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SUBSTOCHASTIC TRANSITION OPERATORS ON TREES AND THEIR ASSOCIATED POISSON INTEGRALS

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0. Introduction. Let P be a transitive nearest neighbor transition operator on a tree T. We do not assume that T is homogeneous, i.e., that the same number of edges touch each vertex. For most of this paper* our only assumption on P is that it is substochastic (which is to say: submarkovian), i.e., the transition probabilities are positive numbers whose sum at each vertex is less than or equal to one. P acts on functions defined on T by the rule

$$PF(u) = \sum_{v} p(u, v) F(v).$$

Functions F such that $PF = F$ are called *P-harmonic*. Harmonic functions on trees and their boundary integral representation have been studied in [Ca]: they can be expressed as "Poisson integrals", over the natural boundary of T, of martingales, i.e., finitely additive measures. In the special case of stochastic operators more explicit representation formulas have been obtained in [KPT] (in particular in the Appendix). The martingale associated with a given harmonic function is explicitly constructed by means of an "inverse Poisson transform". Explicit formulas of this type are the main tools for the study of H^p spaces on trees [KPT]. In order to extend the H^p theory to the more general setting of substochastic operators it is necessary to derive an explicit representation for their harmonic functions. The aim of this paper is to obtain the Poisson representation by reducing the study of substochastic operators to the stochastic case.

There is a very natural and well-known instance of a reduction of this type: For each vertex v let $a(v)$ be the sum of the probabilities of jumping out of v. Since P is substochastic, $a(v) \leqslant 1$. Then we can modify the operator P by regarding v as an absorbing vertex with probability $1-a(v)$. In other words, we extend the tree T to a larger tree T' by associating (in a one-to-one way)

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a new transition operator P' by $P|_T = P$, and $p'(v, v') = 1 - a(v)$, $p'(v', v) = 0$. Then P' is stochastic and its harmonic functions are exactly those that are P-harmonic on T and that vanish on $T' \backslash T$. However, no boundary representation theory is available for this stochastic operator. Indeed, a basic assumption of both [Ca] and the Appendix of [KPT] is that the transition probabilities along each edge are positive. There is also a way of enlarging a tree T to another tree T' without introducing absorbing vertices: We attach to each vertex v of T an infinite "hair", that is, a (countable) half-line of vertices. P can be extended to a stochastic operator P' on T' by assigning the probability $1 - a(v)$ of jumping from v to the first vertex of the attached hair, and arbitrary probabilities of sum one at the vertices along the hair. The P-harmonic functions on T are the restrictions of the P'-harmonic functions on T' that vanish at the first vertex of each hair. This yields a Poisson representation of the P-harmonic functions on T, but the Poisson integrals are now taken over the boundary of T' rather than of T. The principal problem is that these types of reductions, when considered on the original tree, leave things essentially unchanged. In particular, they fail to provide for any privileged positive harmonic functions. If a stochastic extension P' has positive probabilities along all edges, then the constants are P'-harmonic.

The stochastic reduction studied in the present paper associates with each substochastic operator P a renormalized operator $\tilde{P}$ in a completely different way. A reference vertex o is chosen and fixed. The transition probabilities of the operator $\tilde{P}$ are obtained from those of P by multiplication by suitable weights which are determined by recurrence as we move out from the reference vertex o. The weights associated with a given edge in the outward and inward directions are reciprocals so that even when the tree is homogeneous and the substochastic operator is isotropic its associated stochastic process is not. It will be isotropic only in the forward direction (see Section 5).

We construct a positive P-harmonic function whose value at a given vertex is the product of the weights of the edges of the geodesic arc that starts at o and ends as the vertex. Multiplication by this function transports, bijectively, the class of P-harmonic functions onto the class of $\tilde{P}$-harmonic functions, and the Poisson kernel K of P to the Poisson kernel $\tilde{K}$ of $\tilde{P}$. (See Proposition 1 and Theorem 2 for precise statements.) Finally, every P-harmonic function is expressed as the K-Poisson transform (i.e., the Poisson integral over the boundary Ω with integral kernel K) of the distribution on Ω whose $\tilde{K}$-Poisson transform is the associated $\tilde{P}$-harmonic function; the distributions are in one-to-one correspondence with the class of martingales on the boundary which in turn are identified with the class of finitely additive measures on Ω.

These results will hold provided that the renormalized stochastic operator is transient. This is a natural condition since a boundary value theory of this type for $\tilde{P}$ would not otherwise be available and our reduction to the stochastic case would be fruitless. Clearly, the renormalization depends on the choice of the reference vertex. In Section 4 we will show that under typical uniform bounds on the rate of escape to the boundary the boundary measures associated with different reference vertices are equivalent.

In some cases the renormalization can be carried through for transition operators that are not substochastic. As an application, in Section 5 we consider the boundary representation theorem for eigenfunctions of the nearest neighbor isotropic operator on a homogeneous tree, originally obtained in [MZ]. We give a simple proof for all eigenvalues outside the so-called "principal series". The nonisotropic boundary representation studied in [FS] could also be derived by our approach.

Several questions that we consider on trees are the discrete analogue of problems originally formulated for the disc. It would be interesting to consider a continuous analogue (for differential operators) of the renormalization procedure studied in this paper. To our knowledge no such question has been considered in the literature. If we limit our attention to the substochastic operators on homogeneous trees that are isotropic, then the corresponding operator on the disc is the Laplace operator (or the Laplace–Beltrami operator, see [FP]). Thus we are led to consider the solutions of the problem $\Delta F = cF$ in the unit disc, where c is a positive sufficiently smooth function. Our approach suggests that there could be a bijective correspondence between solutions of this problem and a class of "harmonic" functions on the disc with respect to some elliptic operator. This correspondence would then give rise to a boundary representation of the solutions as "Poisson integrals" (with a suitable Poisson kernel) of distributions on the circle. If c is positive and we restrict attention to positive solutions, then the boundary data will yield positive Borel measures. For a related (probabilistic) approach to the Dirichlet problem for the operator $\Delta F - cF$ with $c > 0$ and continuous boundary values, see Section 2.2 of [Fr].

