 1} |f_n(k)|^p \leqslant \varepsilon/2 \quad \text{for } n \geqslant n_0.$$

Hence $\varrho_{p,h}(f_n) < \varepsilon$ for $n \geqslant n_0$, which shows that in this case $\Lambda(G_h^p) = l^0$, the space of all sequences with the topology of pointwise convergence. Since l^0 is nuclear, G_h^p is then an example of a nuclear AWL-group.

3. Classes of realizations of AWL-groups. In this section we shall investigate realizations of a given connected AWL-group.

3.1. Theorem. *The following are equivalent*:

(a) *G has a realization X/Γ with discrete Γ.*

(b) *$K(G)$ is discrete.*

(c) *G has no small circle subgroups (i.e., there exists a neighbourhood U of e in G in which there is contained no closed subgroup isomorphic to T^1).*

(d) *For each realization X/Γ of G, the subgroup Γ is discrete.*

Moreover, if either of (a), (b), (c), (d) *is satisfied, then any two realizations of G are isomorphic, i.e., $R(G) = \{\Lambda(G)/K(G)\}$.*

Proof. (a)$\Rightarrow$(b). Let U be a neighbourhood of 0 in X such that

$$\Gamma \cap (U-U) = \{0\}$$

and let $p\colon X \to X/\Gamma = G$ be the canonical projection. Then $p|U\colon U \to G$ as a homeomorphism of U onto $p(U)$ is an injective continuous and open map. Take $\phi \in p(U) \cap K(G)$. Since

$$\phi(t) \in p(U) \text{ for } t \in [-1, 1] \quad \text{and} \quad \phi(1) = e,$$

we have $\phi(t) \in p(U)$ for all $t \in \boldsymbol{R}$. The mapping

$$\boldsymbol{R} \ni t \to x_t = (p|U)^{-1}(\phi(t))$$

is a one-parameter subgroup of X such that $x_t = tx$ for some $x \in X$. But $x_1 = 0$, so $x = 0$ and $\phi = 0$, i.e., $K(G)$ is discrete.

(b)$\Rightarrow$(c). There exists a neighbourhood U of 0 in $\Lambda(G)$ such that $K(G) \cap (U-U) = \{0\}$. Thus $\operatorname{Exp}|U\colon U \to \operatorname{Exp}(U)$ is an isomorphism of local topological groups. Since U contains no circle subgroup, $\operatorname{Exp}(U)$ contains no circle subgroup.

(c)$\Rightarrow$(d). Let X/Λ be a realization of G with Γ non-discrete and let $p\colon X \to X/\Gamma$ be the canonical projection. For U being a star-shaped neighbourhood of 0 in X, we can find $\gamma \in \Gamma$ such that $U \ni \gamma \neq 0$. Then

$$\boldsymbol{R} \ni t \to p(t \cdot \gamma) \in X/\Gamma$$

is a circle subgroup in $\hat{U}$.

(d)$\Rightarrow$(a) is obvious.

Assume now that $X/\Gamma \in R(G)$ and suppose that (d) is satisfied. Since

$$p\colon X \to X/\Gamma = G \quad \text{and} \quad \operatorname{Exp}\colon \Lambda(G) \to G$$

are local homeomorphisms as in the proof of (a)$\Rightarrow$(b), $\hat{p}\colon X \to \Lambda(G)$ is open. By 2.5, $\hat{p}(\Gamma)$ is dense in $K(G)$ and $K(G)$ is discrete, so $\hat{p}(\Gamma) = K(G)$, i.e., X/Γ and $\Lambda(G)/K(G)$ are isomorphic realizations.

3.2. THEOREM. *Let G be a connected AWL-group. Then the following are equivalent*:

(a) *G has a realization X/Γ, where X has no small subgroups and Γ is discrete.*

(b) *G has no small subgroups.*

(c) *For each realization X/Γ of G, X has no small subgroups and Γ is discrete.*

Proof. (a) $\Rightarrow$ (b). Since Γ is discrete, the canonical projection

$$p\colon X \to X/\Gamma = G$$

induces an isomorphism of local topological groups.

(b) $\Rightarrow$ (c). Let X/Γ be a realization of G. Γ is discrete by 3.1, so X has no small subgroups by the same reasoning as in (a) $\Rightarrow$ (b).

(c) $\Rightarrow$ (a) is obvious.

3.3. THEOREM. *If X is a locally bounded Fréchet space and Γ is a closed non-discrete subgroup of X, then $\hat{p}\colon X \to \Lambda(X/\Gamma)$ induced by the canonical projection $p\colon X \to X/\Gamma$ is not a bijection. In particular, $\Lambda(X/\Gamma)$ is not a locally bounded Fréchet space.*

Proof. Since X is a Fréchet space, X/Γ and $\Lambda(X/\Gamma)$ have complete invariant metrics. Thus, by the Banach open map theorem, if $\hat{p}\colon X \to \Lambda(X/\Gamma)$ is "onto", then $\hat{p}$ is a homeomorphism. Let $\gamma_n \in \Gamma$ be such that $\gamma_n \neq 0$, $\gamma_n \to 0$, and let $t_n \in \boldsymbol{R}$ be chosen in such a way that $t_n \gamma_n \not\to 0$. It is easy to see that

$$\hat{p}(t_n \gamma_n) \to 0$$

since $t\phi \in \hat{U}$ if $\phi \in K(G) \cap \hat{U}$ and $t \in \boldsymbol{R}$. Hence $\hat{p}$ is not a homeomorphism.

