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$\Phi V[h]$ AND RIEMANN-STIELTJES INTEGRATION

BY

N. PAUL SCHEMBARI AND MICHAEL SCHRAMM
(SYRACUSE, NEW YORK)

1. Preliminaries. We consider real-valued functions defined on a closed bounded interval $[a,b]$. Given such a function f and any interval $I=[x,y]\subseteq[a,b]$, we write $f(I)=f(y)-f(x)$. We use "$\subseteq$" to denote set containment and "$\subset$" to denote proper containment. Any collection of intervals mentioned below shall be assumed nonoverlapping. Let f be as above and $\Phi=\{\varphi_k\}_{k=1}^{\infty}$ a sequence of convex functions defined on the nonnegative reals such that for each k,

(i) $\varphi_k(0)=0$ and $\varphi_k(x)>0$ for $x>0$,
(ii) φ_k is nondecreasing and $\varphi_k(x)\to\infty$ with x, and
(iii) for each $x>0$, $\varphi_{k+1}(x)\le\varphi_k(x)$.

For each positive integer n, we define the *Φ-modulus of variation of f on* $[a,b]$, $v(n,\Phi,f)=v(n,\Phi,f,[a,b])$ to be the supremum of the sums $\sum_{k=1}^{n}\varphi_k(|f(I_k)|)$, taken over all collections $\{I_k\}_{k=1}^{n}$ of n intervals in $[a,b]$. This generalized modulus of variation has many of the properties of the modulus of variation introduced by Chanturiya [C], for instance:

(i) $v(n,\Phi,f)\le v(n+1,\Phi,f)$,
(ii) $v(m+n,\Phi,f)\le v(m,\Phi,f)+v(n,\Phi,f)$,
(iii) $v(n,\Phi,f,[x,y])\le v(n,\Phi,f,[a,b])$ if $[x,y]\subseteq[a,b]$, and
(iv) $v(n,\Phi,\sum_{k=1}^{n}a_kf_k)\le\sum_{k=1}^{n}a_kv(n,\Phi,f_k)$, $\sum_{k=1}^{n}a_k=1$, $a_k\ge0$.

The last inequality holds since the φ_k are convex and nondecreasing, hence continuous.

1.1. LEMMA. *Let $c\in[a,b]$. For any f, $v(n,\Phi,f,[a,b])\le v(n,\Phi,f,[a,c])$ $+\,v(n,\Phi,f,[c,b])+\varphi_1(B)$, where $B=\sup f-\inf f$.*

Proof. Let $\{I_k\}_{k=1}^{n}$ be a collection of intervals in $[a,b]$. Let $\{j_i\}$ be the set of integers with $j_i\le j_{i+1}$ and $I_{j_i}\subseteq[a,c]$ for all i, and let $\{k_i\}$ be the set of integers with $k_i\le k_{i+1}$ and $I_{k_i}\subseteq[c,b]$ for all i. Let r be the integer, if any, such that c is in the interior of I_r. Then

$$\sum_{k=1}^{n} \varphi_k(|f(I_k)|) = \sum_i \varphi_{j_i}(|f(I_{j_i})|) + \sum_i \varphi_{k_i}(|f(I_{k_i})|) + \varphi_r(|f(I_r)|)$$
$$\leq \sum_i \varphi_{j_i}(|f(I_{j_i})|) + \sum_i \varphi_{k_i}(|f(I_{k_i})|) + \varphi_1(B).$$

Taking suprema gives the result, since v is increasing in n. ∎

Next, we give a sufficient condition for a function to be a Φ-modulus of variation.

1.2. DEFINITION. Let h be a nondecreasing function on the nonnegative integers with $h(0) = 0$, and Φ be as above. Then h is called *Φ-concave* if the sequence $\varphi_k^{-1}(h(k) - h(k-1))$ is nonincreasing.

1.3. THEOREM. *Let h be Φ-concave. Then there is a function f such that $v(n, \Phi, f) = h(n)$.*

Proof. Let $a_k = \varphi_k^{-1}(h(k) - h(k-1))$ and define f on $[0,1]$ by

$$f(x) = \begin{cases} 0, & x = 1, \\ \displaystyle\sum_{k=1}^{n}(-1)^{k+1}a_k, & \dfrac{1}{n+1} \leq x < \dfrac{1}{n}, \quad n = 1, 2, \ldots, \\ \dfrac{1}{2}\left[\limsup\limits_{n\to\infty}\displaystyle\sum_{k=1}^{n}(-1)^{k+1}a_k + \liminf\limits_{n\to\infty}\displaystyle\sum_{k=1}^{n}(-1)^{k+1}a_k\right], & x = 0. \end{cases}$$

Then for any n, $\sup \sum_{k=1}^{n} \varphi_k(|f(I_k)|)$ is achieved by taking I_k with $|f(I_k)| = a_k$, and we have

$$v(n, \Phi, f) = \sum_{k=1}^{n} \varphi_k(a_k) = h(n). \ ∎$$

If h is nondecreasing and concave (i.e., $h(tn + (1-t)m) \geq th(n) + (1-t)h(m)$ as long as h is defined at n, m and $tn+(1-t)m$), Chanturiya [C] has shown that h is a modulus of variation of some function f (this may be seen by taking $\varphi_k(x) = x$ in the above); if h is also Φ-concave, we have shown that h is a Φ-modulus of variation for some function g. However, the functions f and g are usually different.

Let Φ be as above and h a positive nondecreasing Φ-concave function on the positive integers with $h(k) \to \infty$. (For example, $\varphi_k(x) = x/k$ and $h(k) = \sum_{j=1}^{k} 1/j$.) Then there is an f with $v(n, \Phi, f) = h(n)$. It is evident that f is not of Φ-bounded variation, but it is the case that $v(n, \Phi, f) = O(h(n))$. With this in mind, we make the following definitions:

1.4. DEFINITIONS. Let Φ be as above and h a positive nondecreasing function on the positive integers (h need not be Φ-concave) such that

$$(*) \qquad \text{for each } x > 0, \quad \sum_{k=1}^{n} \varphi_k(x)/h(n) \to \infty \quad \text{as } n \to \infty.$$

We define $\Phi V[h]^*$ to be the class of functions f, defined on $[a,b]$, for which $v(n, \Phi, f, [a,b]) = O(h(n))$. We also define $\Phi V[h]$ to be the class of functions f such that $kf \in \Phi V[h]^*$ for some $k > 0$. (We will soon see the reason for condition $(*)$.)

1.5. THEOREM. *Let Φ, h_1, and h_2 be as above.*

(a) *If h_2 is Φ-concave and $h_1(n) = o(h_2(n))$, then $\Phi V[h_1]^* \subset \Phi V[h_2]^*$.*
(b) *If $h_1 \sim h_2$, then $\Phi V[h_1]^* = \Phi V[h_2]^*$.*

Proof. (a) Let $f \in \Phi V[h_1]^*$. Then $v(n, \Phi, f)/h_2(n) \le v(n, \Phi, f)/h_1(n)$ for large enough n. Since $v(n, \Phi, f) = O(h_1(n))$, $f \in \Phi V[h_2]^*$. Next, we must show there is an $f \in \Phi V[h_2]^*$ with $f \notin \Phi V[h_1]^*$. By Theorem 1.3, there is an f with $v(n, \Phi, f) = h_2(n)$. Then $f \in \Phi V[h_2]^*$, but $v(n, \Phi, f)/h_1(n) = h_2(n)/h_1(n) \to \infty$, so $f \notin \Phi V[h_1]^*$.

(b) Let $f \in \Phi V[h_1]^*$ so that there is a positive M with $v(n, \Phi, f) \le M h_1(n)$ for all n. Then $v(n, \Phi, f) \le M(h_2(n)/A)$ for some A, so $f \in \Phi V[h_2]^*$. If $f \in \Phi V[h_2]^*$, there is a positive N with $v(n, \Phi, f) \le N h_2(n)$ for all n, so $v(n, \Phi, f) \le N(B h_1(n))$ for some B and $f \in \Phi V[h_1]^*$. ∎

Many of the generalized bounded variation spaces that have been considered can be obtained by making the proper choices of Φ and h. If $\varphi_k(x) = x$ for all k, and $h(n) = 1$, then $\Phi V[h] = BV$; if $\Lambda = \{\lambda_k\}_{k=1}^{\infty}$ is a Λ-sequence in the sense of Waterman [W], $\varphi_k(x) = x/\lambda_k$, and $h(n) = 1$, then $\Phi V[h] = \Lambda BV$; if $\varphi_k(x) = x$ for all k, and h is concave, then $\Phi V[h]$ is the Chanturiya class $V[h]$.