Unexplained notation, terminology, and preliminaries are as in [KPT] or [FP].

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1. Renormalization to the stochastic setting. P is a substochastic, transitive, nearest neighbor transition operator on an infinite tree T. We assume further that each vertex in T has at least two neighbors. We write $u \sim v$ if the vertices u and v are neighbors. We denote by $a(v)$ the "*mass*" of P at the vertex v:

$$a(v) \equiv \sum_{u \sim v} p(v, u).$$

Finally, we fix a reference vertex o and for every vertex $v \neq o$ we denote by $v-$ its predecessor in the ordering induced by the reference vertex.

PROPOSITION 1. (i) *Let* $b(o) = a(o)$, *and for* $v \neq o$ *define*

$$b(v) = \frac{a(v) - p(v, v-)}{1 - p(v, v-)\, b(v-)}.$$

Then the transition operator $\tilde{P}$ *defined by*

$$\tilde{p}(v-, v) = p(v-, v)/b(v-), \qquad \tilde{p}(v, v-) = b(v-)\, p(v, v-)$$

is stochastic.

(ii) *Let* $B(0) = 1$, $B(v) = b(v-)\, B(v-)$ *for* $v \neq o$. *Then for every function* F *such that* $PF = F$ *the function* $\tilde{F}(v) = B(v)\, F(v)$ *satisfies* $\tilde{P}\tilde{F} = \tilde{F}$.

(iii) *The operator* $F \to \tilde{F} = BF$ *is a bijection of the space of P-harmonic functions onto the space of* $\tilde{P}$*-harmonic functions.*

Proof. Observe first that for a substochastic transition operator P the normalization coefficient $b(v)$ is bounded by the "mass" of P at v and is bounded below by its "forward mass":

$$a(v) - p(v, v-) \leqslant b(v) \leqslant a(v) \qquad \text{for every } v \neq o.$$

It follows that $b(v-) \leqslant 1$, and so the denominator in the expression that defines $b(v)$ is always positive. Otherwise $p(v, v-) = 1$, which implies that $p(u, v) = 0$ if $u- = v$. But this implies that P is not transitive, a contradiction.

(i) We need to show that

$$\sum_v \tilde{p}(o, v) = 1,$$

and for $v \neq o$,

$$\tilde{p}(v, v-) + \sum_{u: u- = v} \tilde{p}(v, u) = 1.$$

The first follows from the definition: $\tilde{p}(o, v) = a(o)\, p(o, v)$. The second identity follows easily from the definition of $b(v)$ by observing that

$$\sum_{u: u- = v} p(v, u) = a(v) - p(v, v-).$$

(ii) We have

$$\sum_{u: u- = v} \tilde{p}(v, u)\, \tilde{F}(u) + \tilde{p}(v, v-)\, \tilde{F}(v-)$$

$$= \sum_{u: u- = v} p(u, v)\, F(u)\, B(v) + p(v, v-)\, F(v-)\, B(v) = F(v)\, B(v) = \tilde{F}(v).$$

(iii) Since B is a positive function, we can divide by it. The calculation in (ii) shows that F is P-harmonic if and only if FB is $\tilde{P}$-harmonic.

Remark 1. The function $1/B(v)$ is a positive P-harmonic function and $(1/B)^{\sim} \equiv 1$. The stochastic process with "law" $\tilde{P}$ is Doob's "associated process" of $(P, 1/B)$ in the sense that

$$\tilde{p}(u, v) = p(u, v)\, B(u)/B(v) \qquad \text{for every } u, v$$

(terminology as in [Do]; in the terminology of [KSK], Section 8.2, this is called a "B-process").

Following [Ca], but adopting the notation of [KPT], we consider the combinatorial version of the "hitting probability kernel" (which is now an abuse of terminology since P is substochastic!) which is defined as follows:

Definition 1 (*hitting probability*). $U(u, v) = \sum \{p(u, v_1)p(v_1, v_2)\dots p(v_{n-1}, v)\}$, where the sum is taken over all finite sequences of successively adjacent vertices $v_1, v_2, \dots, v_{n-1}$ with $u \sim v_1$, $v_{n-1} \sim u$ and $v_j \neq v$ for all $j = 1, \dots, n-1$. Then, as in [Ca],

$$\begin{aligned} U(v, v) &= \sum p(v, w)\, U(w, v), \\ U(v, v-) &= p(v, v-) + \sum_{u:u-=v} p(v, u)\, U(u, v)\, U(v, v-), \end{aligned} \tag{1}$$

which is the combinatoric version of the Strong Markov Property.

Denote by $\tilde{U}$ the kernel defined in the same way, but in terms of the stochastic transition operator $\tilde{P}$. We now establish the relationship between U and $\tilde{U}$. The next statement can be rephrased (and easily proved) in terms of the "associated B-process" (see Remark 1 and [KSK], Section 8.2). In the spirit of this paper we give a combinatoric proof.

Proposition 2. *$U(u, v) = \tilde{U}(u, v)\, B(v)/B(u)$ for every pair u, v in T.*

Proof. As the kernels U and $\tilde{U}$ are multiplicative along rays ([Ca], Proposition 2.6), it is enough to establish the case $u \sim v$. That is, it suffices to show that

$$U(v-, v) = b(v-)\, \tilde{U}(v-, v) \quad \text{and} \quad U(v, v-) = \tilde{U}(v, v-)/b(v-).$$