If $\Lambda(X/\Gamma)$ is a locally bounded Fréchet space, then $K(G)$ has to be discrete, since $\operatorname{Exp}\colon \Lambda(\Lambda(X/\Gamma)) \to \Lambda(X/\Gamma)$ is an isomorphism.

3.4. COROLLARY. (a) *If G is a Banach AWL-group, then $K(G)$ is discrete.*

(b) *An AWL-group G is Banach iff G is an abelian Banach Lie group.*

Proof. (a) follows by 3.3. To prove (b) observe that if G is a Banach AWL-group, then $K(G)$ is discrete by (a), and then $G_0 = \Lambda(G)/K(G)$ is a Banach Lie group. On the other hand, if G is an abelian Banach Lie group, then $\Lambda(G)/K(G) = T_e(G)$ is a Banach space. Moreover, for the Banach Lie group the exponential map is locally open at 0.

If A is a subset of a vector space (respectively, a group) X, then by $\operatorname{span}(A)$ (respectively, $\operatorname{Gr}(A)$) we shall denote the linear subspace (respectively, a subgroup) of X generated by A.

3.5. DEFINITION. A topological group G is said to be *locally generated* if $\operatorname{Gr}(U) = G$ for each neighbourhood U of e in G.

It is easy to see that each connected topological group is locally generated.

3.6. PROPOSITION. *Let X/Γ be a realization of a connected AWL-group G, let $\hat{p}\colon X \to \Lambda(G)$ be the injection induced by the canonical projection*

$$p\colon X \to X/\Gamma = G$$

and let Σ be a basis of symmetric star-shaped neighbourhoods of 0 in X. Then

$$\hat{p}\Big(\bigcap_{U\in\Sigma} \operatorname{span}(U \cap \Gamma)\Big) \subset \bigcap_{U\in\Sigma} \operatorname{conv}(\hat{U}).$$

In particular, if

$$\bigcap_{U\in\Sigma}\operatorname{span}(U\cap\Gamma)\neq\{0\},$$

then $\Lambda(G)$ is not locally convex.

Proof. Take $U\in\Sigma$. For each $\gamma\in U\cap\Gamma$ and each $t\in\boldsymbol{R}$ we have $\hat{p}(t\gamma)\in\hat{U}$, so

$$\operatorname{conv}(\hat{U})\supset\operatorname{span}(\hat{p}(U\cap\Gamma))=\hat{p}(\operatorname{span}(U\cap\Gamma))\supset\hat{p}\Big(\bigcap_{U\in\Sigma}\operatorname{span}(U\cap\Gamma)\Big).$$

3.7. EXAMPLE. Take $L^p=L^p(0,1)$ and $h=1$ (cf. 2.11). It is easy to observe that K_h^p is arcwise connected in this case, so $U\cap K_h^p$ generates the whole K_h^p for each neighbourhood U of 0 in L^p. Thus $\operatorname{span}(U\cap K_h^p)=\operatorname{span}(K_h^p)$ is dense in L^p, $\hat{p}(L^p)$ is dense in $\Lambda(G_h^p)$, so the convex hull of every neighbourhood of 0 in $\Lambda(G_h^p)$ contains a dense linear subspace, i.e., equals $\Lambda(G_h^p)$. In fact, $\Lambda(G_h^p)=L^0(0,1)$ (cf. 2.12).

3.8. LEMMA. *Let $i\colon\Gamma\to G$ be a continuous dense injective homomorphism of topological groups. Then*

(a) *if Γ is locally generated, then G is locally generated;*

(b) *if G is locally generated, then $i(\Gamma)$ (with the induced topology) is locally generated.*

Proof. (a) Let U be a neighbourhood of e in G and let $V=i^{-1}(U)$. Since

$$\bigcup_{n\in N}V^n=\Gamma,\qquad\text{where } V^n=\underbrace{V+\ldots+V}_{n\text{ times}}$$

we have

$$i(\Gamma)=\bigcup_{n\in N}i(V^n)=\bigcup_{n\in N}(i(V))^n\subset\bigcup_{n\in N}U^n.$$

But

$$G'=\bigcup_{n\in N}U^n$$

is an open and hence closed subgroup of G, so $G'\supset\operatorname{cl}(i(\Gamma))=G$.

