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ON WEIGHTED ESTIMATES FOR THE KAKEYA MAXIMAL OPERATOR

BY

DETLEF MÜLLER (PRINCETON, NEW JERSEY)

Introduction. The theory of A_p-weights associated with the Hardy–Littlewood maximal operator has proved to be a powerful tool in analysis, and many authors have contributed to it. Since the monograph [Ga–Ru] presents an excellent and comprehensive introduction to this theory, we shall not further comment on it here, but refer the reader to this book.

In this article, we shall take up the study of weighted L^p-estimates for the Kakeya maximal operator $\mathfrak{M}_N$ in $\mathbf{R}^2$, i.e. the maximal operator associated with the system of all rectangles of eccentricity $N \geq 1$ with arbitrary direction in $\mathbf{R}^2$. More precisely, we are interested in estimates of the form

$$(\mathfrak{A}_p^{\mathfrak{K}}) \qquad \int_{\mathbf{R}^2} (\mathfrak{M}_N f)^p w\, dx \leq C(\log 2N)^\alpha \int_{\mathbf{R}^2} |f|^p w\, dx,$$

$f \in L^p(w\, dx)$, $N \geq 1$. We say that a weight w is an $\mathfrak{A}_p^{\mathfrak{K}}$-*weight* if $(\mathfrak{A}_p^{\mathfrak{K}})$ holds for some constants C, $\alpha \geq 0$, for every $N \geq 1$.

The Kakeya maximal operator was first studied by A. Córdoba [Co 1], [Co 2]. Córdoba gave sharp estimates for $\mathfrak{M}_N$ on $L^2(\mathbf{R}^2)$, and used them to establish a new proof of the Carleson–Sjölin (–Fefferman–Hörmander) theorem on Bochner–Riesz means on $\mathbf{R}^2$ (see e.g. [Co 1] for references), thus making precise an idea of C. Fefferman expressed in [Fe] that the Kakeya maximal operator should exert a crucial control on Bochner–Riesz multiplier operators. The results of Córdoba, which refer to the case of a constant weight w, show that it is natural to allow a growth factor $(\log 2N)^\alpha$ in $(\mathfrak{A}_p^{\mathfrak{K}})$.

Let us mention that the correct L^p-estimates of the Kakeya maximal operator on $\mathbf{R}^n$ for $n \geq 3$ are still unknown. For partial results in this direction, see e.g. [C–D–Ru] and [I].

In the non-weighted case (and $n = 2$), it is crucial to estimate $\mathfrak{M}_N$ on $L^2(dx)$. We shall therefore also mainly concentrate on the case $p = 2$, and only briefly discuss the case $p > 2$ in the last section (see 4.2).

Nevertheless, we state a necessary condition (compare 4.2) for w to be in $\mathfrak{A}_p^{\mathfrak{K}}$ (resp. in a closely related weight space $A_p^{\mathfrak{K}}$), and there is reason to believe that this condition is also sufficient. Notice that, since there is no obvious reason that the $\mathfrak{A}_p^{\mathfrak{R}}$-condition should not be the same for all dimensions $n \geq 2$, an understanding of the weighted L^p-estimates for $\mathfrak{M}_N$ on $\mathbf{R}^2$ could perhaps also be useful in order to understand $\mathfrak{M}_N$ in higher dimensions.

The article is organized as follows: Section 1 is devoted to the study of the maximal operator $M_{<N}$ associated with the system of all rectangles of size $1 \times r$, $1 \leq r \leq N$, and the corresponding weight space $A_2^{\mathfrak{R}}$. The main result of this section (Th. 1.5) is a geometric characterization of $A_2^{\mathfrak{R}}$. In Section 2 we show how this result can be used to characterize $\mathfrak{A}_2^{\mathfrak{R}}$ (see Th. 2.2). The crucial condition (1.12) in Th. 1.5 is somewhat unhandy. But it seems to be closely related to a simpler condition, namely that, roughly speaking, w is uniformly in A_1 on every line, and this relation is discussed in Section 3. In particular, we show that the "A_1 on lines" condition implies the $\mathfrak{A}_2^{\mathfrak{R}}$-condition (see Cor. 3.2 and Remark 3.3), and that for radial and monotonous weights these two conditions are in fact equivalent (Th. 3.4). Finally, in Section 4, we provide some examples and discuss further problems.

1. The maximal operator $M_{<N}$. We retain the notation of [Mü]. So, a rectangle of *dimension* $\delta \times N\delta$, $\delta > 0$, $N \geq 1$, will be any rectangle in $\mathbf{R}^2$ which is congruent to the rectangle $[0, \delta] \times [0, N\delta]$, and $\mathfrak{B}_{\delta,N}$ will denote the collection of all such rectangles. The associated maximal function is given by

$$M_{\delta,N} f(x) := \sup_{x \in R} \frac{1}{|R|} \int_R |f(y)|\, dy,$$

where the supremum is taken over all rectangles R in $\mathfrak{B}_{\delta,N}$ containing x. For the largest part of the paper, we shall freeze δ to be $\delta = 1$.

By a *weight* we shall always mean a measurable function w on $\mathbf{R}^2$ with values in $[0, \infty]$. For the study of weighted estimates, it will be more convenient to consider the bigger maximal operator

$$M_{<N} f := \sup_{1 \leq r \leq N} M_{1,r} f$$

instead of $M_{1,N}$. The main question which we shall address is the following: For which weights w on $\mathbf{R}^2$ are there constants $\alpha = \alpha(w) \geq 0$ and $C > 0$ such that

$$\int_{\mathbf{R}^2} (M_{<N} f)^2 w\, dx \leq C (\log 2N)^\alpha \int_{\mathbf{R}^2} |f|^2 w\, dx \tag{1.1}$$

for all $f \in L^2(w\, dx)$ and $N \geq 1$? Notice that even for the constant weight $w = 1$ (1.1) can only hold if $\alpha \geq 1/2$ (see [Co 2], [Mü]), so that it appears natural to allow a growth factor $(\log N)^\alpha$ in (1.1).

Observe also that, with $L = [\log N / \log 2] + 1$,

$$M_{<N} f \leq 2 \max_{m=0,\ldots,L} M_{1,2^{m+1}} f, \tag{1.2}$$

since for $2^m \leq r < 2^{m+1}$, $M_{1,r} f \leq 2\, M_{1,2^{m+1}} f$. (1.2) shows that an estimate of the form (1.1) for $M_{1,N} f$ instead of $M_{<N} f$ is in fact equivalent to (1.1).