Every path from $v-$ to v that hits v only at the end splits into two segments: the first segment, possibly empty, is a closed loop which begins and ends at $v-$, and may hit $v-$ more than once, but it never hits v. The second segment is the final jump from $v-$ to v. But in a tree closed loops can only be obtained by running through geodesic arcs an even number of times, the same number of times with one orientation as the other. Because of the relation between P and $\tilde{P}$ (Proposition 1 (i)) successive cancellations yield

$$\prod_i p(v_i, v_{i+1}) = \prod_i \tilde{p}(v_i, v_{i+1})$$

on each closed loop. On the other hand, the forward jump has "probability" $p(v-, v) = b(v-)\tilde{p}(v-, v)$, and therefore

$$U(v-, v) = b(v-)\tilde{U}(v-, v).$$

The same argument shows that

$$U(v, v-) = \tilde{U}(v, v-)/b(v-).$$

We remind the reader that we assume that the stochastic transition operator $\tilde{P}$ is transient. This is equivalent to the requirement that $\tilde{U}(u, u) < 1$ for all vertices u. It follows from Proposition 2 that $U(u, u) = \tilde{U}(u, u)$ for all vertices u, and so: *$\tilde{P}$ is transient if and only if $U(u, u) < 1$ for all vertices u.*

2. The martingale associated with a nonmarkovian transition operator. It was pointed out in [KPT], Section 2, that with each stochastic transition operator $\tilde{P}$ we can associate, in a natural way, another stochastic transition operator with "forward only" transition probabilities $\tilde{\pi}(v-, v)$. The weights $\tilde{\pi}(v-, v)$ are the relative volumes of a "martingale structure" (i.e., a family of weighted partitions) on the boundary Ω of the tree, and $\tilde{P}$-harmonic functions on T can be represented as Poisson integrals over Ω of distributions defined on this structure. It turns out ([KPT], Proposition 2) that

$$\tilde{\pi}(v-, v) = \frac{\tilde{p}(v-, v)[1-\tilde{U}(v, v-)]}{\sum_{w:w-=v-} \tilde{p}(v-, w)[1-\tilde{U}(w, v-)]}. \tag{2}$$

The normalization in (2) is chosen so that

$$\sum_{w:w-=v-} \tilde{\pi}(v-, w) = 1.$$

In the present setting we have two transition operators. One of these, $\tilde{P}$, is stochastic and is associated with a family of weights $\tilde{\pi}(v-, v)$ as in (2). The other operator, P, is substochastic. We will see that there is a natural family of weights: $\pi(v-, v)$ associated with P and at each vertex v the "relative forward probabilities" will add up to the substochastic normalization factor $b(v)$. From Proposition 1 and (2) we have

$$\tilde{\pi}(v-, v) = \frac{p(v-, v)[1-\tilde{U}(v, v-)]}{\sum_{w:w-=v-} p(v-, w)[1-\tilde{U}(w, v-)]}. \tag{3}$$

We define

$$\pi(v-, v) = b(v-)\tilde{\pi}(v-, v). \tag{4}$$

We are now ready to introduce martingales associated with the weights $\tilde{\pi}$ and "substochastic martingales" associated with the π's. Following [KPT], consider the boundary Ω of the tree T, i.e., the set of all semi-infinite rays (geodesics) starting at the reference vertex o. For each v in T denote by E_v the subset of Ω that contains v. The collection $\{E_v: v \in T\}$ is the basis of a compact topology on Ω. Finally, equip Ω with the Borel measure $\tilde{v}$ defined as the hitting measure on Ω of the transient process determined by the stochastic operator $\tilde{P}$. Then, as was shown in [KPT], the weights $\tilde{\pi}$ are given by

$$\tilde{\pi}(v-, v) = \tilde{v}(E_{v-})/\tilde{v}(E_v).$$

On the other hand, the weights π do not give rise to a measure on Ω. Instead they determine a subadditive functional v over $\mathscr{C}(\Omega)$ determined by the rule $v(\chi_{E_v}) = B(v)\,\tilde{v}(E_v)$.

The σ-algebra $\mathscr{A}_n$ generated by the sets $\{E_v: \operatorname{dist}(v, o) = n\}$ gives rise to an expectation $\mathscr{E}_n$ on $\mathscr{A}_n$-measurable functions on Ω, defined in terms of the measure $\tilde{v}$. We will consider martingales $\tilde{f} = \{\tilde{f}_k: k = 0, 1, 2, \ldots\}$ on Ω, with respect to the family of expectations $\{\mathscr{E}_k\}$. By definition, $\tilde{f}_k$ is constant on those sets E_v for which $\operatorname{dist}(v, o) \geqslant k$. As in [KPT] we shall realize a martingale $\tilde{f}$ on Ω as a function on the vertices of T which (by a convenient abuse of notation) we denote again by $\tilde{f}$. (Warning: In the terminology of [KPT] this function is denoted by $f^{\dagger}$. The notation $\tilde{f}$ in that reference has a completely different meaning.) This function is defined by

$$\tilde{f}(v) = \tilde{f}_n|_{E_v} = \frac{1}{\tilde{v}(E_v)} \int_{E_v} \tilde{f}_k \, d\tilde{v}$$

for any $k \geqslant n = \operatorname{dist}(v, o)$. In this setting the martingale property is given by the additivity condition

$$(5) \qquad \sum_{w: w- = v} \tilde{\pi}(v, w)\tilde{f}(w) = \tilde{f}(v) \quad \text{for every } v.$$

These relationships can also be interpreted in terms of distributions on the boundary. In this setting a distribution is a finitely additive linear functional on the linear space generated by characteristic functions of the sets E_v. It is easy to check (using $\tilde{f}$ for yet a third construct) that $\tilde{f}(\chi_{E_v}) \equiv \tilde{v}(E_v)\tilde{f}(v)$ is a distribution.

In the spirit of Proposition 1 we now define a "substochastic martingale" associated with the weights $\pi(v-, v)$ (as defined by (4)) as a function f on T such that $f(v) = \tilde{f}(v)/B(v)$ for $\tilde{f}$ a martingale with respect to the weights $\{\tilde{\pi}\}$. It is now obvious that $\tilde{f}$ has the martingale property with respect to the weights $\{\tilde{\pi}\}$ if and only if f satisfies the analogous property with respect to the weights $\{\pi\}$; i.e.,

$$\sum_{w: w- = v} \pi(v, w) f(w) = f(v) \quad \text{for every } v.$$

Indeed,

$$\sum_{w:w-=v} \tilde{\pi}(v, w)\tilde{f}(w)-\tilde{f}(v) = \sum_{w:w-=v} \pi(v, w)f(w)-f(v) \quad \text{for every } v$$

as follows from (4) and (5).