The next result follows easily from Lemma 1.1; the second part since $v(n, \Phi, kf)$ increases as k increases.

1.6. PROPOSITION. *Let $c \in [a,b]$.*

(a) *If $f \in \Phi V[h]^*$ on $[a,c]$ and $f \in \Phi V[h]^*$ on $[c,b]$, then $f \in \Phi V[h]^*$ on $[a,b]$.*

(b) *Part* (a) *holds when the "$*$" is removed.* ∎

1.7. THEOREM. (a) *If $f \notin \Phi V[h]^*$ on $[a,b]$, then there is an $x \in [a,b]$ such that $f \notin \Phi V[h]^*$ on all closed intervals containing a neighborhood of x.*
(b) *Part* (a) *holds when the "$*$" is removed.*

Proof. We will consider (b), as (a) is similar. Assume $f \notin \Phi V[h]$ on $[a,b]$, and divide $[a,b]$ in half. Then, by Proposition 1.6(b), $f \notin \Phi V[h]$ on at least one of these halves. Call such an interval J_1 and divide J_1 in half.

Then $f \notin \Phi V[h]$ on at least one of the halves of J_1. Call this interval J_2 and divide J_2 in half. Continuing in this manner we obtain a sequence of nested intervals $\{J_k\}_{k=1}^{\infty}$ such that $|J_k| \downarrow 0$ and $f \notin \Phi V[h]$ on J_k for each k. The required point is $x \in \bigcap_{k=1}^{\infty} J_k$, since every neighborhood of x contains one of the J_k. ∎

The next result gives the reason for condition (∗) in Definition 1.4.

1.8. THEOREM. *If $f \in \Phi V[h]$, then f is bounded, and has only simple discontinuities.*

Proof. If f is unbounded, pick nonoverlapping intervals $\{I_k\}_{k=1}^{\infty}$ such that $|f(I_k)| \geq 1$ for all k. Then

$$\frac{v(n,\Phi,f)}{h(n)} \geq \frac{\sum_{k=1}^{n} \varphi_k(|f(I_k)|)}{h(n)} \geq \frac{\sum_{k=1}^{n} \varphi_k(1)}{h(n)} \to \infty \qquad \text{as } n \to \infty,$$

and $v(n,\Phi,f) \neq O(h(n))$. Now suppose f has a discontinuity that is not simple. We may assume $f(x+)$ does not exist for some $x \in [a,b]$. Then there is a positive ε and a sequence of nonoverlapping intervals $\{I_k = [a_k, b_k]\}_{k=1}^{\infty}$ with $a_k \downarrow x$ and $|f(I_k)| > \varepsilon$ for all k. As above, we obtain $v(n,\Phi,f)/h(n) \to \infty$. ∎

Recall that $f \in \Phi V[h]^*$ if and only if there is a positive M with $\sum_{k=1}^{n} \varphi_k(|f(I_k)|) \leq M h(n)$ for all collections $\{I_k\}_{k=1}^{\infty}$ and all n (the same M must work for all $\{I_k\}_{k=1}^{\infty}$.) This condition can be weakened:

1.9. THEOREM. *$f \in \Phi V[h]^*$ if and only if for each collection $\{I_k\}_{k=1}^{\infty}$ there is a positive M* (which may depend on $\{I_k\}_{k=1}^{\infty}$) *such that*

$$\sum_{k=1}^{n} \varphi_k(|f(I_k)|) \leq M h(n)$$

for all n.

Proof. The "only if" part is clear from the definition of $\Phi V[h]^*$. Conversely, assume $f \notin \Phi V[h]^*$. If f is unbounded, we may obtain the result by applying the technique used in Theorem 1.8. Otherwise, let $B = \sup f - \inf f$, and let x be the point provided by Theorem 1.7(a). Pick $n_1^* \geq 4$ such that $h(n_1^*) > 2\varphi_1(B)$ and $v(n_1^*) > 2h(n_1^*)$ (since h is increasing). Then there are intervals $\{I_k\}_{k=1}^{n_1^*}$ such that

$$\sum_{k=1}^{n_1^*} \varphi_k(|f(I_k)|) > 2h(n_1^*).$$

Hence, $\sum_{k=1}^{n_1^*} \varphi_k(|f(I_k)|) > h(n_1^*) + 2\varphi_1(B)$. By deleting the intervals containing x, if any, we obtain $\{I_k\}_{k=1}^{n_1}$ with

$$\sum_{k=1}^{n_1} \varphi_k(|f(I_k)|) > h(n_1^*) \geq h(n_1).$$

Note that $n_1 \geq 2$ since $n_1^* \geq 4$.

Assuming we have chosen $n_1 < \ldots < n_r$ and $\{I_k\}_{k=1}^{n_r}$, none containing x, there is a closed interval J containing a neighborhood of x with $J \cap I_k = \emptyset$ for $k = 1, \ldots, n_r$. Then $f \notin \Phi V[h]$ on J, so there is $n_{r+1}^* > \max(n_r^*, n_r + 2)$ and nonoverlapping intervals $\{J_k\}_{k=1}^{n_{r+1}^*}$ such that $h(n_{r+1}^*) > n_r\varphi_1(B)$ and

$$\sum_{k=1}^{n_{r+1}^*} \varphi_k(|f(J_k)|) > (r+3)h(n_{r+1}^*).$$

Deleting the intervals containing x, if any, gives $\{J_k\}_{k=1}^{n_{r+1}}$ (where $n_{r+1} > n_r$) such that

$$\sum_{k=1}^{n_{r+1}} \varphi_k(|f(I_k)|) > (r+2)h(n_{r+1}^*)$$

since $h(n_{r+1}^*) > n_r\varphi_1(B) > n_1\varphi_1(B) > 2\varphi_1(B)$. Let $I_k = J_k$ for $k = n_r + 1, \ldots, n_{r+1}$. Then

$$\begin{aligned}\sum_{k=1}^{n_{r+1}} \varphi_k(|f(I_k)|) > \sum_{k=n_r+1}^{n_{r+1}} \varphi_k(|f(I_k)|) &= \sum_{k=n_r+1}^{n_{r+1}} \varphi_k(|f(J_k)|) \\ &= \sum_{k=1}^{n_{r+1}} \varphi_k(|f(J_k)|) - \sum_{k=1}^{n_r} \varphi_k(|f(J_k)|) \\ &> (r+2)h(n_{r+1}^*) - n_r\varphi_1(B) > (r+2)h(n_{r+1}^*) - h(n_{r+1}^*) \\ &= (r+1)h(n_{r+1}^*) \geq (r+1)h(n_{r+1}).\end{aligned}$$

Hence, by induction, we obtain a sequence $n_r \uparrow \infty$ and a sequence $\{I_k\}_{k=1}^{\infty}$ of intervals such that $\sum_{k=1}^{n_r} \varphi_k(|f(I_k)|) \geq rh(n_r)$ for all positive integers r. Thus for this sequence $\{I_k\}_{k=1}^{\infty}$ there is no $M > 0$ with $\sum_{k=1}^{n} \varphi_k(|f(I_k)|) \leq Mh(n)$ for all n. ∎

We end this section by giving a generalization of a result of Perlman [P]. We assume the sequences $\Lambda = \{\lambda_k\}$ satisfy $\lambda_k \uparrow \infty$ and $\sum(1/\lambda_k) = \infty$. For any sequence $\{a_k\}$, we write Δa_k for $a_k - a_{k+1}$, and let ΨBV^* be the class of functions of Ψ-bounded variation.