Let us fix some further notation. If A is any set of a positive Lebesgue measure $|A|$, we set

$$w_A := \frac{w(A)}{|A|}, \qquad w(A) := \int_A w\, dx.$$

By $\{Q_i\}_{i \in I}$ we shall always denote the family of all unit cubes of the unit lattice in $\mathbf{R}^2$, and we shall use the abbreviations

$$w_i := w_{Q_i}, \qquad (w^{-1})_i := (w^{-1})_{Q_i}.$$

The letter C will usually denote a constant which is independent of the weight w and N, but which may change from statement to statement. The expression $a \sim b$ will be used to indicate that $ca \leq b \leq Ca$, where $0 < c \leq C$ are constants of the type described before. If R is any rectangle and $r > 0$, then rR will denote the rectangle obtained from R by the dilation with scaling factor r and fixed point the center of R. If $r = 4$, we shall also write $\widetilde{R}$ instead of $4R$.

In the sequel, there will frequently arise the situation that to every unit cube Q_i there is associated a rectangle R_i such that $R_i \cap Q_i \neq \emptyset$, and we shall then denote by I_i the index set $I_i := \{ j : Q_j \cap R_i \neq \emptyset \}$, which gives a "discretization" of R_i on the level of the unit scale.

The following lemma describes some results which follow from standard arguments in the theory of weighted estimates for the Hardy–Littlewood maximal function (see e.g. [Ga–Ru], p. 387 ff.).

LEMMA 1.1. *Assume that*

$$\int_{\mathbf{R}^2} (M_{<N} f)^2 w\, dx \leq A_N \int_{\mathbf{R}^2} |f|^2 w\, dx \tag{1.3}$$

for every $f \in L^2(w\, dx)$. Then

(i) $w_R (w^{-1})_R \leq A_N$ *for all rectangles* $R \in \bigcup_{1 \leq r \leq N} \mathfrak{B}_{1,r}$.

(ii) $1 \leq w_i (w^{-1})_i \leq A_N$ *for every* i.

(iii) $w_Q \leq C A_N^2 w_{Q'}$ *if* $Q, Q' \in \mathfrak{B}_{1,1}$ *have distance* $\operatorname{dist}(Q, Q') \leq 10$.

(iv) *If $\{R_i\}$ is a collection of rectangles in $\mathfrak{B}_{1,r}$, $1 \leq r \leq N$, then*

$$\int_{R_i \cap R_k} w^{-1}\, dx \leq A_N \sum_{j \in I_i \cap I_k} w_j^{-1} \leq A_N \int_{\widetilde{R}_i \cap \widetilde{R}_k} w^{-1}\, dx.$$

(v) *If* (1.3) *holds for every* $N \geq 1$, *then either* $w = 0$ *a.e., or* $w = \infty$ *a.e., or* $w > 0$ *a.e. and* w *is locally integrable.*

Proof. (i) follows completely analogous to estimate (1.10) in [Ga–Ru], p. 390 (for $p = 2$), and (v) from the analogue of (1.6), p. 388 in [Ga–Ru].

(ii) follows from (i) and Hölder's inequality (in order to obtain the estimate from below).

(iii) Applying a suitable motion of $\mathbf{R}^2$, we may assume that Q' is a cube in the unit lattice. Since there are at most 16 cubes Q_i in the unit lattice which intersect Q, we may, by looking at estimates for $w(Q_i)$, assume that also Q is a cube in the unit lattice. Then there is a cube Q_j such that the convex hull R of $Q \cup Q_j$ and the convex hull R' of $Q_j \cup Q'$ are both rectangles in $\bigcup_{1 \leq r \leq 10} \mathfrak{B}_{1,r}$. So, by (i), $w(Q) \leq w(R) \leq 10 A_N (w^{-1})_R^{-1}$, and $(w^{-1})_R \geq \frac{1}{10}(w^{-1})_{Q_j}$, hence $w_Q \leq 100 A_N w_{Q_j}$. Similarly, $w_{Q_j} \leq 100 A_N w_{Q'}$, and we obtain (iii).

(iv) We have $R_i \cap R_k \subset \bigcup_{j \in I_i \cap I_k} Q_j$, and, by (ii), $\int_{Q_j} w^{-1}\, dx \leq A_N w_j^{-1}$, which gives the first estimate. The second follows equally easily. ∎

The next proposition provides a necessary condition for (1.1) to hold.

PROPOSITION 1.2. *Assume that*

$$\int_{\mathbf{R}^2} (M_{<4N} f)^2 w\, dx \leq A \int_{\mathbf{R}^2} |f|^2 w\, dx \tag{1.3'}$$

for every $f \in L^2(w\, dx)$. *Then*

(i) $w_R (w^{-1})_R \leq A$ *for every rectangle* $R \in \bigcup_{1 \leq r \leq 2N} \mathfrak{B}_{1,r}$.

(ii) *If* $1 \leq r \leq N$, *and if for each unit cube* Q_i *we devise a rectangle* $R_i \in \mathfrak{B}_{1,r}$ *with* $R_i \cap Q_i \neq \emptyset$, *then for every* k

$$\sum_i w_i \sum_{j \in I_i \cap I_k} w_j^{-1} \leq C A^9 r^2 (\log 2r)^4. \tag{1.4}$$

Proof. By Lemma 1.1, we only have to prove (ii). So, fix r and a rectangle R_k in the given system of rectangles. There is no loss in generality to assume that $r = N$. Because of the observation (1.2), we shall even assume that $N = 2^M$ with $M \in \mathbf{N}$. After a suitable motion of the coordinate system, we can assume that $R_k = [0, N] \times [0, 1]$. Of course, under this motion the unit lattice in $\mathbf{R}^2$ will perhaps not be left invariant. But, by Lemma 1.1 (iii), averages over nearby cubes in both lattices are comparable, which leads at most to an additional factor A^5 in the estimate of the left-hand side of (1.4). So, we shall also assume that we work with the usual unit lattice in $\mathbf{R}^2$.

We now set

$$u(s) := \int_0^1 w^{-1}(s,t)\,dt, \qquad s \in [0,N],$$

and $u(J) = \int_J u\,ds$ for any interval $J \subset [0,N]$. For the remainder of this proof, an interval J in $[0,N]$ will always mean an interval with integer endpoints. For $m = 0,\dots,M$, let

$$\alpha_m := \max\{\, u(J) : J \text{ an interval of length } 2^m \text{ in } [0,N]\,\},$$

and let $\Omega_{m,j}$, $j = 0,\dots,m$, denote the set of all intervals $J \subset [0,N]$ of length $|J| = 2^m$ such that $2^{-j-1}\alpha_m < u(J) \le 2^{-j}\alpha_m$. By A_m we denote the set of all j such that $\Omega_{m,j} \neq \emptyset$. We choose for every $j \in A_m$ intervals $J^-_{m,j}$, $J^+_{m,j}$ of length 2^m in $[0,N]$ such that:

(1.5) $\quad 2^{-j-1}\alpha_m \le u(J^{\pm}_{m,j}) \le 2^{-j}\alpha_m$;

(1.6) $\quad$ all intervals $J \in \Omega_{m,j}$ lie "between" $J^-_{m,j}$ and $J^+_{m,j}$,

i.e. $\min J^-_{m,j} \le \min J$ and $\max J \le \max J^+_{m,j}$.

