THE CONNECTEDNESS OF SYMMETRIC DEGENERACY LOCI: ODD RANKS

Appendix to "The connectedness of degeneracy loci" by Loring W. Tu

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This appendix is in a sense a continuation of [13], in which we proved a conjecture of Fulton and Lazarsfeld ([11, Remark 2, p. 50]) on the connectedness of symmetric and skew-symmetric degeneracy loci, when the rank is even. We now deal with the remaining case of odd-rank symmetric degeneracy loci. We are able to prove the following.

**Theorem.** Let $E$ be a vector bundle of rank $e$ and $L$ a line bundle over an irreducible variety $X$. Suppose $u: E \otimes E \to L$ is a symmetric bundle map and $r$ a positive odd integer $\leq e$. If $(\text{Sym}^2 E^*) \otimes L$ is ample and \( \dim \mathbb{C} X - (e-r^+ 1) \geq e-r \), then the degeneracy locus $D_r(u)$ is connected.

This is not quite the result that we would like, for in the conjecture the dimension hypothesis is that $\dim \mathbb{C} X - (e-r^+ 1) \geq 1$.

The key observation in our proof is the linear-algebra fact that just as the set of symmetric bilinear maps of rank at most an even integer can be characterized by the existence of an isotropic subspace of a suitable dimension, so the symmetric bilinear maps of rank at most an odd integer can be characterized by the existence of a pair of subspaces $V_1 \subset V_2$ of suitable dimensions such that the bilinear map vanishes on $V_1 \times V_2$. Section 7 is devoted to a proof of this characterization. For the proof of the main theorem, we found it useful to introduce for two subspaces $A$, $B$ of a vector space $E$ the concept of symmetric bilinear maps on $A \times B$ and dually that of the symmetric product $\text{Sym}(A, B)$ of $A$ and $B$. This is done in Section 8. Using the linear algebra developed in Sections 7 and 8, it is then possible to represent an odd-rank symmetric degeneracy locus as the image of a zero locus on a flag bundle.
To prove the connectedness of this zero locus, we proceed more or less as in [13], replacing the Grassmann bundle there by the flag bundle. The cohomology comparison lemma ([[13, Lemma 3.6]]) again applies, but curiously the numbers yield only the result above, instead of the full conjecture.

§ 7. Symmetric maps of rank at most an odd integer

Let $E$ be a vector space of dimension $e$ and $\phi: E \times E \to \mathbb{C}$ a symmetric bilinear form. An isotropic subspace of $\phi$ is a subspace $V$ such that $\phi$ vanishes on $V \times V$. Proposition 1.4 says that $\text{rk } \phi \leq 2p$ if and only if $E$ contains such a subspace of dimension $e - p$.

**Proposition 7.1.** The symmetric bilinear form $\phi: E \times E \to \mathbb{C}$ has rank $\leq 2p + 1$ if and only if $E$ contains a pair of subspaces $V_1 \subset V_2$ of dimensions $e - p - 1$ and $e - p$ respectively such that $\phi$ vanishes on $V_1 \times V_2$.

**Proof.** ($\Rightarrow$) If $\text{rk } \phi \leq 2p$, choose $V_2$ to be an isotropic subspace of dimension $e - p$ and $V_1$ to be any subspace of $V_2$ of codimension one. If $\text{rk } \phi = 2p + 1$, then relative to some basis $\{v_1, \ldots, v_e\}$, $\phi$ is represented by the matrix

$$
\begin{bmatrix}
1 & 0_p & I_p \\
0_p & I_p & 0_p \\
I_p & 0_p & 0_{e-2p-1}
\end{bmatrix},
$$

where $0_p$ denotes the $p \times p$ zero matrix, and $I_p$ the $p \times p$ identity matrix. Choose $V_1$ to be the subspace with basis $\{v_{p+1}, \ldots, v_e\}$ and $V_2$ the subspace with basis $\{v_1, v_{p+2}, \ldots, v_e\}$. The matrix of $\phi|_{V_1 \times V_2}$ relative to this basis is

$$
\begin{bmatrix}
1 & 0 \\
0 & 0_{e-p-1}
\end{bmatrix},
$$

which shows that $\phi$ vanishes on $V_1 \times V_2$.