1.10. THEOREM. *If $\sum h(k)\Delta(1/\lambda_k) < \infty$, then $\Phi V[h]^* \subseteq \Psi BV^*$, where $\Psi = \{\psi_k\}$ with $\psi_k(x) = \varphi_k(x)/\lambda_k$.*

Proof. First, $h(n)/\lambda_n = h(n)\sum_{k=n}^{\infty} \Delta(1/\lambda_k) \le \sum_{k=n}^{\infty} h(k)\Delta(1/\lambda_k)$, so $h(n) = o(\lambda_n)$. Next let $f \in \Phi V[h]^*$. For every collection $\{I_k\}_{k=1}^{\infty}$ of intervals and any N,

$$\begin{aligned}\sum_{k=1}^{N} \frac{\varphi_k(|f(I_k)|)}{\lambda_k} &= \sum_{k=1}^{N-1}\sum_{j=1}^{k} \varphi_j(|f(I_j)|)\Delta(1/\lambda_k) + \sum_{k=1}^{N} \frac{\varphi_k(|f(I_k)|)}{\lambda_N} \\ &= \sum_{k=1}^{N-1} \frac{\sum_{j=1}^{k} \varphi_j(|f(I_j)|)}{h(k)} h(k)\Delta(1/\lambda_k) + \frac{\sum_{k=1}^{N} \varphi_k(|f(I_k)|)}{h(N)} \frac{h(N)}{\lambda_N} \\ &\le \sup_n \frac{v(n,\Phi,f)}{h(n)} \Big[\sum_{k=1}^{N-1} h(k)\Delta(1/\lambda_k) + \frac{h(N)}{\lambda_N}\Big] \le C\end{aligned}$$

for some constant C. Thus, $f \in \Psi BV^*$ with $\Psi = \{\varphi_k(x)/\lambda_k\}$. ∎

1.11. THEOREM. *Let $f \notin \Phi V[h]^*$ be bounded. Then there is $\Lambda = \{\lambda_k\}$ with $\lambda_k \uparrow \infty$, $\sum(1/\lambda_k) = \infty$ and $\sum h(k)\Delta(1/\lambda_k) < \infty$ such that $f \notin \Psi BV$ where $\Psi = \{\varphi_k(x)/\lambda_k\}$.*

Proof. There is a collection of intervals $\{I_k\}$ such that $\sum_{k=1}^{n} \varphi_k(|f(I_k)|) \ne O(h(n))$. Let $n_0 = 0$, and choose n_1 such that $\sum_{k=1}^{n_1} \varphi_k(|f(I_k)|) \ge h(n_1)$. Having chosen $n_1, \ldots, n_{r-1}$, choose $n_r > n_{r-1}$ such that $\sum_{k=1}^{n_r} \varphi_k(|f(I_k)|) > 2\sum_{k=1}^{n_{r-1}} \varphi_k(|f(I_k)|)$ and $\sum_{k=1}^{n_r} \varphi_k(|f(I_k)|) > 2r^2h(n_r)$. Then

$$\sum_{k=n_{r-1}+1}^{n_r} \varphi_k(|f(I_k)|) > \frac{1}{2}\sum_{k=1}^{n_r} \varphi_k(|f(I_k)|) > r^2h(n_r).$$

Let $\lambda_k = r^2h(n_r)$ for $n_{r-1} < k \le n_r$. Then $\lambda_k \uparrow \infty$, and

$$\begin{aligned}\sum_{k=1}^{\infty} h(k)\Delta\left(\frac{1}{\lambda_k}\right) &= \sum_{\{k:\lambda_k \ne \lambda_{k+1}\}} \cdots \\ &= \sum_{r=1}^{\infty} h(n_r)\left[\frac{1}{r^2h(n_r)} - \frac{1}{(r+1)^2h(n_{r+1})}\right] \\ &\le \sum_{r=1}^{\infty} h(n_r)\frac{1}{r^2h(n_r)} = \sum_{r=1}^{\infty} \frac{1}{r^2} < \infty.\end{aligned}$$

Also,

$$\sum_{k=1}^{\infty} \frac{\varphi_k(|f(I_k)|)}{\lambda_k} = \sum_{r=1}^{\infty}\sum_{k=n_{r-1}+1}^{n_r} \frac{\varphi_k(|f(I_k)|)}{\lambda_k}$$

$$= \sum_{r=1}^{\infty}\Big[\sum_{k=n_{r-1}+1}^{n_r} \varphi_k(|f(I_k)|)\Big]\frac{1}{r^2 h(n_r)} \ge \sum_{r=1}^{\infty} 1 = \infty$$

and so, $f \notin \Psi BV^*$ with $\Psi = \{\varphi_k(x)/\lambda_k\}$. Note that $\sum(1/\lambda_k) = \infty$ since

$$\varphi_1(\sup f - \inf f)\sum_{k=1}^{N}\frac{1}{\lambda_k} \ge \sum_{k=1}^{N}\frac{\varphi_k(|f(I_k)|)}{\lambda_k}$$

for all N. ∎

The following extension of Perlman's result follows easily from Theorems 1.10 and 1.11.

1.12. THEOREM. *$\Phi V[h]^* = \bigcap\{ f \in \Psi BV^* : f$ is bounded and $\Psi = \{\varphi_k(x)/\lambda_k\}$ with $\sum h(k)\Delta(1/\lambda_k) < \infty \}$.* ∎

The last three theorems remain true when the "*" is removed.

2. The space $\Phi V[h]$. We begin with a definition of generalized variation that will be used to define a norm on $\Phi V[h]$.

2.1. DEFINITION. For f defined on $[a,b]$, we define the *total $\Phi V[h]$ variation of f* to be

$$V(f) = V_{\Phi,h}(f) = V(f;a,b) = V_{\Phi,h}(f;a,b) = \sup_n \frac{v(n,\Phi,f,[a,b])}{h(n)} .$$

It is clear that $f \in \Phi V[h]^*$ if and only if $V(f) < \infty$ and, because of the convexity of φ_k,

$$V\Big(\sum_{k=1}^{n} a_k f_k\Big) \le \sum_{k=1}^{n} a_k V(f_k) \quad \text{for } \sum_{k=1}^{n} a_k = 1,\ a_k > 0.$$

2.2. THEOREM. *$\Phi V[h]$ is a linear space.*

Proof. Let $\alpha \in \mathbf{R}$ and $f, g \in \Phi V[h]$. Then there are $k_1, k_2 > 0$ such that $V(k_1 f) < \infty$ and $V(k_2 g) < \infty$. If $\alpha \neq 0$, then $V((k_1/|\alpha|)\alpha f) = V(k_1 f) < \infty$, so $\alpha f \in \Phi V[h]$. Now let $k = \frac{1}{2}\min(k_1, k_2)$. Then

$$V(k(f+g)) \le V\left(\frac{1}{2}k_1 f + \frac{1}{2}k_2 g\right) \le \frac{1}{2}[V(k_1 f) + V(k_2 g)] < \infty,$$

so $f + g \in \Phi V[h]$. ∎

Let $\Phi V[h]_0 = \{ f \in \Phi V[h] : f(a) = 0 \}$. We follow Musielak and Orlicz [MO] and define a "norm" on $\Phi V[h]_0$ by $\|f\| = \inf\{ r > 0 : V(f/r) \le 1 \}$.

2.3. LEMMA. (a) $V(f/\|f\|) \le 1$.

(b) *If $\|f\| \le 1$, then $V(f) \le \|f\|$.*

Proof. (a) Let $r_n \downarrow \|f\|$ such that $V(f/r_n) \le 1$. Then for any nonoverlapping $\{I_k\}$, and any N, we have

$$\frac{\sum_{k=1}^{N} \varphi_k(|f(I_k)|/\|f\|)}{h(N)} = \lim_{n\to\infty} \frac{\sum_{k=1}^{N} \varphi_k(|f(I_k)|/r_n)}{h(N)} \le 1,$$

since the φ_k are continuous. Thus $V(f/\|f\|) \le 1$.

(b) For any nonoverlapping $\{I_k\}$, and any N,

$$\frac{\sum_{k=1}^{N} \varphi_k(|f(I_k)|)}{h(N)} \le \|f\| \frac{\sum_{k=1}^{N} \varphi_k(|f(I_k)|/\|f\|)}{h(N)},$$

since the φ_k are convex and $\|f\| \le 1$. Thus $V(f) \le \|f\|$ by (a). ∎

We now remove the quotes from "norm".