Define

$$g_m := \sum_{j\in A_m} (\gamma^-_{m,j}\chi_{R^-_{m,j}} + \gamma^+_{m,j}\chi_{R^+_{m,j}}),$$

where $R^{\pm}_{m,j} := J^{\pm}_{m,j} \times [0,1] \subset R_k$, and

$$\gamma^{\pm}_{m,j} := \frac{1}{|J^{\pm}_{m,j}|}[u(J^{\pm}_{m,j})]^{1/2} = 2^{-m}(w^{-1}(R^{\pm}_{m,j}))^{1/2},$$

and set $g := \sum_{m=0}^{M} g_m$.

We claim that for every i

$$\sum_{j\in I_i\cap I_k} w_j^{-1} \le CN^2(M_{1,4N}g(x))^2, \qquad x \in Q_i. \tag{1.7}$$

To see this, assume that $I_i \cap I_k \neq \emptyset$, and choose an interval J in $[0,N]$ such that $J \times [0,1]$ contains the set $J' = \bigcup_{I_i\cap I_k} Q_j$ and $|J| = 2^m < 2|J'|$.

a) If $u(J) < 2^{-m-1}\alpha_m$, then there exists a unit cube $Q_l \subset J^+_{m,0} \times [0,1]$ such that $w^{-1}(Q_l) \ge u(J)$, since otherwise $u(J^+_{m,0}) < 2^m u(J) < \alpha_m/2$.

Given $x \in Q_i$, we can find a rectangle $R' \in \mathfrak{B}_{1,4N}$ with $x \in R'$ such that $|R' \cap Q_l| \ge \frac{1}{2}$ (see Fig. 1). This implies

$$M_{<4N}g(x) \ge M_{1,4N}g_0(x) \ge \gamma^+_{0,0}\frac{1/2}{4N} \ge C\frac{\alpha_0^{1/2}}{N},$$

hence

$$\sum_{j\in I_i\cap I_k} w_j^{-1} \le u(J) < 2^{-m-1}\alpha_m \le \alpha_0/2 \le CN^2(M_{<4N}g(x))^2.$$

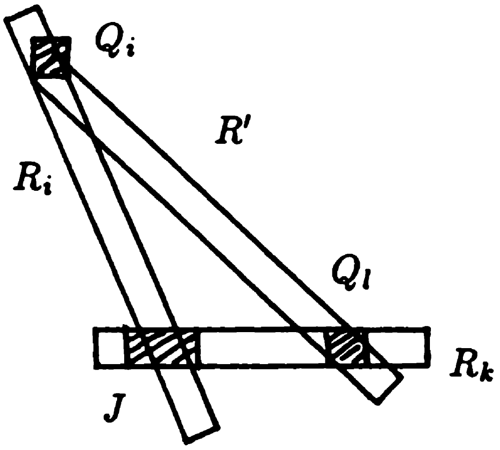

Fig. 1

b) Assume now that $u(J) \in]2^{-j-1}\alpha_m, 2^{-j}\alpha_m]$ for some $j \in \{0,\ldots,m\}$, i.e. $J \in \Omega_{m,j}$. Let $x \in Q_i$. Then J lies between $J^-_{m,j}$ and $J^+_{m,j}$, and assume for instance that $\operatorname{dist}(Q_i, R^+_{m,j}) \geq \operatorname{dist}(Q_i, R^-_{m,j})$.

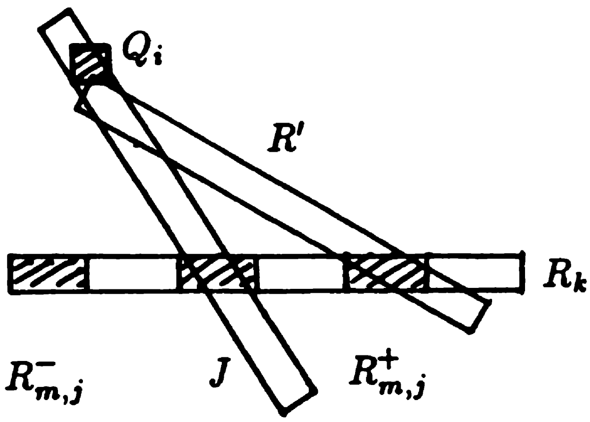

Fig. 2

Then elementary geometric considerations show that there exists a rectangle $R' \in \mathfrak{B}_{1,4N}$ containing x such that $|R' \cap R^+_{m,j}| \geq |R^+_{m,j}|/10 = 2^m/10$ (see Fig. 2). This implies

$$\begin{aligned} M_{<4N}g(x) &\geq M_{1,4N}(\gamma^+_{m,j}\chi_{R^+_{m,j}})(x) \\ &\geq \gamma^+_{m,j}\frac{2^m/10}{4N} \geq C\frac{(2^{-j}\alpha_m)^{1/2}}{N} \geq C\frac{u(J)^{1/2}}{N}, \end{aligned}$$

hence

$$\sum_{j \in I_i \cap I_k} w_j^{-1} \leq u(J) \leq CN^2(M_{<4N}g(x))^2,$$

and (1.7) is proved.

From (1.7) we get

$$(1.8)\qquad \sum_i w_i \sum_{j\in I_i\cap I_k} w_j^{-1} \le CN^2 \sum_i \int_{Q_i} (\min_{Q_i}(M_{<4N}g)^2) w\,dx \le CN^2 \int_{\mathbb{R}^2} (M_{<4N}g)^2 w\,dx.$$

Now, by (1.3′),

$$\int (M_{<4N}g)^2 w\,dx \le A\|g\|^2_{L^2(w\,dx)},$$

and

$$\|g\|_{L^2(w\,dx)} \le \sum_{m=0}^{M} \sum_{j\in A_m} \sum_{\varepsilon=\pm 1} \|\gamma^{\varepsilon}_{m,j}\chi_{R^{\varepsilon}_{m,j}}\|_{L^2(w\,dx)}.$$

But, by Lemma 1.1(i),

$$\|\gamma^{\pm}_{m,j}\chi_{R^{\pm}_{m,j}}\|^2_{L^2(w\,dx)} = (\gamma^{\pm}_{m,j})^2 w(R^{\pm}_{m,j}) = 2^{-2m} w^{-1}(R^{\pm}_{m,j}) w(R^{\pm}_{m,j}) \le A,$$

and thus

$$\|g\|_{L^2(w\,dx)} \le CM^2A^{1/2},$$

hence

$$\sum_i w_i \sum_{j\in I_i\cap I_k} w_j^{-1} \le CA^2N^2(\log 2N)^4. \ \blacksquare$$

Let us remark that the power A^9 in (1.4) is most likely not best possible, but it is certainly sufficient for our purposes.

We now turn to a converse of Prop. 1.2.

PROPOSITION 1.3. *Assume that w is a weight with the following properties:*

(i) $w_{Q_i}(w^{-1})_{Q_i} \le A$ *for all cubes* Q_i.