We finish this section by deriving a more convenient expression for (2) which will be useful in Remark 4 below.

NOTATION. We write $v \to w$ if $w- = v$.

LEMMA 1. *If* $u \to v \to w$, *then*

$$\tilde{\pi}(v, w) = \frac{\tilde{p}(v, w)[1-\tilde{U}(w, v)]}{\tilde{p}(v, u)[1-\tilde{U}(v, u)]}\,\frac{\tilde{U}(v, u)}{b(v)b(u)}.$$

Proof. In view of (2), the statement is equivalent to the identity

$$\sum_{s:s-=v} p(v, s)[1-\tilde{U}(s, v)] = b(u)b(v)[1-\tilde{U}(v, u)]/\tilde{U}(v, u).$$

We now make use of the equalities

$$a(v)-p(v, u) = \sum_{s:s-=v} p(v, s)$$

and

$$\tilde{U}(v, u) = \tilde{p}(v, u)+\sum_{s:s-=v} \tilde{p}(v, s)\tilde{U}(s, v)\tilde{U}(v, u)$$

$$= p(v, u)b(u)+\sum_{s:s-=v} \frac{p(v, s)}{b(v)}\tilde{U}(s, v)\tilde{U}(v, u).$$

Then the identity above reduces to

$$a(v)-p(v, u)-b(v) = -p(v, u)b(v)b(u).$$

This follows, in turn, from the definition of the function b in Proposition 1.

3. P-harmonic functions and their Poisson representations. We start by recalling the probabilistic definition of the Poisson kernel.

DEFINITION 2 (*Poisson kernel*). For v in T and ω in Ω,

$$K(v, \omega) = \lim_{s\to\omega} \frac{U(v, s)}{U(o, s)},$$

where the limit is taken in the natural topology of $T \cup \Omega$.

The limit always exists for nearest neighbor operators on trees because the quotient on the right-hand side stays constant as s moves in the ray ω past the confluence point with the ray "coming from v". It was noted in [Ca] (Theorem 2.1 and Proposition A4) that each P-harmonic function F can be expressed as the Poisson integral of its associated martingale:

For every u in T,

$$F(u) = \int_{\Omega} K(u, \omega)\, d\mu_F(\omega), \tag{6}$$

where μ_F is a martingale on Ω. For the stochastic operator $\tilde{P}$, this martingale is the associated martingale of Section 2. By realizing martingales over Ω as functions on T, as in Section 2, we obtain explicit expressions for $\tilde{f}$ in terms of $\tilde{F}$ and F and conversely:

LEMMA 2 ([KPT], Appendix). *Suppose $\tilde{P}$ is a stochastic transition operator, $\tilde{F}$ is $\tilde{P}$-harmonic, and $\tilde{f}$ is the associated martingale.*

(i) *Inverse Poisson transform*:

$$\tilde{f}(o) = \tilde{F}(o), \quad \tilde{f}(v) = \frac{\tilde{F}(v) - \tilde{U}(v, v-)\tilde{F}(v-)}{1 - \tilde{U}(v, v-)} \qquad \text{for } v \neq o.$$

(ii) *Poisson representation*: *Let* $m = \operatorname{dist}(v, o)$ *and let v_k denote the k-th vertex in the ray from o to v. Then*

$$\tilde{F}(o) = \tilde{f}(o), \quad \tilde{F}(v) = \sum_{k=1}^{m} [1 - \tilde{U}(v_k, v_{k-1})]\, \tilde{U}(v, v_k)\tilde{f}(v_k) + \tilde{U}(v, o)\tilde{f}(o).$$

We want to establish analogous relations in the substochastic setting.

THEOREM 1. *Suppose P is a substochastic operator on a tree, F is P-harmonic, and f is its associated substochastic martingale. Then, with notation as above, we have*:

(i) *Inverse Poisson transform*:

$$f(o) = F(o) \quad f(v) = \frac{F(v) - U(v, v-)F(v-)}{1 - \tilde{U}(v, v-)} \qquad \text{for all } v \neq o.$$

(ii) *Poisson representation*:

$$F(v) = \sum_{k=1}^{m} [1 - \tilde{U}(v_k, v_{k-1})]\, U(v, v_k) f(v_k) + U(v, o) f(o).$$

Proof. To prove (i) note first that $f(o) = \tilde{f}(o) = \tilde{F}(o) = F(o)$. For $v \neq o$, Lemma 2 yields

$$f(v) = \tilde{f}(v)/B(v) = \frac{\tilde{F}(v) - \tilde{U}(v, v-)\tilde{F}(v-)}{B(v)[1 - \tilde{U}(v, v-)]}$$

$$= \frac{F(v) - \tilde{U}(v, v-)F(v-)/b(v-)}{1 - \tilde{U}(v, v-)} = \frac{F(v) - U(v, v-)F(v-)}{1 - \tilde{U}(v, v-)}$$

by Proposition 1 (ii) and Proposition 2. Part (ii) now follows by inverting this formula.

Denote by $K(v, \omega)$ the Poisson kernel of the nonstochastic process P, and by $\tilde{K}(v, \omega)$ the Poisson kernel for the associated stochastic process $\tilde{P}$. We now write the Poisson representation formula as an integral representation over the boundary.