2.4. THEOREM. *$\|\cdot\|$ is a norm on $\Phi V[h]_0$.*

Proof. Clearly $\|0\| = 0$. If $f \neq 0$, then there is an $x \in (a, b]$ such that $f(x) \neq 0$. Then

$$V(f/r) \ge \frac{\varphi_1(|f(x) - f(a)|/r)}{h(1)} = \frac{\varphi_1(|f(x)|/r)}{h(1)} \to \infty \quad \text{as } r \to 0^+.$$

Thus there is an $r_0 > 0$ such that $V(f/r_0) > 1$. By Lemma 2.3(a) and the fact that $V(kf)$ increases as k increases, $\{ r > 0 : V(f/r) \le 1 \} = [\|f\|, \infty)$, so $(0, r_0) \not\subset \{ r > 0 : V(f/r) \le 1 \}$, and $\|f\| > 0$. Now note

$$\begin{aligned} \|kf\| &= \inf\{ r > 0 : V(kf/r) \le 1 \} = \inf\{ r > 0 : V((|k|/r)f) \le 1 \} \\ &= \inf\{ |k|r : r > 0 \text{ and } V(f/r) \le 1 \} = |k|\|f\|. \end{aligned}$$

Lastly, for any nonoverlapping $\{I_k\}$ and any N,

$$\begin{aligned} &\frac{\sum_{k=1}^{N} \varphi_k\left(\frac{|(f+g)(I_k)|}{\|f\| + \|g\|}\right)}{h(N)} \\ &\le \frac{\sum_{k=1}^{N} \varphi_k\left(\frac{\|f\|}{\|f\| + \|g\|}\frac{|f(I_k)|}{\|f\|} + \frac{\|g\|}{\|f\| + \|g\|}\frac{|g(I_k)|}{\|g\|}\right)}{h(N)} \\ &\le \frac{\sum_{k=1}^{N} \left(\frac{\|f\|}{\|f\| + \|g\|}\varphi_k\left(\frac{|f(I_k)|}{\|f\|}\right) + \frac{\|g\|}{\|f\| + \|g\|}\varphi_k\left(\frac{|g(I_k)|}{\|g\|}\right)\right)}{h(N)} \\ &\qquad \text{(by the convexity of the } \varphi_k\text{)} \end{aligned}$$

$$\leq \frac{\|f\|}{\|f\|+\|g\|} + \frac{\|g\|}{\|f\|+\|g\|} \qquad \text{(by Lemma 2.3(a))}$$

$$= 1.$$

Thus $\|f\|+\|g\| \in \{ r > 0 : V((f+g)/r) \leq 1 \}$, and so, $\|f+g\| = \inf\{ r > 0 : V((f+g)/r) \leq 1 \} \leq \|f\| + \|g\|$. ∎

2.5. THEOREM. *$\Phi V[h]_0$ is a Banach space.*

Proof. All that is left to show is that $\Phi V[h]_0$ is complete. Fix $\varepsilon > 0$ and let $\{f_n\}$ be a Cauchy sequence in the norm on $\Phi V[h]_0$, so that $\|f_n - f_m\| < \varepsilon$ for large n and m. Thus, for $x \in [a,b]$ and large n and m,

$$\begin{aligned}\frac{\varphi_1(|f_n(x)-f_m(x)|/\varepsilon)}{h(1)} &= \frac{\varphi_1(|(f_n-f_m)(x)-(f_n-f_m)(a)|/\varepsilon)}{h(1)}\\ &\leq V((f_n-f_m)/\varepsilon) \leq \|(f_n-f_m)/\varepsilon\| \\ &\qquad\qquad \text{(by Lemma 2.3(b))}\\ &= (1/\varepsilon)\|f_n-f_m\| < (1/\varepsilon)\varepsilon = 1.\end{aligned}$$

Hence, $|f_n(x)-f_m(x)| < \varepsilon\varphi_1^{-1}(h(1))$, and we can define $f(x) = \lim_{n\to\infty} f_n(x)$. For nonoverlapping $\{I_k\}$, any N, and large enough n,

$$\frac{\sum_{k=1}^{N}\varphi_k(|(f_n-f)(I_k)|/\varepsilon)}{h(N)} = \lim_{m\to\infty}\frac{\sum_{k=1}^{N}\varphi_k(|(f_n-f_m)(I_k)|/\varepsilon)}{h(N)} \leq 1$$

since $V((f_n-f_m)/\varepsilon) \leq 1$. Thus $V((f_n-f)/\varepsilon) \leq 1$. Since ε was arbitrary, $\|f_n - f\| \to 0$. Now $f(a) = 0$ since $f_n(a) = 0$ for all n; and, for any $\varepsilon > 0$ we can pick n such that $V((f_n-f)/\varepsilon) \leq 1$ and k such that $V(kf_n) \leq 1$. (Recall that $V(kf)$ decreases as k decreases.) Letting $c = \frac{1}{2}\min(1/\varepsilon, k)$, we have

$$\begin{aligned}V(cf) = V(c(f-f_n)+cf_n) &\leq V\left(\frac{1}{2}\left(\frac{f-f_n}{\varepsilon}\right)+\frac{1}{2}(kf_n)\right)\\ &\leq \frac{1}{2}V\left(\frac{f-f_n}{\varepsilon}\right)+\frac{1}{2}V(kf_n) < \infty.\end{aligned}$$

Thus $f \in \Phi V[h]_0$. ∎

Lemma 2.3(b) tells us that if $\|f_n - f\| < \varepsilon < 1$, then $V(f_n - f) < \varepsilon$, so in the above proof, we also have $f_n \to f$ in variation. Also, we see that if $\{f_n\}$ is Cauchy in norm, then $|f_n(x)-f_m(x)| \leq \varepsilon\varphi_1^{-1}(h(1))$ for all x. Thus $f_n \to f$ uniformly, and $\Phi V[h]_0 \cap C$ is a Banach space in this norm.

We can use the above norm to define a norm on $\Phi V[h]$ making it a

Banach space:

$$\|f\|_{\Phi,h} = |f(a)| + \|f - f(a)\|.$$

Our last goal of this section is to show that $\Phi V[h]$ satisfies an analogue of Helly's theorem. We accomplish this by placing $\Phi V[h]$ inside a space for which this is true. In §1, we saw that $\Phi V[h]$ is contained in the spaces ΨBV with $\Psi = \{\varphi_k(x)/\lambda_k\}$ with $\sum h(k)\Delta(1/\lambda_k) < \infty$. However, it is possible that ΨBV does not do what we wish. Schramm has shown [S] that Helly's theorem holds on ΨBV if $\sum \psi_k(x) = \infty$ for each x. We incorporate this condition into our theorem.

2.6. THEOREM. *Let $\{f_n\}$ be a sequence in $\Phi V[h]$ on $[a,b]$. Assume Λ satisfies $\sum h(k)\Delta(1/\lambda_k) < \infty$ and $\sum \varphi_k(x)/\lambda_k = \infty$. If there are numbers $c > 0$ and $M < \infty$ such that $|cf_n(a)| < M$ and $V_{\Phi,h}(cf_n) \le M$ for all n, then there is a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ converging pointwise to $f \in \Phi V[h]$, with $V_{\Phi,h}(cf) \le M$.*

Proof. Note that for each n

$$\begin{aligned}\|cf_n\|_\infty &\le |cf_n(a)| + \sup\{ cf_n(x) : x \in [a,b] \} - \inf\{ cf_n(x) : x \in [a,b] \} \\ &\le M + \varphi_1^{-1}(v(1,\Phi,cf_n)) \le M + \varphi_1^{-1}(h(1)V_{\Phi,h}(cf_n)).\end{aligned}$$

So the functions $cf_n(x)$ are uniformly bounded. For each $\{I_k\}$ and each n,

$$\begin{aligned}\sum_{k=1}^{\infty} \varphi_k(|cf_n(I_k)|)/\lambda_k &= \sum_{k=1}^{\infty}\Big[\sum_{m=1}^{k} \varphi_m(|cf_n(I_m)|)\Big]\Delta(1/\lambda_k) \\ &\le V_{\Phi,h}(cf_n)\sum_{k=1}^{\infty} h(k)\Delta(1/\lambda_k) \\ &\le SM \qquad \text{where } S = \sum h(k)\Delta(1/\lambda_k).\end{aligned}$$

Hence $V_\Psi(cf_n) \le SM$ where $\Psi = \{\varphi_k(x)/\lambda_k\}$. By Helly's theorem for ΨBV, there is a subsequence $\{f_{n_j}\}$ of $\{f_n\}$ converging pointwise to some $f \in \Psi BV$. Now $\sum h(k)\Delta(1/\lambda_k) < \infty$, so $\Psi BV \subseteq \Phi V[h]$ and $f \in \Phi V[h]$. Finally, for any collection $\{I_k\}$ of nonoverlapping intervals and any N,

$$\frac{\sum_{k=1}^{N} \varphi_k(|cf_{n_j}(I_k)|)}{h(N)} \le V_{\Phi,h}(cf_{n_j}) \le M.$$

Letting $j \to \infty$ gives $\sum_{k=1}^{N} \varphi_k(|cf(I_k)|)/h(N) \le M$. Thus $V_{\Phi,h}(cf) \le M$. ∎

3. Fourier series. The Dirichlet–Jordan theorem states that the Fourier series of a function of bounded variation converges pointwise, and converges uniformly on any closed interval of points of continuity of the function. Waterman has shown [W] that this result is also true for the functions of harmonic bounded variation, but that this is the largest ΛBV space where the theorem holds in the sense that, if $HBV \subset \Lambda BV$, there is a continuous function in ΛBV whose Fourier series diverges at a point. We now give a similar result for $\Phi V[h]$.