(ii) *For every choice of rectangles* $R_i \in \mathfrak{B}_{1,N}$ *with* $R_i \cap Q_i \ne \emptyset$ *one has*

$$\sum_i w_i \sum_{j\in I_i\cap I_k} w_j^{-1} \le B$$

for every k, where B does not depend on the choice of R_i's.

Then

$$(1.9)\qquad \int_{\mathbb{R}^2} (M_{1,N}f)^2 w\,dx \le \frac{AB}{N^2} \int_{\mathbb{R}^2} |f|^2 w\,dx$$

for every $f \in L^2(w\,dx)$.

P r o o f. By the usual linearization method (see [Co 1] or [Mü]), it suffices to prove that

$$\int (\tilde{T}g)^2 w\, dx \leq \frac{AB}{N^2} \int g^2 w\, dx$$

for every $g \geq 0$ in $L^2(w\, dx)$, where $\tilde{T}$ is the linearization

$$\tilde{T}g := \sum_i \frac{1}{|R_i|} \Big(\int_{R_i} g\, dx \Big) \chi_{Q_i}$$

of $M_{1,N}$ associated with an arbitrary system of rectangles $\{R_i\}$ as in (ii). Since $|R_i| = N$ for every i, we set

$$Tg := \sum_i \Big(\int_{R_i} g\, dx \Big) \chi_{Q_i},$$

and have to show that $\|Tg\|^2_{L^2(w\, dx)} \leq AB \|g\|^2_{L^2(w\, dx)}$.

To this end, let $\omega = \{w_i\}_i$, let l^2_ω be the Hilbert space $l^2_\omega := \{ \{x_j\}_j \in \mathbf{R}^I : \|\{x_j\}\|^2_{l^2_\omega} = \sum_{j \in I} |x_j|^2 w_j < \infty \}$, and identify any infinite matrix $U = (u_{ij})_{ij}$ with the kernel operator $U(x)_i := \sum_j u_{ij} x_j w_j$. Then we have

$$\|Tg\|^2_{L^2(w\, dx)} = \sum_i \Big(\int_{R_i} g\, dx \Big)^2 w_i \tag{1.10}$$

$$\leq \sum_i \Big(\sum_{j \in I_i} y_j \Big)^2 w_i = \|U(y)\|^2_{l^2_\omega},$$

where $y_j := \int_{Q_j} g\, dx$ and $u_{ij} = s_{ij} w_j^{-1}$, with

$$s_{ij} = \begin{cases} 1 & \text{if } j \in I_i, \\ 0 & \text{if } j \notin I_i. \end{cases}$$

Then the adjoint operator U^* of U is given by the kernel $U^* = (u_{ji})_{ij}$, so that the kernel of UU^* is

$$(UU^*)_{ik} = \sum_j u_{ij} u_{kj} w_j = \sum_j s_{ij} w_j^{-1} s_{kj} w_j^{-1} w_j = \sum_{j \in I_i \cap I_k} w_j^{-1}.$$

Since UU^* is symmetric, by Young's inequality this implies

$$\|U\|^2 = \|UU^*\| \leq \sup_i \sum_k \Big(\sum_{j \in I_i \cap I_k} w_j^{-1} \Big) w_k \leq B.$$

Together with (1.10) we therefore obtain

$$\|Tg\|^2_{L^2(w\, dx)} \leq B \|y\|^2_{l^2_\omega}.$$

Moreover, since

$$\Big(\int_{Q_i} g\,dx\Big)^2 = \Big(\int_{Q_i} (gw^{1/2})w^{-1/2}\,dx\Big)^2 \le \int_{Q_i} w^{-1}\,dx \int_{Q_i} g^2 w\,dx,$$

we finally obtain, invoking also (i),

$$\|Tg\|^2_{L^2(w\,dx)} \le B\sum_i \Big(\int_{Q_i} g\,dx\Big)^2 w_i \le B\sum_i (w^{-1})_i w_i \int_{Q_i} g^2 w\,dx$$

$$\le AB\sum_i \int_{Q_i} g^2 w\,dx = AB\|g\|^2_{L^2(w\,dx)}\,. \blacksquare$$

DEFINITION 1.4. We shall denote by $A_2^{\mathfrak{K}} = A_2^{\mathfrak{K}}(\mathbf{R}^2)$ the convex cone of all weights w on $\mathbf{R}^2$ for which there are constants $A \ge 0$ and $\alpha \ge 0$ such that

$$\int_{\mathbf{R}^2} (M_{<N}f)^2 w\,dx \le A(\log 2N)^\alpha \int_{\mathbf{R}^2} |f|^2 w\,dx \tag{1.11}$$

for every $f \in L^2(w\,dx)$ and every $N \ge 1$.

Notice that because of (1.2) we could also replace $M_{<N}$ by $M_{1,N}$ in (1.11).

The following main result, which gives a geometric characterization of $A_2^{\mathfrak{K}}$-weights, follows immediately from (1.2) and the preceding propositions.

THEOREM 1.5. *Let w be a weight on $\mathbf{R}^2$. Then $w \in A_2^{\mathfrak{K}}$ if and only if there are constants $B, \beta \ge 0$ such that the following hold:*

(i) *$w_Q(w^{-1})_Q \le B$ for all cubes $Q \in \mathfrak{B}_{1,1}$.*

(ii) *For every $N \ge 1$ and any choice of rectangles $R_i \in \mathfrak{B}_{1,N}$ such that $R_i \cap Q_i \ne \emptyset$ one has*

$$\sum_i w_i \sum_{j\in I_i\cap I_k} w_j^{-1} \le BN^2(\log 2N)^\beta, \tag{1.12}$$

for every k.

2. The maximal operator $\mathfrak{M}_{<N}$. In this section, we shall consider maximal functions where only the eccentricity N is kept fixed, but δ varies. So, let

$$\mathfrak{M}_N f(x) := \sup_{\delta>0} M_{\delta,N} f(x) = \sup_{x\in R} \frac{1}{|R|}\int_R |f(y)|\,dy,$$

where the supremum is taken over all rectangles in $\mathfrak{B}_N = \bigcup_{\delta>0} \mathfrak{B}_{\delta,N}$ containing x. Also, define

$$\mathfrak{M}_{<N} f := \sup_{1\le r\le N} \mathfrak{M}_r f.$$

DEFINITION 2.1. $\mathfrak{A}_2^{\mathfrak{R}} = \mathfrak{A}_2^{\mathfrak{R}}(\mathbf{R}^2)$ will denote the convex cone of all weights w on $\mathbf{R}^2$ for which the analogue of (1.11), with $M_{<N}$ replaced by $\mathfrak{M}_{<N}$ (or, equivalently, by $\mathfrak{M}_N$), holds.