DEFINITION 3 (Poisson integral). The *stochastic Poisson integral* of the (stochastic) martingale (distribution) $\tilde{f}$ on Ω is the function on T given by

$$\mathscr{K}\tilde{f}(v) \equiv \int_{\Omega} K(v, \omega)\, d\tilde{f}(\omega) = \lim_{k \to \infty} \int_{\Omega} K(v, \omega)\tilde{f}_k(\omega)\, d\tilde{v}(\omega) = \tilde{f}[K(v, \cdot)],$$

where in the last expression $\tilde{f}$ is being interpreted as a distribution and, as usual, the integral in the third term is independent of k as soon as $k \geqslant \operatorname{dist}(v, o)$. Similarly, let

$$\tilde{\mathscr{K}}\tilde{f}(v) = \int_{\Omega} \tilde{K}(v, \omega)\, d\tilde{f}(\omega) = \tilde{f}[\tilde{K}(v, \cdot)].$$

THEOREM 2. (i) *For every u in T and ω in Ω, $K(v, \omega) = \tilde{K}(v, \omega)/B(v)$.*

(ii) *Let F be P-harmonic, and denote by $\tilde{F}$ its associated $\tilde{P}$-harmonic function and by $\tilde{f}$ its (stochastic) martingale. Then $F = \mathscr{K}\tilde{f}$. In particular, every positive P-harmonic function is the Poisson integral of a positive Borel measure on Ω.*

(iii) *Denote by B the "stochastic renormalization" of Sections 1 and 2 (that is, multiplication by the function B), and by B^{-1} the operation of multiplication by the function $1/B$. Then the following diagram commutes:*

$$\begin{tikzcd}
F \arrow[r, shift left, "B"] & \tilde{F} \arrow[l, shift left, "B^{-1}"] \\
f \arrow[u, "\mathscr{K}"] \arrow[r, shift left, "B"] & \tilde{f} \arrow[l, shift left, "B^{-1}"] \arrow[u, "\tilde{\mathscr{K}}"'] \arrow[ul, "\mathscr{K}"']
\end{tikzcd}$$

Proof. Once part (i) is proved, the remainder of this theorem is a restatement of the previous results. (For part (ii) observe that if F is positive, then so is $\tilde{F}$ and, consequently, $\tilde{f}$ is a positive and finitely additive function and it follows that $\tilde{f}$ is σ-additive.) To prove (i) denote by c the vertex in the geodesic ω which is closest to v, and let $v = v_0, v_1, v_2, \ldots, v_n = c$ be the finite geodesic arc from v to c. Then $K(v, \cdot)$ and $\tilde{K}(v, \cdot)$ are both constant on the sets $E_{v_j} \subset \Omega$, $j = 0, 1, \ldots, n$. On E_{v_j} we have

$$K(v, \omega) = \frac{U(v, v_j)}{U(o, v_j)} = \frac{\tilde{U}(v, v_j)\, B(v_j)/B(v)}{\tilde{U}(o, v_j)\, B(v_j)/B(o)} = \tilde{K}(v, \omega)/B(v)$$

by Proposition 2.

We observe that an explicit form for the Poisson formula is available in two ways. One of these is by Definition 2, and the other is by an application of Lemma 2. It is easy to see that the function

$$\tilde{k}(v, \omega) = \left(1/\tilde{\nu}(\chi_{E_v})\right)^{-1} \chi_{E_v}(\omega)$$

is a martingale kernel associated with the process $\tilde{P}$ in the obvious sense, that is,

$$\tilde{f}(v) = \int_{\Omega} \tilde{k}(v, \omega)\, d\tilde{f}(\omega) = \tilde{f}[\tilde{k}(v, \cdot)].$$

Observe that $\tilde{k}(\cdot, \omega)$ is the martingale associated with $\tilde{K}(\cdot, \omega)$ so that part (ii) of Lemma 2 provides a formula for the computation of $\tilde{K}$.

Remark 2. The reader should observe that, in the diagram of the boundary representations in Theorem 2 (iii), the arrow: $f \to F$ must be carefully interpreted. Indeed, the seemingly "natural" Poisson representation

$$F = \int_{\Omega} K\, df = \lim_m \int_{\Omega} K f_m\, d\nu$$

does not make sense for nonstochastic martingales, because a martingale of this type does not define an integral over Ω (or, equivalently, a subadditive functional on $\mathscr{C}(\Omega)$ does not give rise to a measure on Ω). Indeed, the purpose of the theorem is to reinterpret this "Poisson representation" in terms of meaningful Poisson integrals. Observe, however, that by lifting integrals over Ω to sums over T, we can give a direct meaning to the expression $\int_{\Omega} K\, df$: the Poisson representation formula of Theorem 1 (ii) should be understood in this sense.

Remark 3. In the proof of Theorem 1 we made use of nontrivial results about the inverse Poisson transform proved in [KPT]. For the reader's benefit we will now give a direct proof of the fact that the formula of Theorem 1 (i) does, indeed, give rise to a substochastic martingale. Let

$$f(v) = \left(F(v) - U(v, v-)\,F(v-)\right)/\left(1 - \tilde{U}(v, v-)\right) \quad \text{for } v \neq o,$$

$$f(o) = F(o).$$

By (4), if $v- = o$, then

$$\pi(o, v) = b(o)\,\pi(o, v) = b(o)\,p(o, v)\,[1 - \tilde{U}(v, o)]/D_o,$$

where

$$D_o = \sum_{w \sim o} p(o, w)\,[1 - \tilde{U}(w, o)].$$

Therefore,

$$\begin{aligned}\sum_{v\sim o} \pi(o,v)f(v) &= \frac{b_o}{D_o}\Big(\sum_{v\sim o} p(o,v)F(v) - \sum_{v\sim o} p(o,v)[1-U(v,o)]F(o)\Big)\\ &= \frac{b_o}{D_o}\Big(1-\sum_{v\sim o} p(o,v)U(v,o)\Big)F(o)\\ &= \left(\frac{1}{D_o}\sum_{v\sim o} p(o,v)[1-\tilde{U}(v,o)]\right)f(o) = f(o),\end{aligned}$$

because

$$b(o) = a(o) = \sum_{v\sim o} p(o,v).$$

For $v \neq o$, we obtain the identity

$$\begin{aligned}&\sum_{w:w-=v} \pi(u,w)f(w) - f(v)\\ &\quad= \frac{\tilde{U}(v,v-)}{b(v-)p(v,v-)[1-\tilde{U}(v,v-)]}\Big(\sum_{s\sim w} p(v,s)F(s) - F(v)\Big) = 0\end{aligned}$$

by making use of Lemma 1.