3.1. THEOREM. *If $HBV \subset \Phi V[h]$ on $[0,2\pi]$, there is a continuous $f \in \Phi V[h]_0$ whose Fourier series diverges at $x = 0$.*

Proof. If $HBV \subset \Phi V[h]$, there is a sequence $\{a_k\}$ with $a_k \downarrow 0$, $\sum_{k=1}^{n} \varphi_k(a_k) = O(h(n))$ and $\sum_{k=1}^{\infty} a_k/k = \infty$. Let b_k be defined by $b_1 = b_2 = a_2$, $b_3 = b_4 = a_4$, ... Then, since $b_k \le a_k$ for each k, $\sum_{k=1}^{n} \varphi_k(b_k) = O(h(n))$, but $\sum_{k=1}^{\infty} b_k/k \ge \sum_{k=1}^{\infty} a_{2k}/(2k) = \infty$. Choose $\varepsilon > 0$ with $\sum_{k=1}^{n} \varphi_k(\varepsilon b_k) \le h(n)$ for all n, and let $c_k = \varepsilon b_k$. Then $c_k \downarrow 0$, $c_{2k-1} = c_{2k}$ for $k = 1,2,\ldots$, $\sum_{k=1}^{n} \varphi_k(c_k) \le h(n)$ for all n, and $\sum_{k=1}^{\infty} c_k/k = \infty$. Define

$$f_n(x) = \begin{cases} c_i & \text{if } (2i-2)\pi < (n+1/2)x < (2i-1)\pi \text{ for } i = 1,\ldots,n+1, \\ 0 & \text{otherwise,} \end{cases}$$

Then

$$V_{\Phi,n}(f_n) = \sup_N \frac{v(N,\Phi,f_n)}{h(N)} = \sup_N \frac{\sum_{k=1}^{N} \varphi_k(c_k)}{h(N)} \le 1,$$

so $f_n(x) \in \Phi V[h]$. We also have $\|f_n\| = \inf\{ r > 0 : V_{\Phi,h}(f_n/r) \le 1 \} \le 1$. Now let $S_n(f)$ be the nth partial sum of the Fourier series of f at $x = 0$ and

$$D_n(t) = \frac{\sin(n+1/2)t}{2\sin t/2}$$

be the Dirichlet kernel. Then

$$\begin{aligned}
\pi S_n(f_n) &= \int_0^{2\pi} f_n(t)D_n(t)\,dt \ge \int_0^{2\pi} f_n(t)\frac{\sin(n+1/2)t}{t}\,dt \\
&= \sum_{k=1}^{n+1} \int_{(2k-2)\pi/(n+1/2)}^{(2k-1)\pi/(n+1/2)} f_n(t)\frac{\sin(n+1/2)t}{t}\,dt \\
&= \sum_{k=1}^{n+1} c_k \int_{(2k-2)\pi/(n+1/2)}^{(2k-1)\pi/(n+1/2)} \frac{\sin(n+1/2)t}{t}\,dt \\
&\ge \sum_{k=1}^{n+1} c_k \frac{n+1/2}{(2k-1)\pi} \int_{(2k-2)\pi/(n+1/2)}^{(2k-1)\pi/(n+1/2)} \sin(n+1/2)t\,dt \\
&= \sum_{k=1}^{n+1} c_k \frac{1}{(2k-1)\pi} \int_{(2k-2)\pi}^{(2k-1)\pi} \sin t\,dt
\end{aligned}$$

$$= \sum_{k=1}^{n+1} \frac{2c_k}{(2k-1)\pi} = \frac{1}{\pi}\sum_{k=1}^{n+1} \frac{c_k}{k-1/2} \geq \frac{1}{\pi}\sum_{k=1}^{n+1} \frac{c_k}{k}.$$

Without changing their norms or variations, the functions f_k, $k = 1, 2, \ldots$, may be altered on a set of small enough measure so that they are continuous and the partial sums of their Fourier series differ from those of f_k by an arbitrarily small amount. Thus

$$\|S_n\| \geq \frac{S_n(f_n)}{\|f_n\|} \geq \frac{(1/\pi^2)\sum_{k=1}^{n+1} c_k/k}{\|f_n\|} \to \infty.$$

Hence, by the principle of uniform boundedness (applied to the Banach space $\Phi V[h]_0 \cap C$) there is an $f \in \Phi V[h]_0 \cap C$ such that $\{S_n(f)\}$ does not converge. ∎

4. Riemann–Stieltjes integrals. In this section, the classic theorem on Riemann–Stieltjes integration ("If f is continuous and $g \in BV$, then $\int f\,dg$ exists") will be adjusted in a manner analogous to that of Young [Y] and Leśniewicz and Orlicz [LO], by strengthening the requirements on f and weakening those on g. In what follows, we assume that all nondecreasing functions h are such that $h(1) \geq 1$.

4.1. DEFINITION. Let $\Phi = \{\varphi_n\}$, $\Psi = \{\psi_n\}$, h_1, h_2, and positive constants A and B be given. *L. C. Young's series* for Φ, Ψ, h_1, h_2, A, and B is

$$\sum_{k=1}^{\infty} \varphi_k^{-1}(A/k)\psi_k^{-1}(B/k)h_1(k)h_2(k),$$

which we will denote $LCY(\Phi, \Psi, h_1, h_2, A, B)$.

4.2. LEMMA. *If $LCY(\Phi, \Psi, h_1, h_2, 1, 1)$ converges, then $LCY(\Phi, \Psi, h_1, h_2, A, B)$ converges.*

Proof. Let m be a natural number with $A, B \leq m$. For $im \leq k < (i+1)m$, we have $A/k \leq 1/i$ and $B/k \leq 1/i$, so

$$\sum_{k=1}^{\infty} \varphi_k^{-1}(A/k)\psi_k^{-1}(B/k)h_1(k)h_2(k)$$

$$= \sum_{k=1}^{m-1} \varphi_k^{-1}(A/k)\psi_k^{-1}(B/k)h_1(k)h_2(k)$$

$$+ \sum_{i=1}^{\infty} \sum_{k=mi}^{m(i+1)} \varphi_k^{-1}(A/k)\psi_k^{-1}(B/k)h_1(k)h_2(k)$$

$$\leq C + \sum_{i=1}^{\infty} \sum_{k=mi}^{m(i+1)} \varphi_k^{-1}(1/i)\psi_k^{-1}(1/i)h_1(k)h_2(k)$$

$$\leq C + \sum_{i=1}^{\infty} \sum_{k=mi}^{m(i+1)} \varphi_{m(i+1)}^{-1}\left(\frac{2m}{m(i+1)}\right)\psi_{m(i+1)}^{-1}\left(\frac{2m}{m(i+1)}\right) \times h_1(m(i+1))h_2(m(i+1)).$$

Since k no longer appears in this sum, the above

$$\leq C + m\sum_{i=1}^{\infty} \varphi_{m(i+1)}^{-1}\left(\frac{2m}{m(i+1)}\right)\psi_{m(i+1)}^{-1}\left(\frac{2m}{m(i+1)}\right) \times h_1(m(i+1))h_2(m(i+1)).$$

For any convex φ and $a > 1$, we have $\varphi^{-1}(ax) < a\varphi^{-1}(x)$. Consequently the above

$$\leq C + 4m^3\sum_{i=1}^{\infty} \varphi_{m(i+1)}^{-1}\left(\frac{1}{m(i+1)}\right)\psi_{m(i+1)}^{-1}\left(\frac{1}{m(i+1)}\right) \times h_1(m(i+1))h_2(m(i+1)) < \infty.$$

If $\Phi = \{\varphi_n\}$ is as above and F is convex, we may consider the sequence of convex functions $F\Phi = \{F\varphi_n\}$ and the set $F\Phi V[h]$ (which may or may not have the properties we have examined).