Since the family $\mathfrak{B}_N$ is scale-invariant, it is easy to see that also $\mathfrak{A}_2^{\mathfrak{R}}$ is scale-invariant, i.e. if $w \in \mathfrak{A}_2^{\mathfrak{R}}$, then also $w(r\cdot) \in \mathfrak{A}_2^{\mathfrak{R}}$ for every $r > 0$, with constants A and α in (1.11) not depending on r. Moreover, since $M_{<N}$ is dominated by $\mathfrak{M}_{<N}$, $\mathfrak{A}_2^{\mathfrak{R}} \subset A_2^{\mathfrak{R}}$. These observations show that, if $w \in \mathfrak{A}_2^{\mathfrak{R}}$, then $w(r\cdot) \in A_2^{\mathfrak{R}}$, "uniformly" for every $r < 0$. In fact, also the converse holds:

THEOREM 2.2. *Let w be a weight on $\mathbf{R}^2$. Then $w \in \mathfrak{A}_2^{\mathfrak{R}}$ if and only if $w(r\cdot) \in A_2^{\mathfrak{R}}$, uniformly for every $r > 0$, i.e. if and only if* (i) *and* (ii) *in Theorem* 1.5 *hold for $w(r\cdot)$ for every $r > 0$, with constants B and β independent of r.*

Remark 2.3. Observe that $\mathfrak{M}_N$ is comparable to the Hardy–Littlewood maximal operator HL as follows:

$$\frac{1}{N} HL(f) \leq \mathfrak{M}_N(f) \leq N\, HL(f). \tag{2.1}$$

This clearly implies that $\mathfrak{A}_2^{\mathfrak{R}}$ is contained in the classical Muckenhoupt class A_2. Correspondingly, Muckenhoupt's A_2-condition is equivalent to the condition that condition (i) in Th. 1.5 holds uniformly for all weights $w(r\cdot)$, $r > 0$.

Proof of Theorem 2.2. Let w be a weight, and assume that

$$\int_{\mathbf{R}^2} (M_{<N} f)^2 w(r\cdot)\, dx \leq A(\log 2N)^{\alpha} \int_{\mathbf{R}^2} |f|^2 w(r\cdot)\, dx \tag{2.2}$$

for every $f \in L^2(w(r\cdot)\, dx)$, $N \geq 1$ and $r > 0$. We shall prove that this implies $w \in \mathfrak{A}_2^{\mathfrak{R}}$. The remaining statements in the theorem will then follow from the preceding discussion and Th. 1.5. So we have to prove that

$$\int_{\mathbf{R}^2} (\mathfrak{M}_N f)^2 w\, dx \leq B(\log 2N)^{\beta} \int_{\mathbf{R}^2} |f|^2 w\, dx, \tag{2.3}$$

for some constants $B, \beta \geq 0$. To this end, we adopt some ideas from [Co 1]. First, as in [Co 1], we need only consider rectangles R with direction in the set $\{ 2\pi k/N : k = 1, \ldots, N \}$, and with δ of the form 2^n, $n \in \mathbf{Z}$, in order to form $\mathfrak{M}_N$. We can also assume that N is of the form $N = 2^M$, with $M \in \mathbf{N}$.

Then

$$\mathfrak{M}_N f \sim \sup_{n\in\mathbf{Z}} M_{2^n,2^M} f$$

$$\sim \sup_{j=0,\dots,M-1} (\sup_{k\in\mathbf{Z}} M_{2^{kM+j},2^M} f) \leq \sum_{j=0}^{M-1} T_j f,$$

where $T_j f = \sup_{k\in\mathbf{Z}} M_{2^{kM+j},2^M} f$. This shows that it suffices to prove that

$$\int_{\mathbf{R}^2} (T_j f)^2 w\, dx \leq B(\log 2N)^\beta \int_{\mathbf{R}^2} |f|^2 w\, dx, \tag{2.4}$$

for $j = 0, \dots, M-1$. We shall do this for T_0, the proof for the other T_j's being identical.

T_0 is the maximal function associated to the family of rectangles $\mathfrak{B} = \bigcup_{k\in\mathbf{Z}} \mathfrak{B}_k$, where $\mathfrak{B}_k$ denotes the set of all rectangles of dimension $N^k \times N^{k+1}$. This family has the following property: If $R \in \mathfrak{B}_j$ and $R' \in \mathfrak{B}_k$, and if $R \cap R' \neq \emptyset$ and $k < j$, then $R' \subset \widetilde{R}$.

Now, given $\lambda > 0$, let

$$E_\lambda := \{\, x : T_0 f(x) \geq 4\lambda \,\}.$$

By the standard covering lemma and the sieve technique described in [Co 1], one can devise a sequence $\mathcal{C}$ of rectangles in $\mathfrak{B}$ with the following properties:

(1°) $|R|^{-1} \int_R |f(y)|\, dy \geq \lambda$ for every $R \in \mathcal{C}$;
(2°) $E_\lambda \subset \bigcup_{R\in\mathcal{C}} \widetilde{R}$;
(3°) $R' \cap R = \emptyset$ whenever $R' \in \mathcal{C} \cap \mathfrak{B}_k$ and $R \in \bigcup_{j>k}(\mathcal{C} \cap \mathfrak{B}_j)$.

Notice that $\mathcal{C} \cap \mathfrak{B}_k = \emptyset$ for k sufficiently large. This is due to the fact that $\mathfrak{M}_N$ is bounded by a multiple of the Hardy–Littlewood maximal function, and that w is in the classical Muckenhoupt A_2-class (see Remark 2.3).

Set

$$E_k := \bigcup_{R\in\mathcal{C}\cap\mathfrak{B}_k} R, \qquad \widetilde{E}_k := \bigcup_{R\in\mathcal{C}\cap\mathfrak{B}_k} \widetilde{R}.$$

Then we know by (2°) that $E_\lambda \subset \bigcup_k \widetilde{E}_k$. Moreover, by (3°) and the property of $\mathfrak{B}$ described before, $E_j \cap E_k = \emptyset$ if $j \neq k$.

Let $f_k := f|_{E_k}$. Then, if $x \in \widetilde{E}_k$, there exists some $R \in \mathcal{C} \cap \mathfrak{B}_k$ such that $x \in \widetilde{R}$, and so

$$M_{4N^k,N} f_k(x) \geq \frac{1}{|\widetilde{R}|} \int_R |f_k(y)|\, dy \geq \frac{1}{16|R|} \int_R |f(y)|\, dy \geq \frac{\lambda}{16}.$$

Moreover, from (2.2) one obtains by scaling

$$\|M_{4N^k,N} f_k\|^2_{L^2(w\,dx)} \leq A(\log 2N)^\alpha \|f_k\|^2_{L^2(w\,dx)}\,.$$

These estimates imply

$$\begin{aligned} w(\widetilde{E}_k) &\le w(\{ x : M_{4N^k,N} f_k \ge \lambda/16 \}) \\ &\le CA(\log 2N)^\alpha \|f_k\|^2_{L^2(w\,dx)}/\lambda^2. \end{aligned}$$

But then

$$\begin{aligned} w(E_\lambda) &\le \sum_k w(\widetilde{E}_k) \le CA(\log 2N)^\alpha \frac{1}{\lambda^2} \sum_k \|f_k\|^2_{L^2(w\,dx)} \\ &\le CA(\log 2N)^\alpha \|f\|^2_{L^2(w\,dx)}/\lambda^2. \end{aligned}$$

This shows that T_0 is of weak type (2, 2) on $L^2(w\,dx)$, with norm bounded by $CA(\log 2N)^\alpha$.