($\Leftarrow$) Suppose $E$ contains such a pair of subspaces $V_1 \subset V_2$. Then $V_1$ is an isotropic subspace of dimension $e - p - 1$, and $\text{rk } (\phi|_{V_1 \times V_2}) \leq 1$. By [12, Prop. 4], whenever the dimension of a subspace drops by 1, the rank of the restriction of a quadratic form to the subspace drops by at most 2, so that $\text{rk } (\phi|_{V_2 \times V_2}) \geq \text{rk } \phi - 2p$. Hence, $\text{rk } \phi \leq 2p + 1$. ■

§ 8. The symmetric product of two subspaces

Given a vector space $E$ and two subspaces $A$ and $B$, we say that a linear map $\phi: A \otimes B \to \mathbb{C}$ is symmetric if it is the restriction of a symmetric linear map: $E \otimes E \to \mathbb{C}$. The space of all symmetric linear maps on $A \otimes B$ is denoted
Hom⁴(\(A \otimes B, C\)). In other words,

\[ \text{Hom}^4(A \otimes B, C) := \text{image}(\text{Sym}^2 E^* \subseteq (E \otimes E)^* \to (A \otimes B)^*). \]

We also define the symmetric product Sym\((A, B)\), a subspace of Sym² \(E\), to be

\[ \text{Sym}(A, B) := \text{image}(A \otimes B \subseteq E \otimes E \xrightarrow{j} \text{Sym}^2 E), \]

where \(j: E \otimes E \to \text{Sym}^2 E\) is the natural projection.

**Proposition 8.1.** The dual of Sym\((A, B)\) is canonically isomorphic to the space of all symmetric linear maps on \(A \otimes B\):

\[ \text{Sym}(A, B)^* \cong \text{Hom}^4(A \otimes B, C). \]

**Proof.** The definition of Sym\((A, B)\) may be rephrased in terms of the following exact commutative diagram:

\[
\begin{array}{ccc}
0 & \to & A \otimes B \to E \otimes E \\
& \downarrow & \downarrow j \\
0 & \to & \text{Sym}(A, B) \to \text{Sym}^2 E \\
& \downarrow & \downarrow \\
& & 0 \\
\end{array}
\]

Dualizing gives

\[
\begin{array}{ccc}
0 & \leftarrow & (A \otimes B)^* \leftarrow (E \otimes E)^* \\
& \uparrow & \uparrow f \\
0 & \leftarrow & \text{Sym}(A, B)^* \leftarrow \text{Sym}^2 E^* \\
& \uparrow & \uparrow \\
& & 0 \\
\end{array}
\]

which shows that

\[ \text{Sym}(A, B)^* = \text{image}(\text{Sym}^2 E^* \subseteq (E \otimes E)^* \to (A \otimes B)^*) \]

\[ = \text{Hom}^4(A \otimes B, C). \]

In general the dimension of \(\text{Sym}(A, B)\) depends on the dimension of \(A \cap B\); in the special case when \(A \subset B\), we have the following formula.

**Lemma.** If \(A \subset B\), then

\[ \dim \text{Sym}(A, B) = (a + 1)a/2 + a(b - a), \]

where \(a = \dim A\) and \(b = \dim B\).

**Proof.** Choose a basis \(v_1, \ldots, v_a\) for \(A\), and extend it to a basis \(v_1, \ldots, v_a, v_{a+1}, \ldots, v_b\) for \(B\). Then a basis for \(A \otimes B\) is

\[ v_i \otimes v_j, \quad 1 \leq i \leq a, 1 \leq j \leq b. \]

The images of these vectors in \(\text{Sym}^2 B\) are

\[ (*) \quad v_i v_j := v_i \otimes v_j + v_j \otimes v_i, \quad 1 \leq i \leq a, 1 \leq j \leq b. \]
Because of redundancies, for example \( v_1 v_2 = v_2 v_1 \), the vectors (*) are obviously not linearly independent in \( \text{Sym}^2 B \), but at least they span \( \text{Sym}(A, B) \). Deleting redundant vectors from (*), we are left with
\[
v_i v_j, \quad 1 \leq i \leq j \leq a,
\]
and
\[
v_i v_j, \quad 1 \leq i \leq a, \quad a + 1 \leq j \leq b,
\]
which are linearly independent in \( \text{Sym}^2 B \). So they form a basis of \( \text{Sym}(A, B) \). Consequently,
\[
\dim \text{Sym}(A, B) = (a + 1)a/2 + a(b - a). \quad \blacksquare
\]

**Proposition 8.2.** Suppose \( A \subset B \). Then there is an exact sequence
\[
0 \to \text{Sym}(A, B) \to \text{Sym}^2 B \to \text{Sym}^2 (B/A) \to 0.
\]