4.3. LEMMA. *If $LCY(\Phi, \Psi, h_1, h_2, 1, 1)$ converges, there is a convex $F : [0, \infty) \to [0, \infty)$ such that $LCY(F\Phi, F\Psi, h_1, h_2, 1, 1)$ converges, $F(0) = 0$, $F(x) > 0$ for $x > 0$, and $F(x) = o(x)$ as $x \to 0$.*

Proof. By Lemma 4.2, $LCY(\Phi, \Psi, h_1, h_2, 3n, 3n)$ converges for $n = 1, 2, \ldots$ We may choose $\{k_n\}$, a sequence of natural numbers, with $k_{n+1} > (1 + 1/n)k_n$ and

$$\sum_{k=k_n}^{\infty} \varphi_k^{-1}(3n/k)\psi_k^{-1}(3n/k)h_1(k)h_2(k) < 1/n^2 \quad \text{for each } n.$$

Set

$$p(t) = \begin{cases} 1/n, & 1/k_{n+1} < t \leq 1/k_n, \\ 1 + t, & 1/k_1 < t. \end{cases}$$

Then $p(t)$ is a positive, increasing function and $p(t) \to 0$ as $t \to 0$. Let $F(x) = \int_0^x p(t)\,dt$ for $x \geq 0$. Then F is convex, $F(0) = 0$, and $F(x) = o(x)$ as $x \to 0$. We now see that $LCY(F\Phi, F\Psi, h_1, h_2, 1, 1)$ converges. If $nk_n < k < nk_{n+1}$,

$$F\left(\frac{3n}{k}\right) = \int_0^{3n/k} p(t)\,dt \geq \int_{n/k}^{2n/k} p(t)\,dt \geq \frac{n}{k}p\left(\frac{n}{k}\right) = \frac{n}{k}\cdot\frac{1}{n} = \frac{1}{k}.$$

Likewise, for $nk_{n+1} \le k < (n+1)k_{n+1}$,

$$F\left(\frac{3n}{k}\right) \ge \int_{(n+1)/k}^{(2n+1)/k} p(t)\,dt \ge \frac{n}{k} p\left(\frac{n+1}{k}\right) = \frac{1}{k},$$

so that $F(3n/k) \ge 1/k$ for $nk_n \le k < (n+1)k_{n+1}$ (and $F^{-1}(1/k) \le 3n/k$ for k in that range). Then

$$\begin{aligned}
&\sum_{k=1}^{\infty} (F\varphi)^{-1}(1/k)(F\psi)^{-1}(1/k)h_1(k)h_2(k) \\
&\qquad = \sum_{k=1}^{\infty} \varphi_k^{-1} F^{-1}(1/k)\psi_k^{-1} F^{-1}(1/k) h_1(k)h_2(k) \\
&\qquad = \sum_{k=1}^{k_1-1} \varphi_k^{-1} F^{-1}(1/k)\psi_k^{-1} F^{-1}(1/k) h_1(k)h_2(k) \\
&\qquad\quad + \sum_{k=1}^{\infty} \sum_{k=nk_n}^{(n+1)k_{n+1}-1} \varphi_k^{-1} F^{-1}(1/k)\psi_k^{-1} F^{-1}(1/k) h_1(k)h_2(k) \\
&\qquad \le C + \sum_{k=1}^{\infty} \sum_{k=nk_n}^{(n+1)k_{n+1}-1} \varphi_k^{-1}(3n/k)\psi_k^{-1}(3n/k) h_1(k)h_2(k) \\
&\qquad \le C + \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty. \ \blacksquare
\end{aligned}$$

Let $\chi_n(\Phi, x) = F(\varphi_n(x))/\varphi_n(x)$ for $x > 0$, and $\chi_n(\Phi, 0) = 0$. Taking into account the properties of the sequence Φ and the function F, we have:

(i) $\chi_n(\Phi, x)$ is nondecreasing as a function of x,
(ii) $\chi_{n+1}(\Phi, x) \le \chi_n(\Phi, x)$ for all n, and
(iii) $\chi_n(\Phi, x) \to 0$ as $x \to 0$.

4.4. LEMMA. *If* $B(f) = \sup f - \inf f$, *then*

(i) $\varphi_1(B(f)) \le V_{\Phi,h}(f)h(1)$,
(ii) $V_{F\Phi,h}(f) \le \chi_1(\Phi, B(f))V_{\Phi,h}(f)$.

Proof. Part (i) is clear. To establish part (ii):

$$V_{F\Phi,h}(f) = \sup \frac{\sum_{k=1}^{n} F\varphi_k(|f(I_k)|)}{h(n)} = \sup \frac{\sum_{k=1}^{n} \chi_k(\Phi, |f(I_k)|)\varphi_k(|f(I_k)|)}{h(n)}$$

$$\leq \chi_1(\Phi, B(f)) \sup \frac{\sum_{k=1}^{n} \varphi_k(|f(I_k)|)}{h(n)} = \chi_1(\Phi, B(f))V(\Phi, h)(f). \blacksquare$$

(Note that part (ii) implies $\Phi V[h] \subseteq F\Phi V[h]$.)

We introduce some notation for the computations that follow. Let $\Phi = \{\varphi_n\}$ be as above, h a nondecreasing function on the positive integers, $a = \{a_n\}$ a sequence of real numbers, and $\delta = \{\delta_n\}$ a sequence of integers with $0 \leq \delta_0 < \delta_1 < \dots$ Define

$$\rho_n(\Phi, h, a) = \frac{\sum_{k=1}^{n} \varphi_k(|a_k|)}{h(n)}, \qquad \delta(a) = (\delta(a)_n) = \Big(\sum_{k=\delta_{n-1}+1}^{\delta_n} a_k \Big),$$

$$\rho_n^*(\Phi, h, a) = \sup_{\delta} \rho_n(\Phi, h, \delta(a)).$$

A sequence $\delta(a)$ is the result of replacing some of the commas in $(a_1, a_2, \dots)$ with $+$, the sequence δ determining which commas should be replaced.

4.5. LEMMA. *For any $\Phi = \{\varphi_k\}$, h, and a as above,*

$$\Big|\prod_{k=1}^{n} a_k\Big|^{1/n} \leq \varphi_n^{-1}\left(\frac{1}{n}\rho_n(\Phi, h, a)\right) h(n), \qquad n = 1, 2, \dots$$

Proof. Using the arithmetic-geometric mean inequality, Jensen's inequality, and the properties of convex functions, we have

$$\varphi_n\left(\frac{|\prod_{k=1}^{n} a_k|^{1/n}}{h(n)}\right) \leq \varphi_n\left(\frac{(1/n)\sum_{k=1}^{n} a_k}{h(n)}\right) \leq \frac{1}{n}\left(\frac{\sum_{k=1}^{n} \varphi_k(a_k)}{h(n)}\right)$$

and the result follows. ∎

4.6. LEMMA. *If $a = \{a_k\}$ and $b = \{b_k\}$ are real sequences, Φ and Ψ as above, and h_1 and h_2 are nondecreasing functions on the positive integers, then for every n there is a k_0 with $1 \leq k_0 \leq n$ such that*

$$|a_{k_0} b_{k_0}| \leq \varphi_n^{-1}\left(\frac{1}{n}\rho_n(\Phi, h_1, a)\right) \psi_n^{-1}\left(\frac{1}{n}\rho_n(\Psi, h_2, b)\right) h_1(n)h_2(n).$$

Proof. Let k_0 be such that $|a_{k_0} b_{k_0}| = \min\{\, |a_k b_k| : 1 \leq k \leq n \,\}$. Then

$$|a_{k_0} b_{k_0}| \leq \Big|\sum_{k=1}^{n} a_k b_k\Big|^{1/n} = \Big|\sum_{k=1}^{n} a_k\Big|^{1/n} \Big|\sum_{k=1}^{n} b_k\Big|^{1/n}$$

and the result follows from Lemma 4.5. ∎

4.7. THEOREM. *Let Φ, Ψ, a, b, h_1, and h_2 be as above. Then*

$$\left|\sum_{k=1}^{n}\sum_{i=1}^{k} a_i b_k\right| \le \varphi_1^{-1}(\rho_1^*(\Phi,h_1,a))\psi_1^{-1}(\rho_1^*(\Psi,h_2,b))h_1(1)h_2(1)$$
$$+\sum_{k=1}^{n-1}\varphi_k^{-1}\left(\frac{1}{k}\rho_k^*(\Phi,h_1,a)\right)\psi_k^{-1}\left(\frac{1}{k}\rho_k^*(\Psi,h_2,b)\right)h_1(k)h_2(k)$$

(*where the sum is taken to be* 0 *when* $n = 1$).