Finally, by an application of the interpolation theorems of Marcinkiewicz and Riesz–Thorin, one concludes that T_0 is even of strong type on $L^2(w\,dx)$, with norm bounded by $C'A(\log 2N)^{\alpha+1}$. ∎

3. A comparison between $\mathfrak{A}_2^{\mathfrak{R}}$ and "A_1 on lines". In this section, we shall discuss the geometric conditions (i) and (ii) in Th. 1.5, which we shall refer to as the *$A_2^{\mathfrak{R}}$-condition* (the corresponding scale-invariant condition will be called the *$\mathfrak{A}_2^{\mathfrak{R}}$-condition*).

LEMMA 3.1. *Assume w is a weight on $\mathbf{R}^2$ such that*

$$\frac{1}{|I_R|} \sum_{i \in I_R} w_i \le A \min_{i \in I_R} w_i \tag{3.1}$$

for every rectangle $R \in \bigcup_{1 \le r \le 4N} \mathfrak{B}_{1,r}$, where $I_R = \{ i : Q_i \cap R \ne \emptyset \}$. Then

$$\sum_i w_i \sum_{j \in I_i \cap I_k} w_j^{-1} \le CA^2 N^2 \log 2N \tag{3.2}$$

for every collection of rectangles $R_i \in \mathfrak{B}_{1,N}$ with $R_i \cap Q_i \ne \emptyset$ and every k.

Proof. Fix R_k. For simplicity, we shall assume that $R_k = [0, N] \times [0, 1]$, the proof for general R_k being essentially the same. Let us introduce a more specific notation here, and denote by Q_{ij} the unit cube with lower left vertex $(i, j) \in \mathbf{Z}^2$. Correspondingly, set $w_{ij} = w_{Q_{ij}}$. Moreover, since I_{ij} (the index set corresponding to the rectangle R_{ij} in our collection associated to Q_{ij}) meets I_k only if (i, j) lies in the region $\Omega = \{-N-1 \le i \le 2N+1, -N-1 \le j \le N+2\}$, *all* pairs (i, j) in the arguments which follow will be meant to lie in this region.

Then $I_k = \bigcup_{i=0}^{N-1} Q_{i0}$. If there is some i' with $w_{i'0} = 0$, then (3.1) implies that $w_{ij} = 0$ for all $(i, j) \in \Omega$, and (3.2) is trivial. Therefore, we may assume without restriction that

$$\min_i w_{i0} = 1.$$

Let $u_j = \min_i w_{ij}$ be the minimum over the jth row in the region Ω. Then, by (3.1),

$$\sum_i w_{ij} \le 4ANu_j,$$

and so

$$\text{(3.3)} \quad \sum_i w_{ij} \sum_{(i',j')\in I_{ij}\cap I_k} w_{i'j'}^{-1} \le \sum_i w_{ij}|I_{ij}\cap I_k| \le C' \sum_i w_{ij} \frac{N}{1+|j|}$$
$$\le CAN^2 \frac{u_j}{1+|j|},$$

since elementary geometric considerations show that

$$|I_{ij}\cap I_k| \le C\frac{N}{1+|j|}.$$

(3.3) implies

$$\text{(3.4)} \qquad \sum_l w_l \sum_{m\in I_l\cap I_k} w_m^{-1} \le CAN^2 \sum_{j=-N-1}^{N+2} \frac{u_j}{1+|j|}.$$

Choose i_0 such that $w_{i_0 0} = u_0 = 1$. Then, by (3.1), for every $0 \le r \le N+2$,

$$\sum_{j=0}^{r} u_j \le \sum_{j=0}^{r} w_{i_0 j} \le A(r+1)w_{i_0 0} = A(r+1),$$

and we obtain

$$\sum_{j=0}^{N+2} \frac{u_j}{1+j} = \sum_{l=0}^{N+2}\Big(\sum_{j'=0}^{l} u_{j'}\Big)\left(\frac{1}{1+l} - \frac{1}{2+l}\right) + \frac{1}{N+4}\sum_{j=0}^{N+2} u_j$$
$$\le A\sum_{l=0}^{N+2}(l+1)\left(\frac{1}{l+1} - \frac{1}{l+2}\right) + Au_0$$
$$\le A\sum_{l=0}^{N+2}\frac{1}{l+1} + A \le 2A\log(N+2).$$

We can estimate $\sum_{j=-N-1}^{-1} u_j/(1+|j|)$ similarly, and therefore obtain from (3.4)

$$\sum_l w_l \sum_{m\in I_l\cap I_k} w_m^{-1} \le CA^2N^2(\log 2N). \ \blacksquare$$

COROLLARY 3.2. *Let w be a weight on $\mathbf{R}^2$.*

(i) *If there are constants $A, \alpha \ge 0$ such that*

$$\text{(3.5.a)} \qquad w_Q(w^{-1})_Q \le A \quad \textit{for all cubes } Q \in \mathfrak{B}_{1,1}$$

and

$$w_R \le A(\log 2N)^\alpha \min_{i\in I_R} w_i \tag{3.5.b}$$

for every $N \ge 1$ and every rectangle $R \in \mathfrak{B}_{1,N}$, then $w \in A_2^{\mathfrak{R}}$.

(ii) *If* (3.5.a) *and* (3.5.b) *hold uniformly for every weight $w(r\cdot)$, $r > 0$, then $w \in \mathfrak{A}_2^{\mathfrak{R}}$. This is in particular true if $w_R \le A\,\mathrm{ess\,inf}_R w$ for every rectangle R in $\mathbf{R}^2$.*

Proof. By Th. 1.5 and Lemma 3.1 we only have to show that (3.1) holds, with $A = A(N) \le C(\log 2N)^\alpha$. But, by (3.5.b), if $R \in \mathfrak{B}_{1,N}$,

$$\frac{1}{|I_R|}\sum_{i\in I_R} w_i \le 4w_{\tilde R} \le 10A(\log 2N)^\alpha \min_{i\in I_{\tilde R}} w_i \le 10A(\log 2N)^\alpha \min_{i\in I_R} w_i. \blacksquare$$

Remark 3.3. The condition $w_R \le A\,\mathrm{ess\,inf}_R w$ for all rectangles R in $\mathbf{R}^2$ is equivalent to the following condition ("A_1 on lines"):

(3.6) *For every rotation $\sigma \in SO_2$, the weight $w_\sigma^x(y) := w(\sigma(x,y))$ is in the Muckenhoupt class $A_1(\mathbf{R})$, uniformly for a.e. $x \in \mathbf{R}$ and every σ.*

In fact, if this condition holds, then a two-fold application of Fubini's theorem shows that there is a constant $A > 0$ such that $w_R \le Aw_Q$ for every rectangle R and every cube $Q \subset R$ with sides parallel to the sides of R. But, for instance by Lebesgue's differentiation theorem,

$$\operatorname*{ess\,inf}_R w = \inf_{Q\subset R} w_Q\ ,$$

where the infimum is taken over all cubes Q described before. So we get $w_R \le A\,\mathrm{ess\,inf}_R w$. The proof of the converse follows by similar arguments and is left to the reader.