**Proof.** Tensoring the exact sequence
\[
0 \to A \to B \to B/A \to 0
\]
by \( B \) yields the exact sequence
\[
0 \to A \otimes B \to B \otimes B \to (B/A) \otimes B \to 0,
\]
which fits into the commutative diagram
\[
\begin{array}{cccccc}
0 & \to & A \otimes B & \to & B \otimes B & \to & (B/A) \otimes B & \to & 0, \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \text{Sym}(A, B) & \to & \text{Sym}^2 B & \to & \text{Sym}^2 (B/A) & \to & 0.
\end{array}
\]
A little diagram-chasing shows that \( \text{Sym}(A, B) \) is contained in the kernel of the natural surjection \( \alpha : \text{Sym}^2 B \to \text{Sym}^2 (B/A) \). Since
\[
\dim \ker \alpha = \frac{1}{2} (b + 1)b - \frac{1}{2} (b - a + 1)(b - a)
= \frac{1}{2} (a + 1)a + a(b - a) = \dim \text{Sym}(A, B),
\]
the two spaces \( \text{Sym}(A, B) \) and \( \ker \alpha \) are actually equal. This proves the exactness of the sequence in the proposition. \( \blacksquare \)

**Proposition 8.3.** Suppose \( A \subset B \) and \( \phi \in \text{Sym}^2 B \). Let \( \tilde{\phi} : B^* \otimes B^* \to C \) be the symmetric linear map associated to \( \phi \). Then \( \phi \) lies in \( \text{Sym}(A, B) \) if and only if \( (B/A)^* \) is an isotropic subspace of \( \tilde{\phi} \).

**Proof.** By the exact sequence of Proposition 8.2, an element \( \phi \) of \( \text{Sym}^2 B \) lies in \( \text{Sym}(A, B) \) if and only if its image in \( \text{Sym}^2 (B/A) \) is zero if and only if it is zero as a symmetric map: \( (B/A)^* \times (B/A)^* \to C \) if and only if \( (B/A)^* \) is an isotropic subspace of \( \phi \). \( \blacksquare \)

If \( E \) is a vector space of dimension \( e \) and \( \psi : E \times E \to C \) is a symmetric bilinear map, we define the isotropic Grassmannian \( G_\psi(k, E) \), sometimes written \( G_\psi(k, e) \), to be
\[
G_\psi(k, E) := \{ V \subset G(k, E) | V \text{ is an isotropic subspace of } \psi \}.
\]
Note that for $\dim \mathbb{C} E \geq 2$, $G_{\psi}(1, E)$ is precisely the quadric defined by $\psi$ in the projective space $\mathbb{P}(E)$.

**Proposition 8.4.** Let $B$ be a vector space of dimension $b$, $\phi \in \text{Sym}^2 B$, and $\tilde{\phi}: B^* \otimes B^* \to \mathbb{C}$ the symmetric linear map associated to $\phi$. Then the variety $W$ of all $a$-dimensional subspaces $A$ of $B$ such that $\phi \in \text{Sym}(A, B)$ is isomorphic to the isotropic Grassmannian $G_\mathbb{A}(b-a, B^*)$.

**Proof.** First observe that every subspace of $B^*$ is of the form $(B/A)^*$ for some subspace $A$ of $B$. By Proposition 8.3, the map: $W \to G_\mathbb{A}(b-a, B^*)$ defined by $A \mapsto (B/A)^*$ is an isomorphism.

§ 9. A flag bundle construction

We now begin the proof of the main theorem, assuming $r$ to be an odd integer, say $2p + 1$. By the argument of [13, Section 5], we may take $X$ to be a smooth irreducible projective variety and $L$ to be the trivial line bundle over $X$.

Using the characterization in Proposition 7.1 of symmetric maps of rank at most an odd integer, one can represent an odd-rank symmetric degeneracy locus as the image of a zero locus on a flag bundle, as follows. If $V$ is a vector space of dimension $e$, let $F(a_1, a_2, V)$ be the flag manifold