Remark 4. It is interesting to compare our Poisson representation with that of [Ca]. It was shown in [KPT], Proposition 2, that the martingale μ_F in (5), regarded as a finitely additive measure on Ω, is given by

$$\mu_F(E_v) = U(o,v)\frac{F(v) - U(v,v-)F(v-)}{1-U(v,v-)U(v-,v)}.$$

Actually, this formula was proved in [KPT] for stochastic operators only. However, the proof relies on the combinatorial Strong Markov Property and holds for all transient operators. By making use of Proposition 1 (ii) and Proposition 2, it now follows that

$$\mu_F(E_v) = \tilde{U}(o,v)\frac{\tilde{F}(v) - \tilde{U}(v,v-)\tilde{F}(v-)}{1-\tilde{U}(v,v-)\tilde{U}(v-,v)} = \tilde{\mu}_{\tilde{F}}(E_v).$$

Therefore the nonstochastic Poisson representation of [Ca] is actually given by the Poisson integral of the same (stochastic) martingale which arises in the renormalization procedure studied in this paper.

4. Change of reference vertex and regularity. The renormalized stochastic operator $\tilde{P}$ depends not only on P but also on the choice of reference vertex. Therefore, for different choices of origin we have different boundary representations for the same P-harmonic function F; of course, the Poisson kernels also vary so that the Poisson integrals coincide.

Consider a new choice of origin o' and the associated renormalizing function B'. On an elementary open set E_v of Ω, the ratio of the martingales associated with the origins o and o' is $B(v)/B'(v)$. It is conceivable that this ratio is bounded, which would imply that the associated finitely additive measures on Ω are equivalent. We do not know if this is true for a general substochastic operator. On the other hand, the question concerns the asymptotic ratio of the rates of outward dissipation of mass for the process P, for two different choices of outward orientation. Therefore, it is natural to restrict attention to processes with uniform outward rate of escape. A natural condition of this type, considered in [KPT], is that in the ordering induced by some choice of origin, and for some $\delta > 0$, $p(v, v-) \leqslant \frac{1}{2}-\delta$ for every $v \neq o$. We now show that, under a relativized version of this condition: $p(v, u) \leqslant a(v)(\frac{1}{2}-\delta)$, we have $B(v)/B'(v) \sim 1$ for all v, with constants that depend only on $m = \operatorname{dist}(o, o')$.

Choose a ray $\omega = (o, v_1, v_2, \ldots, v_j, \ldots)$, and let

$$b_k = b(v_k), \qquad b'_k = b'(v_k), \qquad \text{and} \qquad p_k = p(v_k, v_{k-1}).$$

The b_k satisfy the recurrence relation $b_k = (a_k - p_k)/(1 - p_k b_{k-1})$, and if $k > m$, the b'_k satisfy the same recurrence. We observed in Section 1 that $a_k - p_k < b_k < a_k$. This, together with $p_k \leqslant a_k(\frac{1}{2}-\delta)$, yields $\frac{1}{2} \leqslant b_k/b'_k \leqslant 2$ for all k. Now take a $k \geqslant m$. Write $b'_k = (1+\varepsilon_k) b_k$. Then

$$\frac{b'_{k+1}}{b_{k+1}} = \frac{1-p_{k+1}b_k}{1-p_{k+1}b'_k} = \frac{1-p_{k+1}b_k}{1-p_{k+1}(1+\varepsilon_k)b_k} = 1 - \varepsilon_k \frac{p_{k+1}b_k}{1-p_{k+1}(1+\varepsilon_k)b_k}.$$

In other words, $b'_{k+1}/b_{k+1} = (1+\varepsilon_{k+1})$, and

$$0 < \varepsilon_{k+1} = \varepsilon_k \frac{p_{k+1}b_k}{1-p_{k+1}(1+\varepsilon_k)b_k} = \varepsilon_k \frac{p_{k+1}b_k}{1-p_{k+1}b'_k}.$$

We have $p_{k+1}b_k \leqslant (\frac{1}{2}-\delta)a_{k+1}b_k \leqslant \frac{1}{2}-\delta$. Similarly, $1-p_{k+1}b'_k \geqslant \frac{1}{2}+\delta$. Consequently, $|\varepsilon_{k+1}| \leqslant \lambda|\varepsilon_k|$, where $\lambda = (\frac{1}{2}-\delta)/(\frac{1}{2}+\delta) < 1$, and so

$$\sum_{k \geqslant m} |\varepsilon_k| \leqslant \lambda/(1-\lambda),$$

since $|\varepsilon_m| \leqslant 1$. Thus

$$B'(v)/B(v) \leqslant 2^m \prod_{k \geqslant m} (1+\varepsilon_k) \leqslant C \cdot 2^m$$

for a constant $C > 0$ independent of v. By symmetry, $B'(v)/B(v) \geqslant (1/C) \cdot 2^{-m}$. This completes the proof.

An examination of the proof shows that the restriction on the relative size of the transition probabilities can be weakened if we have particular information on the size of the $a(v)$ or other information about the process P.

5. Applications.

1. *Substochastic H^p theory on trees.* H^p spaces related to nearest neighbor stochastic transition operators on trees have been introduced and studied in [KPT], under suitable regularity assumptions, similar to those considered in Section 4. Namely, the stochastic operator $\tilde{P}$ should satisfy $\tilde{p}(v, v-) \leqslant \frac{1}{2}-\delta$, $\tilde{p}(v, u) \geqslant \delta$ for all pairs of neighbors u, v and for some positive δ independent of u and v. Given a substochastic operator P, we can use the stochastic normalization process to transfer to P the H^p theory of [KPT]. In this way, all the results of [KPT] extend to the substochastic operators P for which $\tilde{P}$ satisfies the above bounds. By making use of Proposition 1 it is straightforward to check when these conditions are satisfied. An instance is given in the next subsection.