It would be interesting to know whether the rather unhandy condition (1.12) in Th. 1.5 is in fact equivalent to the much simpler condition (3.5.b). Until now, we have not been able to prove this. However, as the next result shows, it is at least true for radial, monotonous weights. Let $|x|$ denote the Euclidean norm on $\mathbf{R}^2$.

THEOREM 3.4. *Assume that $w = w(|x|)$ is a radial, monotonous weight. Then*

(i) *$w \in A_2^{\mathfrak{R}}$ if and only if there are constants $A, \alpha \ge 0$ such that*

$$\frac{1}{N}\int_a^{a+N} w(t)\,dt \le A(\log 2N)^\alpha \operatorname*{ess\,inf}_{[a,a+N]} w \tag{3.7.a}$$

for every $a \geq 1$, $N \geq 1$, *and*

$$\int_0^1 w(t)t\,dt \int_0^1 w(t)^{-1}t\,dt \leq A; \tag{3.7.b}$$

(ii) $w \in \mathfrak{A}_2^{\mathfrak{R}}$ *if and only if*

$$\frac{1}{N}\int_a^{a+N} w(t)\,dt \leq A(\log(2+N/a))^{\alpha} \operatorname*{ess\,inf}_{[a,a+N]} w \tag{3.8.a}$$

for every $a \geq 0$, $N \geq 0$, *and*

$$\int_0^r w(t)t\,dt \int_0^r w(t)^{-1}t\,dt \leq Ar^4 \tag{3.8.b}$$

for every $r > 0$.

Remark 3.5. As one can easily see, condition (3.8.b) holds automatically if one can choose $\alpha = 0$ in (3.8.a).

With regard to Th. 2.2, it is easy to see that part (ii) in Th. 3.4 follows from part (i) by scaling. And part (i) is a straightforward consequence of Th. 1.5, Corollary 3.2 and the lemmas to follow.

LEMMA 3.6. *Let* w *be as in Th.* 3.4. *Then condition* (3.7.a) *is equivalent to condition* (3.5.b).

Proof. One direction is clear. So, assume conversely that (3.7.a) holds, and let us derive (3.5.b). We assume w non-increasing (the other case can be handled analogously).

Let $R \in \mathfrak{B}_{1,N}$ be without restriction of the form $R = [a, a+1]\times[b-N, b]$, with $a \geq 0$, $b \geq 0$.

If $b \geq N$, we set $\xi = (a,b)$, $\xi' = |\xi|^{-1}\xi$. Then one checks that

$$|\xi - t\xi'| \leq |\xi - (0,t)|, \qquad 0 \leq t \leq N,$$

hence

$$w(a, b-t) \leq w(\xi - t\xi') = w(|\xi| - t).$$

This implies

$$w_R \leq \frac{1}{N}\int_0^N w(a, b-t)\,dt \leq \frac{1}{N}\int_0^N w(|\xi| - t)\,dt$$
$$\leq A(\log 2N)^{\alpha} \operatorname*{ess\,inf}_{[|\xi|-N,|\xi|]} w.$$

Moreover, (3.7.a) also implies that $w(t) \sim w(t')$ if $t, t' \geq 1$ and $|t - t'| < 10$. Then we obtain (3.5.b).

If $0 < b < N$, one can reduce matters to the previous case by splitting R into two rectangles along the first coordinate axis and applying the above estimate to both pieces. ∎

LEMMA 3.7. *Assume that w is radial, monotonous, and satisfies*

$$\sum_i w_i \sum_{j\in I_i\cap I_k} w_j^{-1} \leq Ar^2(\log 2r)^\alpha$$

for every $1 \leq r \leq 2N$ and every choice of rectangles $R_i \in \mathfrak{B}_{p,1}$ with $R_i \cap Q_i \neq \emptyset$. Then

$$w_R \leq CA(\log 2N)^\alpha \min_{i\in I_R} w_i$$

for every rectangle $R \in \mathfrak{B}_{1,N}$.

Proof. Choose $R \in \mathfrak{B}_{1,N}$. An adaptation of the argument in the proof of Lemma 3.6 shows that we may assume w.r. that the longer axis of R points towards the origin. So, let us assume that R has the form

$$R = [a, a+N] \times [0,1], \qquad a > 0,\ a \in \mathbf{N}.$$

As in the proof of Lemma 3.1, we denote by Q_{ij} the unit cube with lower left vertex (i,j).

(a) If w is non-increasing, choose a constellation of rectangles $\{R_{ij}\} \subset \mathfrak{B}_{1,2N}$ with $R_{ij} \cap Q_{ij} \neq \emptyset$ such that $R_{i0} = [a, a+2N] \times [0,1]$ for $i = a, \ldots, a+2N$. Then

$$\begin{aligned}\sum_{(i,j)} w_{ij} \sum_{(k,l)\in I_{ij}\cap I_{a0}} w_{kl}^{-1} &\geq \sum_{i=a}^{a+N-1} w_{i0} \sum_{k=a+N+1}^{a+2N} w_{k0}^{-1} \\ &\geq \Big(\int_R w\, dx\Big) N w_{a+N+1,0}\,,\end{aligned}$$

since, by monotony, $w_{k0} \leq w_{a+N+1,0}$ for $k \geq a+N+1$. By our assumption, this implies

$$w_R \leq 4A(\log 4N)^\alpha w_{a+N+1,0} \leq 4A(\log 4N)^\alpha \min_{i\in I_R} w_i\,.$$

(b) If w is non-decreasing, choose $\{R_{ij}\} \subset \mathfrak{B}_{1,2N}$ such that R_{ij} meets Q_{a0} for $a \leq i \leq a+N$ and $0 \leq j \leq N$, and such that $R_{a0} \supset R$ (see Fig. 3). Then, by monotony,

$$\begin{aligned}\sum_{(i,j)} w_{ij} \sum_{(k,l)\in I_{ij}\cap I_{a0}} w_{kl}^{-1} &\geq \sum_{\substack{i=a,\ldots,a+N\\ j=0,\ldots,N}} w_{ij} w_{a-1,0}^{-1} \\ &\geq \sum_{\substack{j=a,\ldots,a+N\\ j=0,\ldots,N}} w_{i0} w_{a-1,0}^{-1} \geq N\Big(\int_R w\, dx\Big) w_{a-1,0}^{-1}\,,\end{aligned}$$