(b) Let U be a neighbourhood of e in G and let $V=U\cap i(\Gamma)$. Clearly, $U\subset\operatorname{cl}(V)$. The group

$$H=\bigcup_{n\in N}V^n$$

is a closed subgroup of $i(\Gamma)$, i.e., $\operatorname{cl}(H)\cap i(\Gamma)=H$, where cl denotes the closure in G. On the other hand,

$$\operatorname{cl}(H)=\operatorname{cl}\Big(\bigcup_{n\in N}V^n\Big)\supset\bigcup_{n\in N}\operatorname{cl}(V^n)\supset\bigcup_{n\in N}(\operatorname{cl}(V))^n\supset\bigcup_{n\in N}U^n=G.$$

3.9. THEOREM. *Let G be a connected AWL-group. The following are equivalent:*

(a) *G has a realization X/Γ with Γ locally generated.*

(b) *$K(G)$ is locally generated.*

(c) *For each $X/\Gamma \in R(G)$, Γ is locally generated.*

Proof. (a) $\Rightarrow$ (b). From the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & K(G) \arrow[r] & \Lambda(G) \arrow[dr] & \\ & & & G \\ 0 \arrow[r] & \Gamma \arrow[r] \arrow[uu, "\hat{p}|\Gamma"] & X \arrow[ur, "p"'] \arrow[uu, "\hat{p}"] & \end{tikzcd}$$

where $\hat{p}|\Gamma \colon \Gamma \to K(G)$ is a continuous dense injective homomorphism, we conclude by Lemma 3.8 (a) that $K(G)$ is locally generated.

(b) $\Rightarrow$ (c). Let X/Γ be a realization of G. Choose a neighbourhood W of 0 in X and take a neighbourhood U of 0 in X such that

$$U^3 = U+U+U \subset W.$$

Take $\phi \in \hat{p}(U)$, $\phi = \hat{p}(\gamma)$, $\gamma \in \Gamma$. Hence $\phi(t) \in p(U)$ for all $t \in \boldsymbol{R}$, i.e., $p(t\gamma) \subset p(U)$ for all $t \in \boldsymbol{R}$. For each $t \in \boldsymbol{R}$ we can find $\gamma_t \in \Gamma$ such that $t\gamma - \gamma_t \in U$. We can also choose $\gamma_1 = \gamma$. Let n be a natural number such that $t\gamma \in U$ for $-1 \leqslant nt \leqslant 1$. Since

$$\gamma = (\gamma_1 - \gamma_{n-1/n}) + \ldots + (\gamma_{1/n} - 0),$$

where

$$\gamma_{k+1/n} - \gamma_{k/n} \in \Gamma,$$

and

$$(\gamma_{k+1/n} - \gamma_{k/n}) = \left(\gamma_{k+1/n} - \frac{k+1}{n}\gamma\right) + \left(\frac{k+1}{n}\gamma - \frac{k}{n}\gamma\right) + \left(\frac{k}{n}\gamma - \gamma_{k/n}\right) \subset U+U+U \subset W,$$

we have

$$\gamma \in (U^3 \cap \Gamma)^n \subset (W \cap \Gamma)^n.$$

Therefore $\gamma \in \mathrm{Gr}(W \cap \Gamma)$, and thus $\phi \in \hat{p}(\mathrm{Gr}(W \cap \Gamma))$. This shows that

$$\hat{p}(U) \cap \hat{p}(\Gamma) \subset \hat{p}(\mathrm{Gr}(W \cap \Gamma)) = \mathrm{Gr}(\hat{p}(W \cap \Gamma)),$$

and hence

$$\mathrm{Gr}(\hat{p}(U) \cap \hat{p}(\Gamma)) \subset \mathrm{Gr}(\hat{p}(W \cap \Gamma)).$$

By Lemma 3.8 (b), $\mathrm{Gr}(\hat{p}(U) \cap \hat{p}(\Gamma)) = \hat{p}(\Gamma)$. Hence

$$\hat{p}(\Gamma) \subset \mathrm{Gr}(\hat{p}(W \cap \Gamma)) \subset \hat{p}(\Gamma)$$

and

$$\mathrm{Gr}(\hat{p}(W \cap \Gamma)) = \hat{p}(\mathrm{Gr}(W \cap \Gamma)) = \hat{p}(\Gamma),$$

so $\mathrm{Gr}(W \cap \Gamma) = \Gamma$.

4. Concluding remarks. Theorems 3.1 and 3.9 provide examples of situation when topological properties of a quotient group $G = X/\Gamma$ are described in terms of the corresponding closed subgroup Γ of a linear topological space X. We do not know explicitly such a description of G in the case of Γ locally generated. It would be interesting to have a more complete list of types of Γ from the point of view of resulting quotient groups. Observe that by the result of [1] such a classification would be simpler for nuclear AWL-groups than for Banach ones.

Theorem 3.3 implies that, for an AWL-group G with Fréchet locally bounded Lie algebra $\Lambda(G)$, the exponential map is a local isomorphism. Hence two Fréchet locally bounded AWL-groups with the same Lie algebra $\Lambda(G)$ are locally isomorphic. A similar conclusion does not hold in general – the groups discussed in Example 2.12 are not locally isomorphic with their Lie algebras. We do not know how two AWL-groups having the same Lie algebra are related in general.

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INSTITUTE OF MATHEMATICS
UNIVERSITY OF WARSAW
PKiN IXp., 00-901 WARSAW, POLAND

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