$$\{V_1 \subset V_2 \subset V | \dim \mathbb{C} V_i = a_i\}.$$ 

The dimension of this flag manifold is easily shown to be

$$(9.1) \quad a_1(a_2 - a_1) + a_2(e - a_2).$$

Now let $E \to X$ be a vector bundle of rank $e$, and let $\pi: F(e-p-1, e-p, E) \to X$ be its associated flag bundle. Over $F := F(e-p-1, e-p, E)$ there are two universal subbundles $S_1$ and $S_2$ of ranks $e-p-1$ and $e-p$ respectively. By the construction of Section 8, $\text{Sym}(S_1, S_2)$ is a subbundle of $\pi^* \text{Sym}^2 E$ and therefore, $\text{Sym}(S_1, S_2)^*$ is a quotient bundle of $\pi^* \text{Sym}^2 E^*$. The section $u$ of $\text{Sym}^2 E^*$ pulls back under $\pi$ to a section $\pi^* u$ of $\pi^* \text{Sym}^2 E^*$ over $f$, which in turn projects to a section $t$ of $\text{Sym}(S_1, S_2)^*$:

$$t(x, V_1 \subset V_2 \subset E_x) = u(x)_{V_1 \times V_2}.$$ 

By Proposition 7.1, $\pi$ maps the zero locus $Z(t)$ in $F$ surjectively onto the degeneracy locus $D_{2p+1}(u)$ in $X$. Hence, it suffices to prove the connectedness of $Z(t)$. This we do by following the same strategy as in Section 5.

Let $P = \mathbb{P}(\text{Sym}(S_1, S_2))$ and $P' = \mathbb{P}(\text{Sym}^2 E)$. There is a natural map $h: P \to P'$ defined by

$$h(x, V_1 \subset V_2 \subset E_x, \phi \in \text{Sym}(V_1, V_2)) = (x, \phi \in \text{Sym}^2 E_x).$$
We then have the diagram

\[(9.2)\]

\[
\begin{array}{ccc}
P-Z(t^*) \subset P = P(\text{Sym}(S_1, S_2)) & \xrightarrow{h} & F(\text{Sym}^2 E) \\
\downarrow & & \downarrow \\
\text{some} & \text{cohomology} & \text{cohomology} \\
\end{array}
\]

\[
\begin{array}{ccc}
P-Z(t^*) \subset P = P(\text{Sym}^2 E) & \xrightarrow{\pi^*} & \text{Sym}(S_1, S_2)^* \\
\downarrow & & \downarrow \\
\pi^* & \text{cohomology} & \pi^* \\
\end{array}
\]

\[
\begin{array}{ccc}
P-Z(t^*) \subset P = P(\text{Sym}^2 E) & \xrightarrow{\pi^*} & \text{Sym}(S_1, S_2)^* \\
\downarrow & & \downarrow \\
\pi^* & \text{cohomology} & \pi^* \\
\end{array}
\]

**Proposition 9.3.** The natural map \(h: P \rightarrow P'\) sends \(P-Z(t^*)\) to \(P'-Z(u^*)\).

**Proof.** Since

\[
t^*(x, V_1 \subset V_2 \subset E_x, \phi \in \text{Sym}(V_1, V_2)) = t(x, V_1 \subset V_2 \subset E_x)^*(\phi) = (u(x)|_{V_1 \times V_2})^*(\phi) = u(x)^*(\phi) = u^*(x, \phi) = u^*(h(x, V_1 \subset V_2, \phi)),
\]

\(t^*(\cdot) \neq 0\) iff \(u^*(h(\cdot)) \neq 0\). Hence \(h\) sends \(P-Z(t^*)\) to \(P'-Z(u^*)\). \(\blacksquare\)

To apply the cohomology lemma (Lemma 5.4) it is now necessary to compute the fiber dimension of \(h\).

**§ 10. The fibers of \(h\)**

The map \(h: P(\text{Sym}(S_1, S_2)) \rightarrow P(\text{Sym}^2 E)\) can be factored into a composition of two natural maps \(h_1\) and \(h_2:\)

\[
\begin{array}{ccc}
P(\text{Sym}(S_1, S_2)) & \xrightarrow{h_1} & P(\text{Sym}^2 S) \\
\downarrow & & \downarrow \\
F(e-p-1, e-p, E) & \xrightarrow{h_2} & X,
\end{array}
\]

where \(S\) is the universal subbundle over the Grassmann bundle \(G(e-p, E)\),

\[
h_1(x, V_1 \subset V_2 \subset E_x, \phi \in \text{Sym}(V_1, V_2)) = (x, V_2 \subset E_x, \phi \in \text{Sym}^2 V_2),
\]

and

\[
h_2(x, V_2 \subset E_x, \phi \in \text{Sym}^2 V_2) = (x, \phi \in \text{Sym}^2 E).
\]