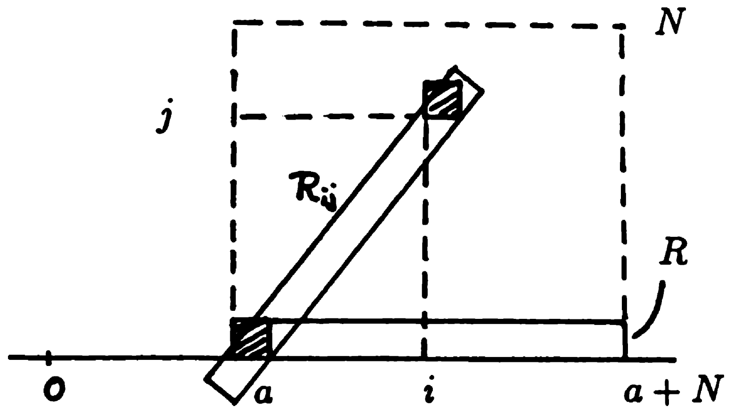

Fig. 3

and our assumption implies

$$w_R \leq 4A(\log 4N)^\alpha w_{a-1,0} = 4A(\log 4N)^\alpha \min_{i\in I_R} w_i \ . \ \blacksquare$$

Notice that our arguments have in fact made use of the assumption only for $r = 2N$. We have nevertheless stated the lemma in this form, since, as it stands, it might perhaps be true even without the assumption "w radial and monotonous".

4. Examples, remarks and open questions

4.1. The following results can easily be derived from Th. 3.4:

(a) $w(x) = |x|^\alpha$ is in $\mathfrak{A}_2^{\mathfrak{K}}$ if and only if $-1 \leq \alpha \leq 0$.

(b) If $a > 1$, then $w(x) = \log^\alpha(a + |x|)$ is in $\mathfrak{A}_2^{\mathfrak{K}}$ for every $\alpha \geq 0$.

(c) $w(x) = \log^\alpha(1 + |x|)$ $(\alpha \geq 0)$ is in $\mathfrak{A}_2^{\mathfrak{K}}$ if and only if $\alpha < 2$. In fact, w satisfies (3.8.a) for every $\alpha \geq 0$, but (3.8.b) only if $\alpha < 2$, since $\log^\alpha(1 + t) \sim t^\alpha$ as $t \to 0$.

4.2. $\mathfrak{A}_p^{\mathfrak{K}}$-*theory.* For $1 < p < \infty$, one can define weight spaces $A_p^{\mathfrak{K}}$ respectively $\mathfrak{A}_p^{\mathfrak{K}}$ by replacing the $L^2(w\,dx)$-norm in Def. 1.4 respectively Def. 2.1 by the $L^p(w\,dx)$-norm.

Now, for $p < 2$, there will be no reasonable $\mathfrak{A}_p^{\mathfrak{K}}$-theory. For, if $w = 1$, then $\mathfrak{M}_{<N}$ has an operator norm on $L^p(dx)$ which is at least of order $N^{(2-p)/p}$ (see e.g. [Mü]), and the same is true for arbitrary w, if only w is comparable to a constant weight on some open set.

On the other hand, in view of Th. 1.5 and the classical A_p-condition of Muckenhoupt, one is tempted to *conjecture* that for $p \geq 2$, $w \in A_p^{\mathfrak{K}}$ if and only if there are constants $B, \beta \geq 0$ such that:

(i) $w(Q)(w^{-1/(p-1)}(Q))^{p-1} \leq B$ for all cubes $Q \in \mathfrak{B}_{1,1}$;

(ii) for every $N \geq 1$ and any choice of rectangles $R_i \in \mathfrak{B}_{1,N}$ such that $R_i \cap Q_i \neq \emptyset$ one has

$$\sum_i w_i \Big(\sum_{j \in I_i \cap I_k} w_j^{-1/(p-1)} \Big)^{p-1} \leq BN^p (\log 2N)^\beta. \tag{4.1}$$

In fact, the necessity of these two conditions follows by an easy modification of the proof of Prop. 1.2.

4.3. *Higher dimensions.* There is no obvious reason to assume that the $\mathfrak{A}_p^{\mathfrak{K}}$-condition should depend on the dimension of the space. So, if the conjecture in the previous subsection is true, then most likely the conditions (i) and (ii), in particular (4.1), will also describe the $\mathfrak{A}_p^{\mathfrak{K}}$-weights for the Kakeya maximal operator in $\mathbf{R}^n$, $n \geq 2$.

Indeed, this would match very well with the conjecture that the Kakeya maximal operator on $\mathbf{R}^n$ behaves "well" on L^p if and only if $p \geq n$ (see e.g. [Mü]). For, if $w = 1$, then the left-hand side of (4.1) can always be estimated by

$$C \sum_{d=1}^{N} N d^{n-2} (N/d)^{p-1} \sim N^p \sum_{d=1}^{N} d^{n-p-1},$$

which is of order $N^p(\log 2N)^\beta$ for some $\beta \geq 0$ if and only if $p \geq n$.

4.4. Can one dispense with "w radial and monotonous" in Lemma 3.7, so that $\mathfrak{A}_2^{\mathfrak{K}}$ is in fact equivalent to an "A_1 on lines" condition?

4.5. An important fact in the "classical" A_p-theory is that for every positive Borel measure μ s.t. $HL(\mu) < \infty$ a.e. (HL = Hardy–Littlewood maximal operator) $HL(\mu)^\gamma \in A_1 \subset A_p$ for every $0 < \gamma < 1$ (see [Ga–Ru], p. 158, Th. 3.4). Now, it is certainly not true that $\mathfrak{M}_{<N}(\mu)^\gamma \in \mathfrak{A}_2^{\mathfrak{K}}$ in general; for instance $\mathfrak{M}_{<N}(\chi_{\{|x|\leq 1\}})(y) \sim |y|^{-2}$ for $|y| \geq 2N$ and therefore is not in $\mathfrak{A}_2^{\mathfrak{K}}$. However, it could be true of $\widetilde{\mathfrak{M}}(\mu)^\gamma$, $0 < \gamma \leq 1$, where $\widetilde{\mathfrak{M}}f := \sup_{N \geq 1} \mathfrak{M}_{<N} f$ is the maximal function associated to the system of all rectangles. But the example of a Kakeya set in [Co 1], p. 8 ff. can be used to prove that $\widetilde{\mathfrak{M}}$ is unbounded on every L^p-space, $1 \leq p < \infty$ (and the same holds true of $\widetilde{\mathfrak{M}}_q f := [\widetilde{\mathfrak{M}}(f^q)]^{1/q}$, $q \geq 1$), in contrast to the Hardy–Littlewood maximal operator. Therefore such a result, if true, would probably be of little importance. But there may be other, canonical constructions of $\mathfrak{A}_p^{\mathfrak{K}}$-weights.

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THE INSTITUTE FOR ADVANCED STUDY
SCHOOL OF MATHEMATICS
PRINCETON, NEW JERSEY 08540
U.S.A.

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