2. *Boundary representation of eigenfunctions of the isotropic Laplace operator on a homogeneous tree.* So far we have limited attention to substochastic transition operators. However, the stochastic renormalization can work for other operators as well. The necessary conditions for this method are that the renormalization weights be nonzero and that the renormalized stochastic operator be transient. In this section we apply our method to obtain a boundary representation for eigenfunctions of the isotropic nearest neighbor transient operator on a homogeneous tree. Let T be a homogeneous tree with $q+1$ edges touching each vertex, and denote by R the nearest neighbor isotropic operator on T: $r(u, v) = 1/(q+1)$ if $u \sim v$. We shall obtain, for some eigenvalues, a variant of the following representation theorem proved in [MZ] (see also Chapters 4 and 6 of [FP]):

Let F be a function on T such that $RF = \gamma F$ for some complex number γ. Denote by ν the "harmonic" or "equidistributed" measure on Ω, that is, $\nu(\Omega) = 1$ and $\nu(E_v) = \nu(E_w)$ if $\operatorname{dist}(v, o) = \operatorname{dist}(w, o)$. *Choose a complex number z such that*

$$\gamma = \frac{1}{q+1}(q^z + q^{1-z}) \quad \text{and} \quad z \neq ik\pi/\log q,\ k \in \mathbf{Z}.$$

(If $\gamma^2 = 4q/(q+1)^2$, there is one such z, and two otherwise.) Then there exists a martingale $f = \{f_j\}$ on Ω such that, for every v in T,

$$F(v) = \int_\Omega K^z(v, \omega)\, df(\omega) \equiv \lim_j \int_\Omega K^z(v, \omega)\, d\nu(\omega).$$

A boundary representation theorem as complete as this cannot follow by our renormalization method. Indeed, our approach reduces attention to the study of P-harmonic functions; the problem $RF = \gamma F$ is transformed into $PF = F$ for $P = \gamma^{-1} R$. Clearly, this restatement of the problem is meaningless for the eigenvalue $\gamma = 0$. However, it will be necessary to discard other eigenvalues as well, in order to make sure that the renormalized weights do not vanish and that $\tilde{P}$ is transient, as we now show.

By applying the transformation of Proposition 1 (i), we generate a sequence of weights b_j. Because of isotropy the weight depends only on the distance of the vertex from o. The recurrence relation is

$$b_{j+1} = \frac{q}{(q+1)\gamma - b_j}, \; j > 0, \qquad b_0 = \frac{1}{\gamma}.$$

This recurrence relation fails to generate a sequence $\{b_j\}$ only if $b_j = (q+1)\gamma$ for some j. For the moment we assume that does not happen. If the sequence $\{b_j\}$ converges to a limit b, then b satisfies the condition $b^2 - (q+1)\gamma b + q = 0$. By setting $b = q^z$, it follows that

$$\gamma = \frac{q^z + q^{1-z}}{q+1}.$$

Moreover, the renormalizing function B is radial, that is, it depends only on the distance of the vertex from o. If $\operatorname{dist}(v, o) = n$, then

$$b(v) = \prod_{j=0}^{n-1} b_j.$$

Moreover, $1/B$ is P-harmonic, by the proof of Proposition 1, provided only that $b_j \neq 0$ for all j. Then $1/B$ is a radial γ-eigenfunction of R, and $1/b(0) = 1$. Therefore, by the results of Chapter 3 of [FP], $1/B$ coincides with the "spherical function" ϕ_z associated with the eigenvalue γ. The recurrence relation fails (i.e., $(q+1)\gamma = b_j$ for some j) exactly for those values of z for which ϕ_z has zeros. With an abuse of notation, let us denote by $\phi_z(n)$ the value of ϕ_z on vertices at a distance n from o. Then the formula $P_n(z) = \phi_z(n)$ defines a polynomial of degree n and gives rise to the sequence of orthogonal polynomials for the Plancherel measure of the convolution algebra of the radial functions (see [FP] and [Ca]). It is well known that the zeros of these polynomials are contained in the support of the Plancherel measure. In the parameterization used above, this is the line $\operatorname{Re} z = \frac{1}{2}$. In fact, ϕ_z does not necessarily have zeros for *every* z in this line, but for the sake of implicity, we shall exclude all the eigenvalues corresponding to this line (the so-called "principal series" in the terminology of [FP]). The explicit formulas of Chapter 3 of [FP] show that, outside the principal series, the sequence $\phi_z(n)$ is asymptotic to an exponential sequence, and therefore $b_n = \phi_z(n)/\phi_z(n+1)$ does have a limit for the range of eigenvalues that we are considering. The excluded set is the real line segment $(-2\sqrt{q}/(q+1), 2\sqrt{q}/(q+1))$. This range is much larger than the set of eigenvalues which make $P = \gamma^{-1}R$ substochastic. The latter set is simply

$$\{z\colon \gamma > 1\} = \{z\colon \operatorname{Im} z = 2k\pi/\log q \text{ for some } k \in \boldsymbol{Z}, |\operatorname{Re} z - \tfrac{1}{2}| > \tfrac{1}{2}\}.$$

However, to make use of the renormalization method we first must show that $\tilde{P}$ is transient. By Proposition 1, if $v \neq o$ and $\operatorname{dist}(v, o) = n$, then

$$\tilde{p}(v-, v) = \frac{1}{(q+1)\gamma b(v-)} = \frac{\phi_z(n+1)}{(q+1)\gamma\phi_z(n)},$$

whereas

$$\tilde{p}(v, v-) = \frac{\phi_z(n)}{(q+1)\gamma\phi_z(n+1)}.$$

Therefore, the explicit formulas for ϕ_z (see Chapter 3 of [FP]) show that for large n

$$\tilde{p}(v, v-) \simeq \begin{cases} q^{1-s}/(q^s+q^{1-s}) & \text{if } z = s+it \text{ with } s > \frac{1}{2}, \\ q^s/(q^s+q^{1-s}) & \text{if } z = s+it \text{ with } s < \frac{1}{2}. \end{cases}$$

It follows that if $\operatorname{Re} z \neq \frac{1}{2}$, there is a constant $0 < \delta < \frac{1}{2}$, which depends only on z, such that $\tilde{p}(v, v-) \leqslant \frac{1}{2} - \delta$ for all but, at most, a finite number of vertices. This implies that $\tilde{P}$ is transient (see the appendix to [KPT]).