Proof. By induction: For $n = 1$, we have

$$\begin{aligned}|a_1b_1| &= \frac{\varphi_1^{-1}(\varphi_1(|a_1|))}{h_1(1)}\frac{\psi_1^{-1}(\psi_1(|b_1|))}{h_2(1)}h_1(1)h_2(1)\\ &\le \varphi_1^{-1}\left(\frac{\varphi_1(|a_1|)}{h_1(1)}\right)\psi_1^{-1}\left(\frac{\psi_1(|b_1|)}{h_2(1)}\right)h_1(1)h_2(1)\\ &\le \varphi_1^{-1}(\rho_1^*(\Phi,h_1,a))\psi_1^{-1}(\rho_1^*(\Psi,h_2,b))h_1(1)h_2(1).\end{aligned}$$

For $n > 1$, define a sequence $a' = \{a'_k\}$ by $a'_k = a_{k+1}$. By Lemma 4.6, there is a k_0 with $1 \le k_0 \le n-1$ so that

$$\begin{aligned}|a'_{k_0}b_{k_0}| = |a_{k_0+1}b_{k_0}| &= \varphi_{n-1}^{-1}\left(\frac{1}{n-1}\rho_{n-1}(\Phi,h_1,a')\right)\\ &\quad\times\psi_{n-1}^{-1}\left(\frac{1}{n-1}\rho_{n-1}(\Psi,h_2,b)\right)h_1(n-1)h_2(n-1)\\ &\le \varphi_{n-1}^{-1}\left(\frac{1}{n-1}\rho_{n-1}^*(\Phi,h_1,a)\right)\\ &\quad\times\psi_{n-1}^{-1}\left(\frac{1}{n-1}\rho_{n-1}^*(\Psi,h_2,b)\right)h_1(n-1)h_2(n-1).\end{aligned}$$

Let δ be given by $\delta_k = 1$ for $k \ne k_0$ and $\delta_{k_0} = 2$, and let $c = \delta(a)$ and $d = \delta(b)$. Then

$$\sum_{k=1}^{n-1}\sum_{i=1}^{k} c_i d_k = a_{k_0+1}b_{k_0} + \sum_{k=1}^{n}\sum_{i=1}^{k} a_i b_k$$

so

$$\left|\sum_{k=1}^{n}\sum_{i=1}^{k} a_i b_k\right| \le \left|\sum_{k=1}^{n-1}\sum_{i=1}^{k} c_i d_k\right| + |a_{k_0+1}b_{k_0}|$$

and the result follows by the induction hypothesis. ∎

We turn now to Riemann–Stieltjes sums. Let $P = \{[x_{k-1},x_k] : k = 1,\dots,n\}$ be a partition of $[a,b]$, with $\Xi = \{\xi_k : 1 \le k \le n,\ x_{k-1} \le \xi_k \le x_k\}$ a collection of intermediate points. We denote the Riemann–Stieltjes sum of f with respect to g constructed from these elements by

$S(f,g,P,\Xi) = \sum_{k=1}^{n} f(\xi_k)(g(x_k) - g(x_{k-1}))$.

4.8. THEOREM. *Let Φ, Ψ, h_1, and h_2 be given, P and Ξ be as above, and $f(a) = 0$. Then*

$$|S(f,g,P,\Xi)| \le 2\sum_{k=1}^{n-1} \varphi_k^{-1}\left(\frac{1}{k}V_{\Phi,h_1}(f)\right)\psi\left(\frac{1}{k}V_{\Psi,h_2}(g)\right)h_1(k)h_2(k).$$

Proof. Take $\xi_0 = a$. Then

$$\begin{aligned}|S(f,g,P,\Xi)| &= \left|\sum_{k=1}^{n} f(\xi_k)(g(x_k) - g(x_{k-1}))\right| \\ &= \left|\sum_{k=1}^{n}\sum_{i=1}^{k}(f(\xi_i) - f(\xi_{i-1}))(g(x_k) - g(x_{k-1}))\right|.\end{aligned}$$

Let $\alpha_i = f(\xi_i) - f(\xi_{i-1})$ for $i = 1,\dots,n$, $\alpha_i = 0$ for $i > n$, $\beta_k = g(x_k) - g(x_{k-1})$ for $k = 1,\dots,n$, $\beta_i = 0$ for $k > n$, $\alpha = \{\alpha_i\}$, and $\beta = \{\beta_k\}$. Then the above

$$\begin{aligned}&= \left|\sum_{k=1}^{n}\sum_{i=1}^{k}\alpha_i\beta_k\right| \\ &\le \varphi_1^{-1}(\rho_1^*(\Phi,h_1,\alpha))\psi_1^{-1}(\rho_1^*(\Psi,h_2,\beta))h_1(1)h_2(1) \\ &\quad + \sum_{k=1}^{n-1}\varphi_k^{-1}\left(\frac{1}{k}\rho_k^*(\Phi,h_1,\alpha)\right)\psi_k^{-1}\left(\frac{1}{k}\rho_k^*(\Psi,h_2,\beta)\right)h_1(k)h_2(k) \\ &\le \varphi_1^{-1}(V_{\Phi,h_1}(f))\psi_1^{-1}(V_{\Psi,h_2}(g))h_1(1)h_2(1) \\ &\quad + \sum_{k=1}^{n-1}\varphi_k^{-1}\left(\frac{1}{k}V_{\Phi,h_1}(f)\right)\psi_k^{-1}\left(\frac{1}{k}V_{\Psi,h_2}(g)\right)h_1(k)h_2(k) \\ &\le 2\sum_{k=1}^{n-1}\varphi_k^{-1}\left(\frac{1}{k}V_{\Phi,h_1}(f)\right)\psi_k^{-1}\left(\frac{1}{k}V_{\Psi,h_2}(g)\right)h_1(k)h_2(k). \quad\blacksquare\end{aligned}$$

We now establish our main result:

4.9. THEOREM. *Suppose $LCY(\Phi,\Psi,h_1,h_2,1,1)$ converges, $f \in \Phi V[h_1]_0 \cap C$, and $g \in \Psi V[h_2]$ on $[a,b]$. Then $\int_a^b f\,dg$ exists.*

Proof. We may assume that $V_{\Phi,h_1}(f) < \infty$ and $V_{\Psi,h_2}(g) < \infty$. Let $P_1 = \{[x_{k-1}^1, x_k^1]\}_{k=1}^{n_1}$ and $P_2 = \{[x_{k-1}^2, x_k^2]\}_{k=1}^{n_2}$ be partitions of $[a,b]$ with intermediate points $\Xi_1 = \{\xi_k^1\}$ and $\Xi_2 = \{\xi_k^2\}$ respectively. Let $P = \{[x_{k-1}, x_k]\}_{k=1}^{n}$ be the common refinement of P_1 and P_2. Define step functions f_1 and f_2 by

$$f_i(x) = f_i(x,P_i,\Xi_i) = \begin{cases} 0, & x = a, \\ f(\xi_k^i), & x_{k-1}^i < x \le x_k^i, \end{cases} \qquad \text{for } i = 1,2.$$

Then, for $i = 1, 2$,

$$S(f, g, P_i, \Xi_i) = \sum_{k=1}^{n_i} f(\xi_k^i)(g(x_k^i) - g(x_{k-1}^i)) = \sum_{k=1}^{n_i} f_i(x_k^i)(g(x_k^i) - g(x_{k-1}^i))$$
$$= \sum_{k=1}^{n} f_i(x_k)(g(x_k) - g(x_{k-1})),$$

so that the difference

$$S(f, g, P_1, \Xi_1) - S(f, g, P_2, \Xi_2) = 2\sum_{k=1}^{n} \frac{1}{2}(f_1(x_k) - f_2(x_k))(g(x_k) - g(x_{k-1}))$$

(the 2 and 1/2 are for later convenience). Note that the above sum is $2S(\frac{1}{2}(f_1 - f_2), g, P, P)$. Applying Theorem 4.8 to $\frac{1}{2}(f_1 - f_2)$, g, $F\Phi$, $F\Psi$, h_1, and h_2 (F having been chosen in Lemma 4.3), we have

$$|S(f, g, P_1, \Xi_1) - S(f, g, P_2, \Xi_2)|$$
$$\leq 4\sum_{k=1}^{n} (F\varphi_k)^{-1}\left(\frac{1}{k}V_{F\Phi, h_1}\left(\frac{1}{2}(f_1 - f_2)\right)\right)$$
$$\times (F\psi_k)^{-1}\left(\frac{1}{k}V_{F\Psi, h_2}(g)\right) h_1(k) h_2(k).$$