In [13, Section 3] we analyzed the fibers of \(h_2\), and found that if \((x, \phi \in \text{Sym}^2 E_x) \in P(\text{Sym}^2 E)\), then
\[ h_2^{-1}(x, \phi) \simeq \{ V_2 \in G(e-p, E_x) | \text{im} \phi \subset V_2 \subset E_x \} \]
\[ \simeq G(e-p-rk \phi, e-rk \phi). \]

We now analyze the fibers of \( h_1 \). Let \((x, V_2 \subset E_x, \phi \in \text{Sym}^2 V_2)\) be an element of \( P(\text{Sym}^2 S) \). Denote by \( \tilde{\phi} : V_2^* \otimes V_2^* \to C \) the symmetric linear map associated to \( \phi \). Then
\[
h_1^{-1}(x, V_2, \phi) = \{ V_1 \in G(e-p-1, V_2) | \phi \in \text{Sym}(V_1, V_2) \} \]
\[ \simeq G_{\tilde{\phi}}(1, V_2^*) \quad \text{(by Prop. 8.4)} \]
\[ \simeq \text{a quadric in } P^{e-p-1}. \]

Therefore, for \((x, \phi) \in P(\text{Sym}^2 E)\),
\[
\dim_c h_1^{-1}(x, \phi) = \dim_c G_{\tilde{\phi}}(1, e-p) + \dim_c G(e-p-rk \phi, e-rk \phi)
\]
\[ = e-p-2 + (e-p-rk \phi) p \]
\[ = (e-p-rk \phi)(p+1) + rk \phi - 2. \]

§ 11. Completing the proof

Returning to Diagram 9.2, our goal now is to compute the cohomology of \( P - Z(t^*) \) by applying Lemma 5.4. Stratifying \( U = P' - Z(u^*) \) by rank, we let \( Y_k = U_{e-p-k} = P(D_{e-p-k}(\text{Sym}^2 E)) - Z(u^*) \) be the locus of rank \( \leq e-p-k \) in \( U \) as in Section 3. Since \( \text{Sym}^2 E^* \) is ample, \( U \) is affine, and each \( Y_k \), being a closed subvariety of \( U \), is also affine. Then
\[ \ldots \subset Y_{k+1} \subset Y_k \subset \ldots \subset Y_0 \]
and if \((x, \phi) \in Y_k - Y_{k+1}\),
\[ \dim_c h_1^{-1}(x, \phi) = k(p+1) + e-p-k-2 = (k-1)p + e - 2 \]
by (10.1). In the cohomology comparison lemma (5.4) set \( d(k) = (k-1)p + e - 2. \) Then
\[
R = \max_{k \geq 0} \{ \dim_c Y_k + 2(k-1)p + 2e - 4 \}
\]
\[ = \max_{k \geq 0} \left\{ \dim_c P' - \frac{p+k+1}{2} + 2(k-1)p + 2e - 4 \right\} \]
\[ = \max_{k \geq 0} \left\{ \dim_c P' - \frac{p-k}{2} + 2e - 3p - 4 \right\} \]
\[ = \dim_c P' + 2e - 3p - 4 \]
\[ = \dim_c X + \frac{e+1}{2} + 2e - 3p - 5 \]
\[ = \dim_c X + \frac{e^2 + 5e}{2} - 3p - 5. \]
By Lemma 5.4,

\[ H^q(P - Z(t^*); Z) = 0 \quad \text{for} \quad q \geq \dim_c X + \frac{e^2 + 5e}{2} - 3p - 4. \]

By (9.1) the dimension of the flag bundle \( F = F(e - p - 1, e - p, E) \) is

\[
\dim_c F = \dim_c X + (e - p - 1)1 + (e - p)p \\
= \dim_c X + (e - p)(p + 1) - 1.
\]

A straightforward computation shows that

\[
\dim_c X \geq \binom{e - 2p}{2} + e - 2p - 1 \iff 2 \dim_c F - 1 \geq \dim_c X + \frac{e^2 + 5e}{2} - 3p - 4.
\]

By hypothesis, \( \dim_c X \geq (e - (2p + 1)^2 + 1) + e - (2p + 1) \). Hence,

\[ H^q(F - Z(t); Z) = H^q(P - Z(t^*); Z) = 0 \]

for \( q = 2 \dim_c F, 2 \dim_c F - 1 \). As in Section 5 this implies that \( Z(t) \) and hence \( D_{2p+1}(u) \) is connected.

References