Since the nearest neighbor transition probabilities are constant for the operator P and the b_j converge to a nonzero limit, the lower bound condition is met trivially.

Thus the renormalization applies for every eigenvalue outside the principal series and leads to the representation theorem for the corresponding eigenfunctions of R as boundary integrals of martingales. It is interesting to compare our boundary representation and that of [MZ]. Both these integral representations are of the type

$$F(v) = \int_{\Omega} K(v, \omega) f(\omega)\, d\nu(\omega),$$

where ν is a positive Borel measure on Ω, K is an integral kernel and f is a martingale (regarded as a sequence of locally constant functions on Ω). Let us denote by (ν_R, K_R, f_R) the measure, kernel, and martingale of [MZ], and by $(\nu_{\tilde{P}}, K_P, f_{\tilde{P}})$ the triple of this paper. Observe that we now write K_P instead of $K_{\tilde{P}}$ for consistency of notation; indeed, our integral kernel is the Poisson kernel for the nonstochastic operator $P = \gamma^{-1}R$. On the contrary, we adopt the notation $(\nu_{\tilde{P}}, f_{\tilde{P}})$ because our boundary representation involves the hitting distribution and the martingale on Ω associated with the renormalized operator $\tilde{P}$ (see Theorem 2 and Remark 2).

However, the measures ν_R and $\nu_{\tilde{P}}$ coincide. Indeed, ν_R is the equidistributed probability measure on Ω: $\nu_R(E_v) = \nu_R(E_w)$ if $\operatorname{dist}(v, o) = \operatorname{dist}(w, o)$. On the other hand, the operator $P = \gamma^{-1}R$ is isotropic, and therefore the weights π of Section 2 are isotropic in the forward directions. Therefore $\nu_{\tilde{P}}$ is also equidistributed. On the contrary, $K_R^z \neq K_{\tilde{P}}$. Indeed, the Poisson kernel K_R is given by (see [FP], Chapter 3) $K_R(v, \omega) = q^{-H(v,\omega)}$, where $H(v, \omega)$ is the "horocycle number", which is the number of edges shared by the arc from o to

v and the infinite ray (see [Ca] for more details about horocycles on trees). Let us now compute $K_{\tilde{P}}$. By Theorem 2 (i),

$$K_{\tilde{P}}(v, \omega) = K_P(v, \omega)/\phi_z(v).$$

By Definition 2, $K_P(v, \omega) = U_P(v, c)/U_P(o, c)$, where c is the confluence point of the ray ω with the arc coming from v, that is, the vertex in ω at distance $H(v, \omega)$ from o. Let now $u_P = U_P(u, v)$ for $u \sim w$. (Remember that P is isotropic.) As U_P is multiplicative, it follows that $K_P(v, \omega) = u_P^{H(v,\omega)}$. The number u_P can be computed by using (1):

$$u_P^2 - (1+1/q)\gamma u_P + 1/q = 0.$$

The corresponding equation for the operator R is

$$u_R^2 - (1+1/q) u_R + 1/q = 0,$$

whose solutions are $u_R = 1/q$ and $u_R = 1$. The former solution leads to the Poisson kernel K_R, the latter to the trivial positive harmonic function 1.

Returning to the operator P, we obtain the two solutions in the form

$$u_P = (1/2q)\left((q+1)\gamma \pm \sqrt{\gamma^2(q+1)^2 - 4q}\right).$$

One of these two numbers, according to the value of γ (and so of z), is used to form the Poisson kernel (see [FS] for more details). It is easy to see that, for generic γ, $K_R^z \neq K_P$.

3. Boundaries of graphs with tree-like foliations. Consider the Cartesian product of a homogeneous tree T and a finite set. For simplicity we restrict attention to a set with two elements. The resulting graph can be visualized as a "sandwich" of two copies of T: a "lower" tree T_0 and an "upper" tree T_1. In other words, this graph Γ is a "fiber bundle" with base T and fibers all equal to $\mathbf{Z}_2 = \mathbf{Z} \pmod 2$. Let P be a nearest neighbor, stochastic transition operator on Γ which is a convex combination of the isotropic, nearest neighbor, stochastic transition operator on the base T and of the symmetric operator with probabilities $\frac{1}{2}$ on the fiber $\mathbf{Z}_2$. A study of the Poisson and Martin boundaries associated with the operator P will appear in a forthcoming paper [PTW]. Denote by F_i ($i = 0, 1$) the restrictions of a P-harmonic function F to the lower and upper trees T_i ($i = 0, 1$), respectively. The determination of the boundary of P on Γ can be reduced to a problem on the tree T by considering, instead of F, two new functions on T defined by $F_+ = F_0 + F_1$ and $F_- = F_0 - F_1$. It is easy to see that F_+ is R-harmonic, whereas F_- is an eigenfunction of R for the eigenvalue 3. In fact, F is P-harmonic on Γ if and only if F_+ and F_- have the properties described. We can then get a Poisson representation theorem for P-harmonic functions on Γ on a boundary which is a Cartesian product of the boundary of T with $\mathbf{Z}_2$. The boundary values are vector pairs of distributions on the boundary of T. It turns out that if F is a P-harmonic function that is either nonnegative or bounded, then $F_- \equiv 0$. This says that the Martin and Poisson boundaries of (P, Γ) are both one-dimensional, being isomorphic to the boundary of a single tree, while the full boundary is two-dimensional. Details will appear in [PTW].

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