By Lemma 4.4,

$$V_{F\Phi,h}(\tfrac{1}{2}(f_1 - f_2)) \leq \chi_1(\Phi, B(\tfrac{1}{2}(f_1 - f_2)))V_{\Phi,h}(\tfrac{1}{2}(f_1 - f_2))$$
$$\leq \tfrac{1}{2}\chi_1(\Phi, B(\tfrac{1}{2}(f_1 - f_2)))(V_{\Phi,h}(f_1) + V_{\Phi,h}(f_2))$$
$$\leq \chi_1(\Phi, B(\tfrac{1}{2}(f_1 - f_2)))V_{\Phi,h}(f).$$

Since f is uniformly continuous, for given $\varepsilon > 0$ we may choose partitions P_1 and P_2 sufficiently fine so that $B(\frac{1}{2}(f_1 - f_2)) < \varepsilon$. Then

$$|S(f, g, P_1, \Xi_1) - S(f, g, P_2, \Xi_2)| \leq 4\sum_{k=1}^{n} (F\varphi_k)^{-1}\left(\frac{1}{k}\chi_1(\Phi, \varepsilon)V_{F\Phi, h_1}(f)\right)$$
$$\times (F\psi_k)^{-1}\left(\frac{1}{k}V_{F\Psi, h_2}(g)\right) h_1(k) h_2(k),$$

which is bounded for $\varepsilon > 0$ by Lemmas 4.2 and 4.3. Since $\chi_1(\Phi, \varepsilon) \to 0$ as $\varepsilon \to 0$, we may make this sum as small as we like by making ε small. ∎

5. Generalized variations. While the classes $V_{\Phi,h}(f)$ encompass a large number of those previously examined, they all depend on the expressions $|f(I_k)|$ for their definition. Concepts of variation not involving this expression have also been studied:

1. Brown [B] considers the *n-th variation*, based on sums of the form $\sum_{r=0}^{n-k}\binom{n-r-1}{k-1}|\Delta_n^k f(x_r)|$ where $a = x_0 < x_1 < \ldots < x_n \le b$, $x_k - x_{k-1} = h$ for $k = 1, \ldots, n$, $b - x_n < h$, and $\Delta_n^k f(x) = \sum_{r=0}^{k}(-1)^{k-r}\binom{k}{r} f(x + rh)$.

2. Russell [R] considers the sums $\sum_{i=0}^{n-k}(x_{i+k} - x_i)|Q_k(f; x_i, \ldots, x_{i+k})|$ where $a \le x_0 < x_1 < \ldots < x_n \le b$ and $Q_k(f; y_0, \ldots, y_k) = \sum_{i=0}^{k}[f(y_i)/P(i,k)]$ with $P(i,k) = \prod_{j=0, j\ne i}^{k}(y_i - y_j)$.

3. Let ω be a nondecreasing function defined on $[a,b]$ and let $S \subseteq [a,b]$ be the set of continuity points of ω. Let f be a function that is continuous at each point of S such that $\lim_{x\to c+, x\in S} f(x)$ and $\lim_{x\to c-, x\in S} f(x)$ exist finitely for $c \in [a,b]\setminus S$. Bhakta [Bh] considers the *total ω-variation* of such an f in sums of the form $\sum_{i=0}^{n}|f(x_i+) - f(x_i-)|$ where $\omega(a) = \omega(x_0) < \omega(x_1) < \ldots < \omega(x_n) = \omega(b)$.

In view of this variety, we attempt to prove the following result in the most general terms possible:

5.1. DEFINITION. Let F be the class of (finite-valued) functions defined on $[a,b]$. A *variation function* is any function $G : F \to \{r \in \mathbf{R} : r \ge 0\} \cup \{\infty\}$. GBV is the class of all functions f satisfying $G(f) < \infty$.

5.2. THEOREM. *Let G be a variation function such that GBV satisfies an analogue of Helly's theorem* (i.e., if $\{f_n(a)\}$ is bounded and $\{f_n\}$ are of uniform bounded G-variation, then there is a subsequence $\{f_{n_j}\}$ of $\{f_n\}$ and $f \in GBV$ with $f_{n_j} \to f$ everywhere). *Assume L_n is a sequence of linear operators on GBV satisfying*

(i) *$L_n(g) \to g$ everywhere.*

(ii) *For every n, $G(L_n(g)) \le CG(g)$ for some constant C which may depend on g. Then $f \in GBV$ if and only if $L_n(f)$ have uniformly bounded G-variations.*

Proof. Assume $f \in GBV$. Then $G(L_n(f)) \le CG(f) < \infty$, so $L_n(f)$ have uniformly bounded G-variations. Conversely, assume $L_n(f)$ have uniformly bounded G-variations. Since $L_n(f)(a) \to f(a)$, $\{L_n(f)(a)\}$ is bounded. Hence, by the Helly analogue, there is $n_j \uparrow \infty$ and $g \in GBV$ such that $L_{n_j}(f) \to g$ everywhere. Since $L_n(f) \to f$ everywhere, $f \equiv g$, and thus $g \in GBV$. ∎

This theorem has applications to many spaces—we give two. The first is a result of Zygmund [Z, p. 138].

5.3. COROLLARY. *Let f be regulated and periodic. Then f is of bounded variation if and only if the $(C,1)$ means of the Fourier series of f, $\sigma_n(f)$, have uniformly bounded variations.*

Proof. Let $V(f)$ be the variation of f. Note that V satisfies Helly's theorem. Since f is regulated and periodic, Fejér's theorem [Z, p. 189] gives $\sigma_n(f) \to f$. Finally, let $K_n(t)$ be the linear means of $\frac{1}{2} + \sum_{k=1}^{\infty} \cos kt$ [Z, pp. 84–85]. For any partition $\{x_k\}_{k=1}^{n}$ of $[a, b]$,

$$\sum_{k=1}^{n} |\sigma_n(x_k) - \sigma_n(x_{k-1})| = \frac{1}{\pi} \sum_{k=1}^{n} \left| \int_{-\pi}^{\pi} [f(x_k + t) - f(x_{k-1} + t)] K_n(t)\, dt \right|$$

$$\leq \frac{1}{\pi} \int_{-\pi}^{\pi} \left[\sum_{k=1}^{n} |f(x_k + t) - f(x_{k-1} + t)| \right] K_n(t)\, dt$$

(since K_n is positive)

$$\leq V(f) \qquad \left(\text{since } f \text{ is periodic and } \int_{-\pi}^{\pi} K_n(t)\, dt = \pi\right).$$

Thus $V(\sigma_n(f)) \leq V(f)$, and the theorem applies. ∎

5.4. COROLLARY. *Let f be regulated and periodic. $V_{\Phi,h}(f) < \infty$ if and only if $V_{\Phi,h}(\sigma_n(f))$ are uniformly bounded.*

Proof. $V_{\Phi,h}$ satisfies Helly's theorem and $\sigma_n(f) \to f$ by Fejér's theorem. For any sequence of nonoverlapping intervals,

$$(\dagger) \qquad \frac{\sum_{k=1}^{N} \varphi_k(|\sigma_n(I_k)|)}{h(N)} = \frac{\sum_{k=1}^{N} \varphi_k\left(\left|\pi^{-1} \int_{-\pi}^{\pi} f(I_k + t) K_n(t)\, dt\right|\right)}{h(N)}$$

$$\leq \frac{\sum_{k=1}^{N} \varphi_k\left(\pi^{-1} \int_{-\pi}^{\pi} |f(I_k + t)| K_n(t)\, dt\right)}{h(N)}$$

(since φ_k is nondecreasing for each k)

$$\leq \frac{\pi^{-1} \sum_{k=1}^{N} \int_{-\pi}^{\pi} \varphi_k(|f(I_k + t)|) K_n(t)\, dt}{h(N)}$$

(by Jensen's inequality)

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{\sum_{k=1}^{N} \varphi_k(|f(I_k + t)|)}{h(N)} K_n(t)\, dt \leq V_{\Phi,h}(f).$$

Thus $V_{\Phi,h}(\sigma_n) \leq V_{\Phi,h}(f)$. (In (†), $f(I+t) = f(y+t) - f(x+t)$ for $I = [x, y]$.)

∎

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N. Paul Schembari

DEPARTMENT OF MATHEMATICS
SYRACUSE UNIVERSITY
SYRACUSE, NEW YORK 13214
U.S.A.

Michael Schramm

DEPARTMENT OF MATHEMATICS
LE MOYNE COLLEGE
SYRACUSE, NEW YORK 13214-1399
U.S.A.

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